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Uncertainty Quantification of Radio Propagation Using Polynomial Chaos

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Abstract—In this paper we demonstrate how so-called polynomial chaos expansions can be used to create efficient algorithms for uncertainty quantification in some classes of problems related to wave propagation in stochastic environment. We provide an example from telecommunication.

1. INTRODUCTION

Briefly, uncertainty quantification (UQ) is a scientific discipline that aims at quantifying uncertainties in various applications. Hence, it can be used to characterize how likely an output is and therefore, as an example, be an important support in decision-making. See [14] for an introduction and further references.

In most fields, there exists a widespread use of modeling and simulation tools. In spite of this, to provide, in some sense, objective confidence intervals for the numerical predictions is in general a difficult task. High-impact decisions require rigorous estimates of the confidence. To the greatest extent possible, we try to represent real world systems, and hence we aim at getting the best model possible which in general implies that we get complex models which even in the ideal situation can require a lot of resources to run.

When a model is verified, a part of the Validation and Verification (V&V) process is to compare numerical results with physical observations. Rigorous estimates of the uncertainties are important to be able to estimate the predictive quality of a model. Thus, we see that UQ is an important part of V&V since any kind of comparison between physical observations and output from associated numerical models must include the quantification of uncertainty.

From a computational point of view, the technique we use to propagate the uncertainty is highly interesting. Many techniques exist, and these techniques can be binary classified into *intrusive methods* and *nonintrusive methods*. Nonintrusive methods require a deterministic model, where, of course, the model should be well defined. Nonintrusive models require the deterministic model to be run multiple times. Monte Carlo (MC) methods are examples of nonintrusive methods [4, 13]. Intrusive methods require a stochastic formulation of the problem. An example here is Stochastic Galerkin type methods, and we will discuss this type of methods further in Section 2.

The last step in a normal UQ-process is quantification of the confidence. The way we do this will be strongly dependent on the way we propagate the uncertainty and the measure we consider to be relevant. We provide an example regarding how it can be done in Section 2.

The idea of incorporating and quantifying uncertainty in our analysis is nontrivial in practice since we, in general, require efficient ways of calculating and evaluating uncertainties. When it comes to uncertainties in, for example, parameters within certain models we might use various families of orthogonal polynomials to find efficient ways to quantify the uncertainty of certain calculated results. In Section 2, we introduce polynomial chaos and relate them to the construction of efficient algorithms for numerical solutions of certain partial differential equations.

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More precisely, we will, in Section 3, consider a scenario within the field of radio propagation. We assume that we have a limited amount of information in the initial field. This uncertainty can for example come from algorithmic uncertainties in previous calculations or model uncertainties. Furthermore, when the radio wave propagates we have a section in the environment where we do not have complete information on some of the dielectric parameters. We use a paraxial approximation as model for the propagation of the wave in this environment. In [3] and [1], a similar technique was applied on the Maxwell equations.

2. A STOCHASTIC GALERKIN METHOD

We derive things in a formal way in this section, and note that we need to make assumptions and derive consequences to put this section on a solid mathematical ground. The main objective is to introduce the idea.

Let $(\Omega, \mathcal{M}, R)$ be an abstract probability space, where Ω is the sample space, $\mathcal{M}$ the set of events, and R the probability measure. Assume that $Y : \Omega \to Y(\Omega) \stackrel{\text{def}}{=} \Upsilon \subset \mathbb{R}^d$ is a random variable and denoted by μ_Y, the distribution of Y.

In a standard way, we define the space $L^2(\Upsilon, \mathcal{M}_Y, \mu_Y)$ equipped with the inner-product

$$\langle f, g\rangle \stackrel{\text{def}}{=} \int_\Upsilon fg \mathrm{d}\mu_Y \tag{1}$$

and for simplicity we write $L^2(\Upsilon, \mu_Y)$.

Now, consider the kth order partial differential equation

$$\mathcal{E}\left(x, Y(\omega), D_x^k u(x, Y(\omega)), \ldots, u(x, Y(\omega))\right) = 0, \tag{2}$$

where $(x, Y(\omega)) \in \mathcal{D} \times \Upsilon$ and the equation is interpreted in a classical sense for each fixed ω with $\mathcal{D}$ an open set in $\mathbb{R}^N$. We assume that this equation is supported with appropriate boundary conditions on $\partial\mathcal{D}$. Here the mapping $\mathcal{E}$ is an appropriate functional and u a sufficiently smooth function on $\mathcal{D} \times \Upsilon$.

Define

$$S_P \stackrel{\text{def}}{=} \operatorname{Span}\{\Psi_0(y), \ldots, \Psi_P(y)\} \subset L^2(\Upsilon, \mu_Y), \tag{3}$$

where we want to choose $\{\Psi_p\}_{p=1}^P$ as an ON-set in $L^2(\Upsilon, \mu_Y)$. The idea is to solve Eq. (2) restricted to S_P. In other words, we project the coefficients of the PDE onto S_P and solve the associated equation. Let $\mathbb{P}_k$ denote the polynomials with degree less than or equal to k and $\hat{\mathbb{P}}_k$ the set of polynomials in $\mathbb{P}_k$ orthogonal to $\mathbb{P}_{k-1}$. Cameron-Martin theorem [2, 18] states (for standard Gaussians) that

$$u = u_0 P_0 + \sum_{i_1=1}^{\infty} u_{i_1} \hat{P}_1(Q_{i_1}) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty} u_{i_1,i_2} \hat{P}_2(Q_{i_1}, Q_{i_2}) + \ldots \tag{4}$$

holds true for any $u \in L^2(\Omega, \mathcal{M}, R)$, and the convergence is strong in norm. Here $Q_{i_1}, Q_{i_2}, \ldots$ are stochastic variables defined on Ω. The efficiency and accuracy of any algorithm involving expansions of this type will depend on the creation of S_P.

For the highest order polynomial in Eq. (4), n, and the number of stochastic processes, d, we can write

$$u \approx u_P \stackrel{\text{def}}{=} \sum_{p=0}^{P} u_p \Psi_p(y), \quad \text{where} \quad P = \frac{(n+d)!}{n!d!} - 1. \tag{5}$$

If the solution is not sufficiently smooth in y, see for example [17].

We can now, using Eq. (2), write

$$\langle \mathcal{E}|_{S_P}, \Psi_p\rangle = 0, \quad \forall p, \tag{6}$$

with the corresponding operations for the boundary. Hence, the name Galerkin. Depending on the structure of the equation there will be different ways of solving this equation.

As we mentioned in Section 1, the final part is to quantify the confidence. This is the calculation of

$$\mathbf{E}\left[f(u)\right] \approx \int_{\Upsilon} f\left(\sum_{p=0}^{P} u_p \Psi_p\right) \mathrm{d}\mu_Y, \tag{7}$$

for an appropriately chosen function f, depending on the measure we are interested in.

A final note is that we can actually derive approximate realizations of the true distribution, and hence if we have found a sufficiently good approximation, we may use this as an algorithm for sampling or as a basis for a detection algorithm.

3. STOCHASTIC WAVE PROPAGATION

Computational electromagnetic modelling (CEM) [9, 15] of the interaction between electromagnetic fields and an environment will in general involve parameters associated with a limited degree of information. These parameters are, according to our belief, in general represented in terms of probabilities. Obviously these uncertainties will imply uncertainties in the results. One way of handling these uncertainties is introduced in this section.

Let $u \in C^\infty(\Xi)$, where $\Xi := \{(x, z) : x > 0 \text{ and } z > 0\}$. Fock and Leontovich introduced, in [7, 10], a model that can be used for wave propagation, see also [11]. We use

$$D \stackrel{\text{def}}{=} \frac{1}{4}\frac{1}{k^2}\partial^3_{z^2x} - \frac{i}{2k}\partial^2_z + \frac{n^2+3}{4}\partial_x - ik\frac{n^2-1}{2}, \tag{8}$$

where $n \in \mathbb{C}$ will be defined later in this section, and k is the (angular) wavenumber as the governing operator. There are many different approximations, based on the theory of pseudo-differential operators, but this one is quite common to work with in telecommunication and propagation in unguided medium. Parabolic approximations are used in many different fields for example underwater acoustics [16] and geophysics [5]. Note here that this model, although extensively used for deterministic scenarios, introduces model uncertainty into our analysis. This equation has also previously been studied, see for example [8] and the references therein. To the best of the authors' knowledge, this equation has previously not been studied in a stochastic framework. Consider

$$\begin{cases} Du = 0, \\ \lim_{|(x,z)|\to\infty} |u(x, z)| = 0, \\ \partial_z u|_{z=0} = -\Gamma u|_{z=0}, \\ u|_{x=0} = f(z), \end{cases} \tag{9}$$

where $f : \mathbb{R}_+ \to \mathbb{C}$ is the initial condition generated from a source. Here

$$\Gamma(x) \stackrel{\text{def}}{=} \begin{cases} ik\sqrt{\epsilon(x)-1}, & \text{HED}, \\ ik\dfrac{\sqrt{\epsilon(x)-1}}{\epsilon(x)}, & \text{VED}. \end{cases} \tag{10}$$

Define a grid, $G(\Delta x, \Delta z)$ with $u^m_l \stackrel{\text{def}}{=} u(x_m, z_l)$ constant, Δx the size of the grid in x and Δz in z. Let $x_m \stackrel{\text{def}}{=} (m + \frac{1}{2})\Delta x$, $z_l \stackrel{\text{def}}{=} (l + \frac{1}{2})\Delta z$ and let m, l be nonnegative integers. We note that depending on our position in the grid we will get different refraction indices with different degrees of information. Therefore, we introduce the notation below. Put

$$M \stackrel{\text{def}}{=} \left\{(m, l) \in \mathbb{N}^2 : \; a_x \le m \le A_x \text{ and } a_z \le l \le A_z\right\}, \tag{11}$$

where $a_x \le A_x$ and $a_z \le A_z$ denote an area in the grid with an uncertainty in the imaginary part of n. If (m, l) or $(m+1, l)$ belongs to M and (m, l) or $(m+1, l)$ not belong to M (in the exclusive sense) we say that the element (m, l) belongs to M_b. When (m, l) and $(m + 1, l)$ belong to M, we say that the element belongs to M_u, and when (m, l) and $(m + 1, l)$ do not belong to M, we say that the element belongs to M_d. Put

$$n^m_l(X)^2 \stackrel{\text{def}}{=} \epsilon + d^m_l X, \quad \text{where} \quad d^m_l \stackrel{\text{def}}{=} \frac{i}{\omega\epsilon_0} \cdot \begin{cases} 0, & (m, l) \in M_d, \\ 1, & (m, l) \in M_u \cup M_b. \end{cases} \tag{12}$$

Here $X \sim \mathcal{U}(a,b)$ is the stochastic variable representing the uncertainty associated with the imaginary part of the permittivity.

We use the Crank-Nicolson method [6] to find the approximate solution and hence introduce an algorithmic uncertainty into the propagation. This results in a system of the form

$$Au^{m+1} = Bu^m \tag{13}$$

for a fixed m. In other words, there will be a system of deterministic equations that we need to solve in each step. To be explicit, the system based on Eq. (9) reads

$$u_{l+1}^{m+1}(X) + \alpha_l^{m+\frac{1}{2}}(X)u_l^{m+1}(X) + u_{l-1}^{m+1}(X) = \gamma\left(u_{l+1}^m(X) + \beta_l^{m+\frac{1}{2}}(X)u_l^m(X) + u_{l-1}^m(X)\right), \tag{14}$$

where

$$\gamma \overset{\text{def}}{=} \frac{1 + i2k\Delta x - (k\Delta x)^2}{1 + (k\Delta x)^2}, \tag{15}$$

$$\alpha_l^{m+\frac{1}{2}}(X) \overset{\text{def}}{=} (k\Delta z)^2\left(\xi_l^{m+\frac{1}{2}}(X) - 1\right) + 4ik\frac{\Delta z^2}{1 - ik\Delta x} - 2, \quad l > 0, \tag{16}$$

and

$$\beta_l^{m+\frac{1}{2}}(X) \overset{\text{def}}{=} (k\Delta z)^2\left(\xi_l^{m+\frac{1}{2}}(X) - 1\right) - 4ik\frac{\Delta z^2}{1 - ik\Delta x} - 2, \quad l > 0. \tag{17}$$

Here

$$\xi_l^{m+\frac{1}{2}}(X) \overset{\text{def}}{=} \left(\frac{n_l^m(X) + n_l^{m+1}(X)}{2}\right)^2. \tag{18}$$

In addition, when we take the boundary conditions into account,

$$\alpha_0^{m+\frac{1}{2}} \overset{\text{def}}{=} \frac{1}{2}\left((k\Delta z)^2\left(\xi_0^{m+\frac{1}{2}}(X) - 1\right) + 4ik\frac{\Delta z^2}{1 - ik\Delta x} - 2\right) + \Gamma^{m+1}\Delta z - ik\Delta x\Delta z\Gamma^{m+\frac{1}{2}}, \tag{19}$$

and

$$\beta_0^{m+\frac{1}{2}} \overset{\text{def}}{=} \frac{1}{2}\left((k\Delta z)^2\left(\xi_0^{m+\frac{1}{2}}(X) - 1\right) - 4ik\frac{\Delta z^2}{1 - ik\Delta x} - 2\right) + \Gamma^m\Delta z + ik\Delta x\Delta z\Gamma^{m+\frac{1}{2}}, \tag{20}$$

where $\Gamma^m \overset{\text{def}}{=} \Gamma(x_m)$ and $\Gamma^{m+\frac{1}{2}} \overset{\text{def}}{=} \frac{1}{2}(\Gamma^m + \Gamma^{m+1})$. We initiate the field in u_l^0 defined for all l. The asymptotic condition will be handled by a Hann-Poisson filter.

We will proceed by introducing $L^2(\mathbb{R}, \mathrm{dm})$, where $\mathrm{dm} \overset{\text{def}}{=} \chi_{[-1,1]}\mathrm{d}\mu$ with the inner-product

$$\langle f, g\rangle \overset{\text{def}}{=} \int_{\mathbb{R}} fg\,\mathrm{dm}. \tag{21}$$

Fix $P > 0$ and project $u_l^m(X)$ onto the span of $\{L_p(Z)\}_{p=0}^P$, where $P > 0$ and $Z \sim \mathcal{U}(-1,1)$. Here L_p is the univariate Legendre polynomial of degree p, which, in this case, will correspond to the function Ψ_p in Section 2. This will result in

$$u_l^m(X) \approx \sum_{p=0}^P {}^p u_l^m L_p(Z). \tag{22}$$

Therefore, Eq. (14) can be written as

$${}^q u_{l+1}^{m+1} + \sum_{p=0}^P {}_q^p e_l^{m\,p} u_l^{m+1} + {}^q u_{l-1}^{m+1} = \gamma\left({}^q u_{l+1}^m + \sum_{p=0}^P {}_q^p g_l^{m\,p} u_l^m + {}^q u_{l-1}^m\right), \tag{23}$$

where $q = 0, 1, \ldots, P$, with

$${}_q^p e_l^m \overset{\text{def}}{=} \frac{\left\langle \alpha_l^{m+\frac{1}{2}}(X)L_p, L_q\right\rangle}{\langle L_q, L_q\rangle}, \tag{24}$$

and

$$ {}_{q}^{p}g_{l}^{m} \overset{\text{def}}{=} \frac{\left\langle \beta_{l}^{m+\frac{1}{2}}(X) L_p, L_q \right\rangle}{\langle L_q, L_q \rangle}. \tag{25} $$

In Section 5, we will present a brief analysis of the algorithm.

4. A NUMERICAL EXAMPLE

We consider a scenario with a source located 13 m above ground, vertical polarization, and working at 50 MHz. To initiate the field, we use an approximative solution (by Norton), u_0^l, of the Sommerfeld's half-space problem [12].

The dielectric parameters we used are $\epsilon_{\text{air}} = \epsilon_{\text{vegetation}} = 1.00032$, $\epsilon_{\text{ground}} = 15$, $X_{\text{air}} = 0.0\,[\text{S/m}]$, $X_{\text{ground}} = 5 \cdot 10^{-3}\,[\text{S/m}]$, and $X_{\text{vegetation}} \sim \mathcal{U}(20 \cdot 10^{-6}, 50 \cdot 10^{-6})\,[\text{S/m}]$. We use a grid with $\Delta x = 4.5\,[\text{m}]$ and $\Delta z = 1.5\,[\text{m}]$ for $\Xi = \{(x, z) : 0 < x < 3000 \text{ and } 0 < z < 60\}$. The scenario can be found in Figure 1.

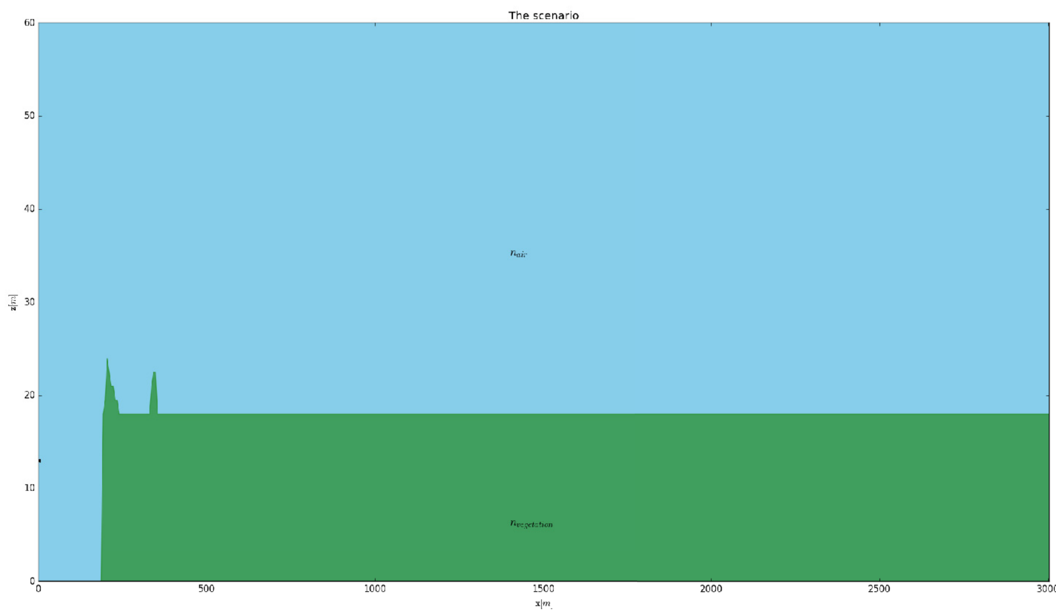

Figure 1. The scenario used in the numerical example. Blue — air with known refraction index. Green — vegetation modelled with a stochastic refraction index. The dielectric parameters we used are $\epsilon_{\text{air}} = \epsilon_{\text{vegetation}} = 1.00032$, $\epsilon_{\text{ground}} = 15$, $X_{\text{air}} = 0.0$ [S/m], $X_{\text{ground}} = 5 \cdot 10^{-3}$ [S/m], and $X_{\text{vegetation}} \sim \mathcal{U}(20 \cdot 10^{-6}, 50 \cdot 10^{-6})$ [S/m].

Introduce the propagation factor, $C_{\text{PF}} := 20 \log |\frac{E}{E_0}|$, where E_0 is the field under free-space conditions. In Figure 2 and Figure 3, the calculated propagation factor can be found for a linear polynomial ($P = 1$) and quadratic polynomial ($P = 2$), respectively. As a comparison, we run a non-intrusive algorithm that consists of simply using randomly sampled realizations of the uncertainty associated with the vegetation and running an implementation of the standard algorithm. In Figure 4 and Figure 5, the calculated propagation factor is given, for 5 realizations and 1000 realizations, respectively. The sample mean and the unbiased sample variance estimator were used as estimators of the associated estimands.

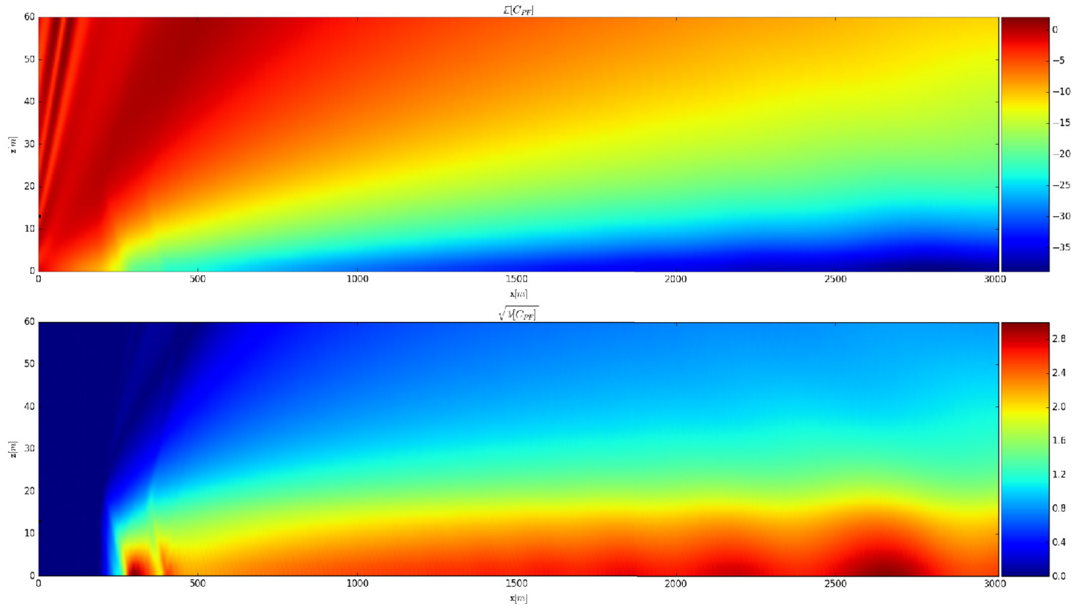

Figure 2. First two moments, expectation value (top) and standard deviation (bottom), of the propagation factor using linear polynomials.

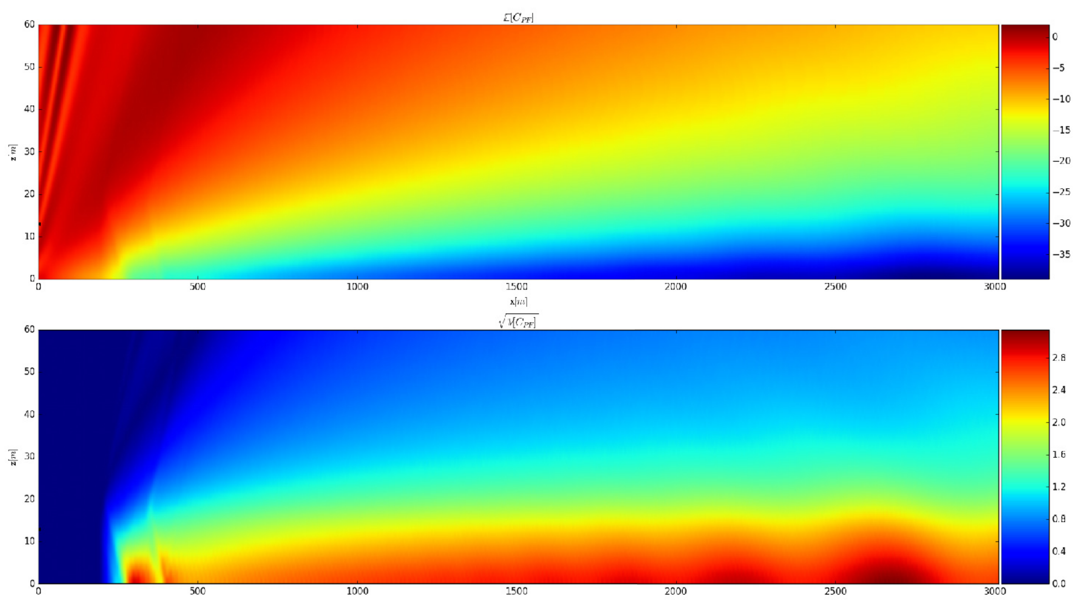

Figure 3. First two moments, expectation value (top) and standard deviation (bottom), of the propagation factor using quadratic polynomials.

5. DISCUSSION OF NUMERICAL RESULTS AND CONCLUSIONS

We will now provide a comparison in performance, with respect to computational effort, between the two different methods studied in this paper along with a discussion of the numerical results.

Recall that P is the degree of the polynomial used in the intrusive method and that the results can be found in Figure 2, Figure 3, Figure 4, and Figure 5.

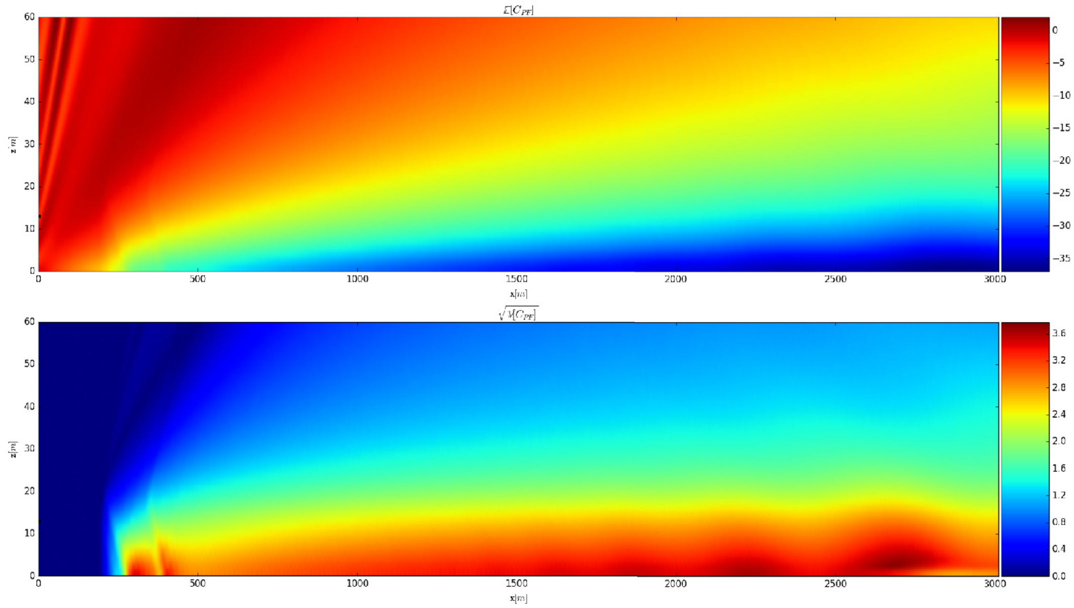

Figure 4. First two moments, expectation value (top) and standard deviation (bottom), of the propagation factor estimated from 5 realizations.

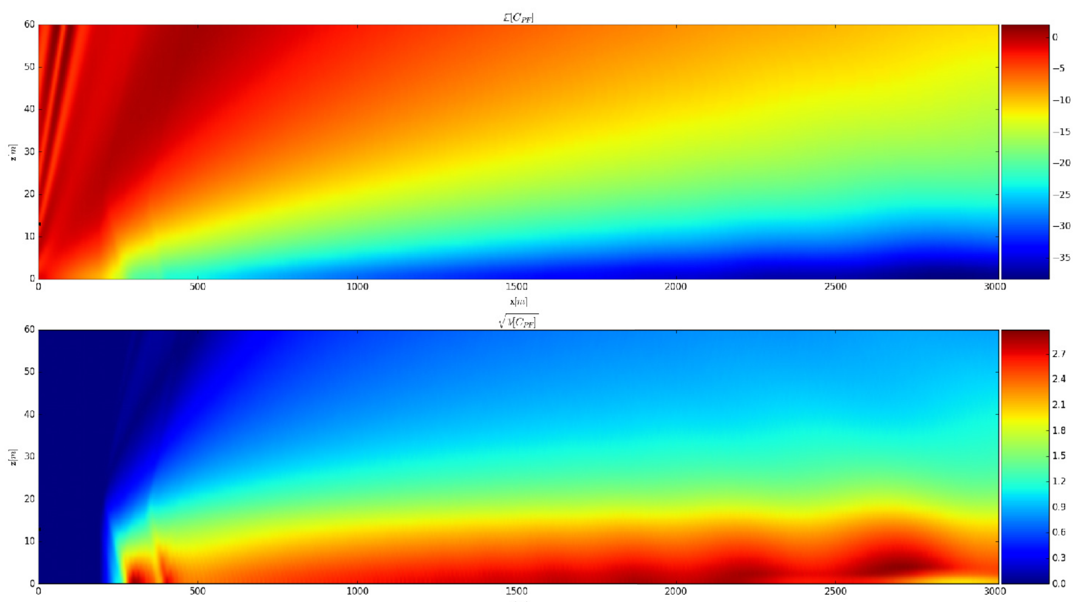

Figure 5. First two moments, expectation value (top) and standard deviation (bottom), of the propagation factor estimated from 1000 realizations.

The first step will be to calculate the matrices A and B in Equation (13). The extra computational effort required by the intrusive method compared to the non-intrusive method is the evaluation of P^2 integrals one time, that is, when the propagation enters the area with a limited degree of information.

Until the propagation enters the area with a limited degree of information, the speed of propagating the solution is equal between the methods. Once we propagate into an area with limited degree of information, the matrix A will get a new structure, and we need to create a new optimized linear solver. Once we again propagate into an area with complete information, the stepping part will require approximately P times the time for the non-intrusive stepping with respect to one sample (the standard case).

Once we are done stepping through the grid, an integral for both the first moments needs to be evaluated in each grid cell. Note that the reason for this requirement is that we are interested in estimating the first two moments of the stochastic variable C_{PF}. If we were interested in moments of other stochastic variables, they could be calculated explicitly in terms of the coefficients associated with S_p, introduced in Eq. (3).

In total, we can see that there is a potential huge speedup using the method suggested in this paper. Further studies with respect to computational effort, accuracy, and other numerical properties need to be conducted but at a given point in the grid the intrusive code (which can be optimized further) for the linear case required roughly the same time as for 5 non-intrusive runs.

The focus of this paper was to demonstrate a possibly efficient method to estimate uncertainties, as discussed above. However, a few words about the results are appropriate. To begin, we note that linear and quadratic polynomial expansions yield similar results, i.e., the intrusive method seems to have converged for a fixed grid. However, for the non-intrusive approach there are what we believe to be nonphysical structures in the standard deviation close to the ground between 2500 and 3000 m. One possibility is that we do not have a sufficient number of samples to get a good estimate with this particular non-intrusive method. Besides that, the expectation values of the propagation factor C_{PF} for the two methods agree well, as well as the standard deviation. We would like to stress that there might be other non-intrusive methods that perform significantly better than the one we used, but this is beyond the scope of this paper.

An obvious observation is that the propagation factor becomes very small in the modelled vegetation but with a significant uncertainty.

We note that there is a significant uncertainty (standard deviation is about 1.6 dB) in C_{PF} in parts of the deterministic part of the domain, which is most pronounced right above the vegetation. In fact, the uncertainty propagates up in the whole computational volume above the vegetation.

A final remark is that quantification of uncertainties related to EM wave propagation is important for, e.g., detection and tracking applications.

Further validation cases will be presented in the future.

REFERENCES

1. Austin, A. C. M. and C. D. Sarris, "Efficient analysis of geometrical uncertainty in the fdtd method using polynomial chaos with application to microwave circuits," *IEEE Transactions on Microwave Theory and Techniques*, Vol. 61, No. 12, 4293–4301, Dec. 2013.
2. Cameron, R. H. and W. T. Martin, "The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals," *Annals of Mathematics*, Vol. 48, No. 2, 385–392, 1947.
3. Chauvire, C., J. S. Hesthaven, and L. Lurati, "Computational modeling of uncertainty in time-domain electromagnetics," *SIAM Journal on Scientific Computing*, Vol. 28, No. 2, 751–775, 2006.
4. Chen, M.-H., Q.-M. Shao, and J. G. Ibrahim, *Monte Carlo Methods in Bayesian Computation*, Springer, New York, 2000.
5. Claerbout, J. F., *Imagining the Earth's Interior*, Blackwell Scientific Pub., Palo Alto, CA, 1985.
6. Crank, J. and P. Nicolson, "A practical method for numerical evaluation of solutions of partial differential equations of the heat conduction type," *Proc. Camb. Phil. Soc.*, Vol. 43, 50–67, 1947.
7. Fock, V. A., *Electromagnetic Diffraction and Propagation Problems*, Pergamon Press, 1965.
8. Holm, P., "Wide-angle shift-map PE for a piecewise linear terrain finite-difference approach," *IEEE Transactions on Antennas and Propagation*, Vol. 55, 2773–2789, 2007.
9. Jin, J. M., *Theory and Computation of Electromagnetic Fields*, Wiley, 2011.
10. Leontovich, M. A. and V. A. Fock, "Solution of propagation of electromagnetic waves along earth's surface by the method of parabolic equations," *J. Physics, USSR*, Vol. 10, 13–23, 1946.
11. Levy, M., "Parabolic equation methods for electromagnetic wave propagation," *IET*, 2000.

12. Norton, K. A., "The propagation of radio waves over the surface of the earth and in the upper atmosphere," *Proceedings of the Institute of Radio Engineers*, Vol. 25, 1203–1236, 1937.
13. Robert, C. P. and G. Casella, *Monte Carlo Statistical Methods (Springer Texts in Statistics)*, Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2005.
14. Smith, R. C., "Uncertainty quantification: Theory, implementation, and applications," *Computational Science and Engineering*, 2013.
15. Taflove, A. and S. C. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, 3rd Edition, Artech House, Jun. 2005.
16. Tapper, F. D., "The parabolic approximation method," *Wave Propagation and Underwater Acoustics*, Vol. 70, 224–287, 1977.
17. Wan, X. and G. E. Karniadakis, "An adaptive multi-element generalized polynomial chaos method for stochastic differential equations," *Journal of Computational Physics*, Vol. 209, No. 2, 617–642, 2005.
18. Wiener, N., "The homogeneous chaos," *American Journal of Mathematics*, Vol. 60, No. 4, 897–936, 1938.