

Modified GO Solutions for the High Frequency Reflected Wave in the Focal Region of a 3D Elliptical Reflector Placed in Isotropic Chiral Medium

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Abstract—High frequency electromagnetic (EM) fields in the focal region of a 3D elliptical reflector placed in a homogeneous and reciprocal chiral background have been analyzed using geometrical optics (GO) approximation and Maslov’s method. The GO solutions becomes invalid at the focal region of a 3D elliptical reflector due to unreal singularities. Therefore, an asymptotic method based on Maslov’s theory has been applied to derive high frequency EM fields, which is also valid at the focal points. Moreover, the effect of chirality parameter of the background medium on the position of focal points for both Left circularly polarized wave (LCP) and right circularly polarized (RCP) wave are described by plotting the derived expressions numerically using MATLAB.

1. INTRODUCTION

Chiral medium has been studied extensively since early nineteenth century due to its potential applications in field optics, material sciences, photonics, chemistry, particle physics, life sciences acoustic, and seismic systems [1–3]. Due to extraordinary characteristics, easy fabrication, availability and potential diverse applications in electromagnetic waves propagation, radiation, scattering, such media have drawn the attention of many researchers over the last decade. Chiral medium can be realized practically by using miniature spiral wires, conducting springs and other planner or non-planner chiral objects. These chiral objects provide cross coupling between electric and magnetic fields inside the medium, and thus the medium exhibits artificially tunable electromagnetic properties which cannot be found in conventional medium. Apart from these general applications, focusing systems, associated with chiral medium and having different potential applications, are illustrated in [4–6]. These focusing systems (lenses and reflectors) with different shapes are suitable to be used in feed arrays in imaging, satellite communication, optical-fiber communications, and design of multiple beam antenna systems ranging from microwave to optical frequency range. The interaction of EM wave with these media has been analyzed using different analytical and computational methods such as Method of Moment (MOM), Finite Difference Time Domain (FDTD), Physical Optics (PO), Genetic Algorithm (GA), Geometric Optics (GO), Kirchhoff-Huygens integral method, Debye Wolf focusing integral method and various other methods and tools which are a combination of [7–11]. In this work, we use Geometrical Optics (GO) which is a ray-based technique to model EM fields for high-frequency field expressions for 3D elliptical reflector with chiral background. A general GO field expression can be calculated using Gauss’ theorem [5], and the relationship is given by

$$u(r) = E(r_0)J^{-1/2} \exp(-jk(s_0 + n^2t)) \quad (1)$$

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In the above equation, $E(r_0)$ is the initial absolute value of field intensity of the wave. The Jacobian is used as a transformation from the ray coordinates (ξ, η, ζ) to Cartesian coordinate (x, y, z) , which can be calculated by $J = D(t)/D(0)$, where $D(t) = \partial(x, y, z)/\partial(\xi, \eta, t)$. The transport equation is used to find the amplitude of the GO expression, and the Eikonal equation is used to find the phase of the wave. The calculated GO approximation fails at the focal points due to shrinking of the ray tube to zero, which is unreal. These points are derived by putting the term $J = 0$. Moreover, these points are often of great practical importance in all practical applications. The GO approximation is amended by an asymptotic method based on Maslov's theory which combine the ray theory and Fourier transform. Maslov's method is applied to find the field around the focal points of different reflectors over the years as given in [12–21]. Fourier integral is used to find the stationary phase points from phase function. The general form of the field expression which is also valid around the focal points using Maslov's method is given in [12]

$$u(r) = \frac{k}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_T(\xi, \eta) \left(\frac{D(t)}{D(0)} \frac{\partial(p_x, p_y)}{\partial(x, y)} \right)^{-\frac{1}{2}} \exp[-jk\{s_0 + n^2t - x(p_x, p_y, z)p_x - y(p_x, p_y, z)p_y + xp_x + yp_y\}] dp_x dp_y \quad (2)$$

The term used $\frac{D(t)}{D(0)} \frac{\partial(p_x, p_y)}{\partial(x, y)}$ in the above equation be derived using the expression given by

$$\frac{D(t)}{D(0)} \frac{\partial(p_x, p_y)}{\partial(x, y)} = \frac{1}{D(0)} \frac{\partial(p_x, p_y, z)}{\partial(\xi, \eta, \zeta)}. \quad (3)$$

In this work, the focusing system is a 3D elliptical reflector with a homogeneous and reciprocal chiral medium as a background medium. Chirality is a structural phenomenon which creates optical activity in chiral medium. Optical activity is the ability of a medium to rotate the polarization plane of lineally polarized light or wave passes through the medium [26]. Due to the birefringence nature of the chiral medium, there exist two different types of waves in a chiral medium, i.e., right circularly polarized (RCP) and left circularly polarized (LCP) EM waves with different refraction indices due to different phase velocities [28]. The optical activity increases with the increase in the concentration of chiral objects into the host medium. Furthermore, optical activity also depends upon the depth of the chiral medium [24]. The interaction of electromagnetic waves with chiral medium has been studied over the years by many authors [18–28]. The constitutive relation used to describe the waves inside the chiral medium is represented by [28]

$$\mathbf{D} = \epsilon(\mathbf{E} + \beta \nabla \times \mathbf{E}), \quad \mathbf{B} = \mu(\mathbf{H} + \beta \nabla \times \mathbf{H}) \quad (4)$$

ϵ , μ , and β are permittivity, permeability and chirality parameter, respectively. ϵ and μ have their usual dimensions, and β has the dimension of length. By using the equations given in Eq. (4), the solution of Maxwell's equations results in coupled differential equations. The equations which make these differential equations for \mathbf{E} and \mathbf{H} uncoupled are given by [28]

$$\mathbf{E} = \mathbf{Q}_L - j\sqrt{\frac{\mu}{\epsilon}}\mathbf{Q}_R, \quad \mathbf{H} = \mathbf{Q}_R - j\sqrt{\frac{\epsilon}{\mu}}\mathbf{Q}_L \quad (5)$$

and \mathbf{Q}_L , \mathbf{Q}_R are RCP and LCP waves respectively and satisfy the following general wave equations

$$(\nabla^2 + n_1^2 k^2)\mathbf{Q}_L = 0, \quad (\nabla^2 + n_2^2 k^2)\mathbf{Q}_R = 0 \quad (6)$$

where, $n_1 = 1/(1 - k\beta)$ and $n_2 = 1/(1 + k\beta)$ are equivalent refractive indices for LCP and RCP waves, respectively, and $k = \omega\sqrt{\epsilon\mu}$. Eq. (6) shows that fields in the chiral medium can be derived similarly as calculated in conventional medium if we use the transformation from coupled differential equations to uncoupled differential equation given in Eq. (5). Therefore, GO approximation for chiral medium for both LCP and RCP waves are solved independently, and the total field can be obtained using the superposition theorem. The expressions for high frequency EM field using GO method and Maslov's method are derived and analyzed in the coming sections.

2. FIELDS AROUND THE CAUSTIC REGION OF THE ELLIPTICAL REFLECTOR USING GO AND MASLOV'S METHOD

In this paper, we want to derive the reflected field around the focal points of a 3D elliptical reflector with chiral background. Two incident waves (RCP and LCP) are assumed because chiral medium exhibits the property of converting a linearly polarized wave to LCP and RCP waves. The amplitudes for both RCP and LCP waves are taken as unity for simplicity, and their phase velocities are ω/kn_2 and ω/kn_1 , respectively. Four waves are reflected when both LCP and RCP waves hit the perfect electric conductor (PEC) 3D elliptical reflector. These waves are represented by RR, RL, LL and LR. RR and RL are RCP and LCP reflected wave components respectively, when RCP is incident on the PEC 3D elliptical surface, and LL and LR are LCP and RCP reflected waves respectively, when LCP wave excites the elliptical reflector antenna. Both RR and LL waves make an angle α along the z -axis, and their amplitudes are given by [12]

$$E_{LL}(r_0) = \frac{\cos \alpha - \cos \alpha_2}{\cos \alpha + \cos \alpha_2}, \quad E_{RR}(r_0) = \frac{\cos \alpha - \cos \alpha_1}{\cos \alpha + \cos \alpha_1} \quad (7)$$

Furthermore, the phase velocities for LL and RR waves are ω/kn_1 and ω/kn_2 , respectively. RL and LR waves have phase function ω/kn_1 and ω/kn_2 , respectively. The initial amplitudes of these reflected rays are given by

$$E_{RL}(r_0) = \frac{2 \cos \alpha}{\cos \alpha + \cos \alpha_1}, \quad E_{LR}(r_0) = \frac{2 \cos \alpha}{\cos \alpha + \cos \alpha_2} \quad (8)$$

RL and LR make angles α_1 and α_2 along z -axis, respectively. The values of these angles are derived from Snell's law and describe by

$$\alpha_1 = \sin^{-1} \left\{ \frac{n_1}{n_2} \sin \alpha \right\}, \quad \alpha_2 = \sin^{-1} \left\{ \frac{n_2}{n_1} \sin \alpha \right\} \quad (9)$$

The RL waves bend away from the 3D elliptical reflector if we consider the value of $\beta > 0$ which means that $n_1 > n_2$ and $\alpha_2 > \alpha$. On the other hand, LR wave bends towards the 3D elliptical reflector for $\beta > 0$. If $\beta = 0$, then only normal reflection takes place, and if the chirality parameter β increases the difference between the angles α and α_1, α_2 increases. If we take the value of $\beta < 0$, the roles of LCP and RCP waves are interchanged. The 3D elliptical reflector with isotropic and homogeneous chiral background is shown in Figure 1 and can be describe by the equation given below

$$\zeta = g(\xi, \eta) = \frac{a}{b} \sqrt{b^2 - \xi^2 - \eta^2} = \frac{a}{b} \sqrt{b^2 - \rho^2} \quad (10)$$

In the above relationship, a and b are the radii of the 3D dielectric elliptical reflector along the major and minor axes, respectively, where (ξ, η, ζ) are the initial values of (x, y, z) , $\rho^2 = \xi^2 + \eta^2$. In this case, the 3D reflector system has been placed in homogeneous and reciprocal chiral medium. Therefore, due to birefringence nature of the chiral medium, two waves (LCP and RCP) propagating along the z -axis

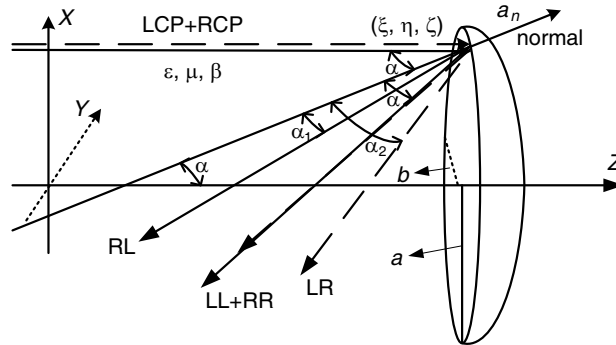


Figure 1. 3D elliptical PEC reflector embedded in chiral medium.

are considered as incident waves on the 3D elliptical shape. The relationship which validates the general wave equation is given by

$$\mathbf{Q}_L = (\mathbf{a}_x + j\mathbf{a}_y) \exp(-jkn_1z), \quad \mathbf{Q}_R = (\mathbf{a}_x - j\mathbf{a}_y) \exp(-jkn_2z) \quad (11)$$

where \mathbf{a}_x and \mathbf{a}_y are the unit vector along x -axis and y -axis, respectively. The polarization of the wave is ignored for the sake of simplicity, and the amplitude of the incident wave is taken as unity and given by

$$\mathbf{Q}_L = \exp(-jkn_1z), \quad \mathbf{Q}_R = \exp(-jkn_2z) \quad (12)$$

These waves make an angle α with the normal to the surface of a 3D elliptical reflector. The unit normal vector to the surface can be describe by

$$\mathbf{a}_n = \sin \alpha \cos \gamma \mathbf{a}_x + \sin \alpha \sin \gamma \mathbf{a}_y + \cos \alpha \mathbf{a}_z \quad (13)$$

where, α is the angle between normal vector and z -axis

$$\sin \alpha = \frac{-\dot{g}(\rho)}{\sqrt{1 + (\dot{g}(\rho))^2}} = \frac{a\rho}{\sqrt{b^2(b^2 - \rho^2) + a^2\rho^2}} \quad (14)$$

$$\cos \alpha = \frac{1}{\sqrt{1 + (\dot{g}(\rho))^2}} = \frac{b\sqrt{b^2 - \rho^2}}{\sqrt{b^2(b^2 - \rho^2) + a^2\rho^2}} \quad (15)$$

$$\tan \gamma = \frac{\eta}{\xi} \quad (16)$$

The prime sign in the above equation is for the derivative. The surface coordinates in term of polar coordinates (α, γ) for the 3D elliptical shape are expressed in the relationship given by

$$\xi = \frac{b^2 \sin \alpha \cos \gamma}{\sqrt{b^2 \sin^2 \alpha + a^2 \cos^2 \alpha}} \quad (17)$$

$$\eta = \frac{b^2 \sin \alpha \sin \gamma}{\sqrt{b^2 \sin^2 \alpha + a^2 \cos^2 \alpha}} \quad (18)$$

$$\zeta = \frac{a^2}{\sqrt{b^2 \tan^2 \alpha + a^2}} \quad (19)$$

The reflected wave vectors from the 3D elliptical reflector placed in homogeneous and reciprocal chiral medium are shown in Figure 1. The wave vector for the reflected wave from the elliptical reflector can be found by Snell's law given by

$$\mathbf{p}^r = \mathbf{p}^i - 2(\mathbf{p}^i \cdot \mathbf{a}_n) \mathbf{a}_n \quad (20)$$

where \mathbf{p}^i and \mathbf{p}^r are the wave vector for the incident wave and reflected wave, respectively. The reflected wave vector for LL, RR, RL and LR rays are calculated and given by the following expressions

$$\mathbf{p}_{LL} = -n_1 \sin 2\alpha \cos \gamma \mathbf{a}_x - n_1 \sin 2\alpha \sin \gamma \mathbf{a}_y - n_1 \cos 2\alpha \mathbf{a}_z \quad (21)$$

$$\mathbf{p}_{RR} = -n_2 \sin 2\alpha \cos \gamma \mathbf{a}_x - n_2 \sin 2\alpha \sin \gamma \mathbf{a}_y - n_2 \cos 2\alpha \mathbf{a}_z \quad (22)$$

$$\mathbf{p}_{RL} = -n_1 \sin(\alpha + \alpha_1) \cos \gamma \mathbf{a}_x - n_1 \sin(\alpha + \alpha_1) \sin \gamma \mathbf{a}_y - n_1 \cos(\alpha + \alpha_1) \mathbf{a}_z \quad (23)$$

$$\mathbf{p}_{LR} = -n_2 \sin(\alpha + \alpha_2) \cos \gamma \mathbf{a}_x - n_2 \sin(\alpha + \alpha_2) \sin \gamma \mathbf{a}_y - n_2 \cos(\alpha + \alpha_2) \mathbf{a}_z \quad (24)$$

The initial phases for LL, RR, RL, LR waves are $s_{LL}(r_0) = n_1\zeta$, $s_{RR}(r_0) = n_2\zeta$, $s_{RL}(r_0) = n_2\zeta$ and $s_{LR}(r_0) = n_1\zeta$. It is worth to note here that the apparent wave numbers of all rays, LL, LR, RL, and RR, make the difference in the reflected wave vector. All these rays have different effects on the focal points due to the difference in their phase velocities. Phase velocity changes with wave number which in turn varies with chirality parameter β of the chiral medium. The Jacobian term in Eq. (1) is used to transform ray coordinates to the Cartesian coordinates. The transformation for LL, LR, RL, and

RR rays reflected from 3D elliptical reflector placed in isotropic and homogeneous chiral medium are calculated as

$$D_{LL}(t) = \begin{vmatrix} 1 + \frac{\partial p_{LLx}}{\partial \xi} t & \frac{\partial p_{LLy}}{\partial \xi} t & \frac{\partial \zeta}{\partial \xi} + \frac{\partial p_{LLz}}{\partial \xi} t \\ \frac{\partial p_{LLx}}{\partial \eta} t & 1 + \frac{\partial p_{LLy}}{\partial \eta} t & \frac{\partial \zeta}{\partial \eta} + \frac{\partial p_{LLz}}{\partial \eta} t \\ p_{LLx} & p_{LLy} & p_{LLz} \end{vmatrix}$$

$$J_{LL} = \frac{D_{LL}(t)}{D_{LL}(0)}$$

$$= \left(\frac{2n_1 X t}{ab^2 \cos \gamma \sin \gamma} \right)^2 - \frac{4n_1 t}{b^2 \cos \alpha} \left(\frac{\sin^2 \alpha \cos^2 \alpha X^{\frac{1}{2}}}{\cos^2 \gamma \sin^2 \gamma} + \frac{4X^{\frac{3}{2}}}{a^2} + X^{\frac{1}{2}} \cos^2 \alpha \cos 2\alpha \right) + 1 \quad (25)$$

$$D_{RR}(t) = \begin{vmatrix} 1 + \frac{\partial p_{RRx}}{\partial \xi} t & \frac{\partial p_{RRy}}{\partial \xi} t & \frac{\partial \zeta}{\partial \xi} + \frac{\partial p_{RRz}}{\partial \xi} t \\ \frac{\partial p_{RRx}}{\partial \eta} t & 1 + \frac{\partial p_{RRy}}{\partial \eta} t & \frac{\partial \zeta}{\partial \eta} + \frac{\partial p_{RRz}}{\partial \eta} t \\ p_{RRx} & p_{RRy} & p_{RRz} \end{vmatrix}$$

$$J_{RR} = \frac{D_{RR}(t)}{D_{RR}(0)}$$

$$= \left(\frac{2n_2 X t}{ab^2 \cos \gamma \sin \gamma} \right)^2 - \frac{4n_2 t}{b^2 \cos \alpha} \left(\frac{\sin^2 \alpha \cos^2 \alpha X^{\frac{1}{2}}}{\cos^2 \gamma \sin^2 \gamma} + \frac{4X^{\frac{3}{2}}}{a^2} + X^{\frac{1}{2}} \cos^2 \alpha \cos 2\alpha \right) + 1 \quad (26)$$

$$D_{RL}(t) = \begin{vmatrix} 1 + \frac{\partial p_{RLx}}{\partial \xi} t & \frac{\partial p_{RLy}}{\partial \xi} t & \frac{\partial \zeta}{\partial \xi} + \frac{\partial p_{RLz}}{\partial \xi} t \\ \frac{\partial p_{RLx}}{\partial \eta} t & 1 + \frac{\partial p_{RLy}}{\partial \eta} t & \frac{\partial \zeta}{\partial \eta} + \frac{\partial p_{RLz}}{\partial \eta} t \\ p_{RLx} & p_{RLy} & p_{RLz} \end{vmatrix}$$

$$J_{RL} = \frac{D_{RL}(t)}{D_{RL}(0)}$$

$$= 1 + \frac{2X_1 \sin(\alpha + \alpha_1)}{\sin 2\alpha (\tan \alpha \sin(\alpha + \alpha_1) + \cos(\alpha + \alpha_1))} \left(\frac{n_1 X t}{ab^2 \cos \gamma \sin \gamma} \right)^2$$

$$- \frac{4n_1 X^{\frac{1}{2}} \sec \alpha t}{\tan \alpha \sin(\alpha + \alpha_1) + \cos(\alpha + \alpha_1)} \left(\frac{\sin^2(\alpha + \alpha_1)}{(b \cos \gamma \sin \gamma)^2} + \frac{2X X_1}{b^2 a^2} + \frac{\cos(\alpha + \alpha_1) \sin(\alpha + \alpha_1)}{b^2 \sin \alpha \sec \alpha} \right) \quad (27)$$

$$D_{LR}(t) = \begin{vmatrix} 1 + \frac{\partial p_{LRx}}{\partial \xi} t & \frac{\partial p_{LRy}}{\partial \xi} t & \frac{\partial \zeta}{\partial \xi} + \frac{\partial p_{LRz}}{\partial \xi} t \\ \frac{\partial p_{LRx}}{\partial \eta} t & 1 + \frac{\partial p_{LRy}}{\partial \eta} t & \frac{\partial \zeta}{\partial \eta} + \frac{\partial p_{LRz}}{\partial \eta} t \\ p_{LRx} & p_{LRy} & p_{LRz} \end{vmatrix}$$

$$J_{LR} = \frac{D_{LR}(t)}{D_{LR}(0)}$$

$$= 1 + \frac{2X_2 \sin(\alpha + \alpha_2)}{\sin 2\alpha (\tan \alpha \sin(\alpha + \alpha_2) + \cos(\alpha + \alpha_2))} \left(\frac{n_2 X t}{ab^2 \cos \gamma \sin \gamma} \right)^2$$

$$- \frac{4n_2 X^{\frac{1}{2}} \sec \alpha t}{\tan \alpha \sin(\alpha + \alpha_2) + \cos(\alpha + \alpha_2)} \left(\frac{\sin^2(\alpha + \alpha_2)}{(b \cos \gamma \sin \gamma)^2} + \frac{2X X_2}{b^2 a^2} + \frac{\cos(\alpha + \alpha_2) \sin(\alpha + \alpha_2)}{b^2 \sin \alpha \sec \alpha} \right) \quad (28)$$

In the above expressions

$$X = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha \quad (29)$$

$$X_1 = \frac{\sqrt{n_1^2 - n_2^2 \sin^2 \alpha} + n_2 \cos \alpha}{\sqrt{n_1^2 - n_2^2 \sin^2 \alpha}} \quad (30)$$

$$X_2 = \frac{\sqrt{n_2^2 - n_1^2 \sin^2 \alpha} + n_1 \cos \alpha}{\sqrt{n_2^2 - n_1^2 \sin^2 \alpha}} \quad (31)$$

The geometrical optics fields for each ray are obtained by putting Eqs. (30)–(36) in Eq. (1) and given by

$$u_{LL}(r) = \frac{\cos \alpha - \cos \alpha_2}{\cos \alpha + \cos \alpha_2} (J_{LL})^{-1/2} \exp -jk\{n_1^2 t + n_1 \zeta\} \quad (32)$$

$$u_{RR}(r) = \frac{\cos \alpha - \cos \alpha_1}{\cos \alpha + \cos \alpha_1} (J_{RR})^{-1/2} \exp -jk\{n_2^2 t + n_2 \zeta\} \quad (33)$$

$$u_{RL}(r) = \frac{2 \cos \alpha}{\cos \alpha + \cos \alpha_1} (J_{RL})^{-1/2} \exp -jk\{n_1^2 t + n_2 \zeta\} \quad (34)$$

$$u_{LR}(r) = \frac{2 \cos \alpha}{\cos \alpha + \cos \alpha_2} (J_{LR})^{-1/2} \exp -jk\{n_2^2 t + n_1 \zeta\} \quad (35)$$

The focal points equations where Jacobian is zero for LL, RR, RL, and LR rays are given as

$$\left(\frac{2n_1 X t}{ab^2 \cos \gamma \sin \gamma} \right)^2 - \frac{4n_1 t}{b^2 \cos \alpha} \left(\frac{\sin^2 \alpha \cos^2 \alpha X^{\frac{1}{2}}}{\cos^2 \gamma \sin^2 \gamma} + \frac{4X^{\frac{3}{2}}}{a^2} + X^{\frac{1}{2}} \cos^2 \alpha \cos 2\alpha \right) + 1 = 0 \quad (36)$$

$$\left(\frac{2n_2 X t}{ab^2 \cos \gamma \sin \gamma} \right)^2 - \frac{4n_2 t}{b^2 \cos \alpha} \left(\frac{\sin^2 \alpha \cos^2 \alpha X^{\frac{1}{2}}}{\cos^2 \gamma \sin^2 \gamma} + \frac{4X^{\frac{3}{2}}}{a^2} + X^{\frac{1}{2}} \cos^2 \alpha \cos 2\alpha \right) + 1 = 0 \quad (37)$$

$$\begin{aligned} & \frac{2X_1 \sin(\alpha + \alpha_1)}{\sin 2\alpha (\tan \alpha \sin(\alpha + \alpha_1) + \cos(\alpha + \alpha_1))} \left(\frac{n_1 X t}{ab^2 \cos \gamma \sin \gamma} \right)^2 \\ & - \frac{4n_1 X^{\frac{1}{2}} \sec \alpha t}{\tan \alpha \sin(\alpha + \alpha_1) + \cos(\alpha + \alpha_1)} \left(\frac{\sin^2(\alpha + \alpha_1)}{(b \cos \gamma \sin \gamma)^2} + \frac{2X X_1}{b^2 a^2} + \frac{\cos(\alpha + \alpha_1) \sin(\alpha + \alpha_1)}{b^2 \sin \alpha \sec \alpha} \right) + 1 = 0 \quad (38) \end{aligned}$$

$$\begin{aligned} & \frac{2X_2 \sin(\alpha + \alpha_2)}{\sin 2\alpha (\tan \alpha \sin(\alpha + \alpha_2) + \cos(\alpha + \alpha_2))} \left(\frac{n_2 X t}{ab^2 \cos \gamma \sin \gamma} \right)^2 \\ & - \frac{4n_2 X^{\frac{1}{2}} \sec \alpha t}{\tan \alpha \sin(\alpha + \alpha_2) + \cos(\alpha + \alpha_2)} \left(\frac{\sin^2(\alpha + \alpha_2)}{(b \cos \gamma \sin \gamma)^2} + \frac{2X X_2}{b^2 a^2} + \frac{\cos(\alpha + \alpha_2) \sin(\alpha + \alpha_2)}{b^2 \sin \alpha \sec \alpha} \right) + 1 = 0 \quad (39) \end{aligned}$$

The volume of the ray tube for these rays vanishes near the caustic region. Therefore, GO solution cannot be applied to find the solution around the caustic region. So we derive the field around the caustic region using Maslov's method. To calculate the expressions for the fields around the caustic region by Eq. (2) we need Eq. (3) for the amplitude of different reflected rays

$$J_{LL} \frac{\partial(p_{LLx}, p_{LLy})}{\partial(x, y)} = \left(\frac{2n_1 X \cos 2\alpha}{ab^2 \cos \gamma \sin \gamma} \right)^2 \quad (40)$$

$$J_{RR} \frac{\partial(p_{RRx}, p_{RRy})}{\partial(x, y)} = \left(\frac{2n_2 X \cos 2\alpha}{ab^2 \cos \gamma \sin \gamma} \right)^2 \quad (41)$$

$$J_{RL} \frac{\partial(p_{RLx}, p_{RLy})}{\partial(x, y)} = \frac{2n_1^2 X^2 X_1 \cos^2(\alpha + \alpha_1) \sin(\alpha + \alpha_1)}{a^2 b^4 \sin 2\alpha \cos^2 \gamma \sin^2 \gamma [\cos(\alpha + \alpha_1) + \tan \alpha \sin(\alpha + \alpha_1)]} \quad (42)$$

$$J_{LR} \frac{\partial(p_{LRx}, p_{LRy})}{\partial(x, y)} = \frac{2n_2^2 X^2 X_2 \cos^2(\alpha + \alpha_2) \sin(\alpha + \alpha_2)}{a^2 b^4 \sin 2\alpha \cos^2 \gamma \sin^2 \gamma [\cos(\alpha + \alpha_2) + \tan \alpha \sin(\alpha + \alpha_2)]} \quad (43)$$

The transformation components for each ray are calculated. Now to calculate the phase function in Eq. (2), x and y are expressed in terms of mixed coordinates (p_x, q_y, z) . Similarly, t is represented in terms of hybrid coordinates as $t = (z - \zeta)/p_z$. The general expression for the phase function is calculated by using hybrid coordinates as

$$S = S_0 + n^2 t - (\xi + q_x t)p_x - (\eta + p_y t)q_y + x p_x + p q_y \quad (44)$$

Putting Eqs. (22)–(24) and the coefficients in relations (26)–(29) in Eq. (49), we get the phase function for all four rays as given by

$$s_{LL} = n_1 \left[2X^{\frac{1}{2}} \cos \alpha - x \sin 2\alpha \cos \gamma - y \sin 2\alpha \sin \gamma - z \cos 2\alpha \right] \quad (45)$$

$$s_{RR} = n_2 \left[2X^{\frac{1}{2}} \cos \alpha - x \sin 2\alpha \cos \gamma - y \sin 2\alpha \sin \gamma - z \cos 2\alpha \right] \quad (46)$$

$$s_{RL} = n_1 \left[\frac{n_2}{n_1} a^2 X^{\frac{-1}{2}} \cos \alpha + a^2 X^{\frac{-1}{2}} \cos \alpha \cos(\alpha + \alpha_1) + b^2 X^{\frac{-1}{2}} \sin \alpha \sin(\alpha + \alpha_1) - x \cos \gamma \sin(\alpha + \alpha_1) - y \sin \gamma \sin(\alpha + \alpha_1) - z \cos(\alpha + \alpha_1) \right] \quad (47)$$

$$s_{LR} = n_2 \left[\frac{n_1}{n_2} a^2 X^{\frac{-1}{2}} \cos \alpha + a^2 X^{\frac{-1}{2}} \cos \alpha \cos(\alpha + \alpha_2) + b^2 X^{\frac{-1}{2}} \sin \alpha \sin(\alpha + \alpha_2) - x \cos \gamma \sin(\alpha + \alpha_2) - y \sin \gamma \sin(\alpha + \alpha_2) - z \cos(\alpha + \alpha_2) \right] \quad (48)$$

The integration variables (dp_x, dq_y) for all four rays are transformed to (ξ, η) , and the following relation is obtained

$$dp_{LLx} dp_{LLy} = \begin{vmatrix} \frac{\partial p_{LLx}}{\partial \xi} & \frac{\partial p_{LLy}}{\partial \xi} \\ \frac{\partial p_{LLx}}{\partial \eta} & \frac{\partial p_{LLy}}{\partial \eta} \end{vmatrix} = \frac{4n_1^2 X^2 \cos 2\alpha}{a^2 b^4 \sin^2 \gamma \cos^2 \gamma} d\xi d\eta \quad (49)$$

$$dp_{RRx} dp_{RRy} = \begin{vmatrix} \frac{\partial p_{RRx}}{\partial \xi} & \frac{\partial p_{RRy}}{\partial \xi} \\ \frac{\partial p_{RRx}}{\partial \eta} & \frac{\partial p_{RRy}}{\partial \eta} \end{vmatrix} = \frac{4n_2^2 X^2 \cos 2\alpha}{a^2 b^4 \sin^2 \gamma \cos^2 \gamma} d\xi d\eta \quad (50)$$

$$dp_{RLx} dp_{RLy} = \begin{vmatrix} \frac{\partial p_{RLx}}{\partial \xi} & \frac{\partial p_{RLy}}{\partial \xi} \\ \frac{\partial p_{RLx}}{\partial \eta} & \frac{\partial p_{RLy}}{\partial \eta} \end{vmatrix} = \frac{2n_1^2 X^2 X_1 \cos(\alpha + \alpha_1) \sin(\alpha + \alpha_1)}{a^2 b^4 \sin 2\alpha \sin^2 \gamma \cos^2 \gamma} d\xi d\eta \quad (51)$$

$$dp_{LRx} dp_{LRy} = \begin{vmatrix} \frac{\partial p_{LRx}}{\partial \xi} & \frac{\partial p_{LRy}}{\partial \xi} \\ \frac{\partial p_{LRx}}{\partial \eta} & \frac{\partial p_{LRy}}{\partial \eta} \end{vmatrix} = \frac{2n_2^2 X^2 X_2 \cos(\alpha + \alpha_2) \sin(\alpha + \alpha_2)}{a^2 b^4 \sin 2\alpha \sin^2 \gamma \cos^2 \gamma} d\xi d\eta \quad (52)$$

Now the conversion factor from the surface coordinates (ξ, η) to angular coordinates (α, γ) is

$$d\xi d\eta = \begin{vmatrix} \frac{\partial \xi}{\partial \alpha} & \frac{\partial \eta}{\partial \alpha} \\ \frac{\partial \xi}{\partial \gamma} & \frac{\partial \eta}{\partial \gamma} \end{vmatrix} = \frac{a^2 b^4 \cos \alpha \sin \alpha}{X^2} d\alpha d\gamma \quad (53)$$

Now putting all the phase terms, transformation terms and amplitude terms in Eq. (2), the finite fields which are valid around the focal points for LL, RR, RL, and LR waves are given by

$$u_{LL}(r) = \frac{n_1 a k}{2\pi} \int_0^H \int_0^{2\pi} \left(\frac{\cos \alpha - \cos \alpha_2}{\cos \alpha + \cos \alpha_2} \right) \frac{b^2 \sin 2\alpha}{X \sin \gamma \cos \gamma} \times \exp \left[-j k n_1 \left\{ 2X^{\frac{1}{2}} \cos \alpha - x \sin 2\alpha \cos \gamma - y \sin 2\alpha \sin \gamma - z \cos 2\alpha \right\} \right] d\alpha d\gamma \quad (54)$$

$$u_{RR}(r) = \frac{n_2 a k}{2\pi} \int_0^H \int_0^{2\pi} \left(\frac{\cos \alpha - \cos \alpha_1}{\cos \alpha + \cos \alpha_1} \right) \frac{b^2 \sin 2\alpha}{X \sin \gamma \cos \gamma} \times \exp \left[-j k n_2 \left\{ 2X^{\frac{1}{2}} \cos \alpha - x \sin 2\alpha \cos \gamma - y \sin 2\alpha \sin \gamma - z \cos 2\alpha \right\} \right] d\alpha d\gamma \quad (55)$$

$$u_{LR}(r) = \frac{n_1 k}{\sqrt{2}\pi} \int_0^H \int_0^{2\pi} \left(\frac{ab^2 \cos \alpha}{\cos \alpha + \cos \alpha_1} \right) \left(\frac{X_1 \sin 2\alpha (\cos(\alpha + \alpha_1) + \tan \alpha \sin(\alpha + \alpha_1))}{\csc(\alpha + \alpha_1) X^2 \sin^2 \gamma \cos^2 \gamma} \right)^{\frac{1}{2}} \times \exp \left[-j k n_1 \left\{ \frac{n_2}{n_1} a^2 X^{\frac{-1}{2}} \cos \alpha + a^2 X^{\frac{-1}{2}} \cos \alpha \cos(\alpha + \alpha_1) + b^2 X^{\frac{-1}{2}} \sin \alpha \sin(\alpha + \alpha_1) - x \cos \gamma \sin(\alpha + \alpha_1) - y \sin \gamma \sin(\alpha + \alpha_1) - z \cos(\alpha + \alpha_1) \right\} \right] d\alpha d\gamma \quad (56)$$

$$u_{LR}(r) = \frac{n_2 k}{\sqrt{2}\pi} \int_0^H \int_0^{2\pi} \left(\frac{ab^2 \cos \alpha}{\cos \alpha + \cos \alpha_2} \right) \left(\frac{X_2 \sin 2\alpha (\cos(\alpha + \alpha_2) + \tan \alpha \sin(\alpha + \alpha_2))}{\csc(\alpha + \alpha_2) X^2 \sin^2 \gamma \cos^2 \gamma} \right)^{\frac{1}{2}} \times \exp \left[-j k n_2 \left\{ \frac{n_1}{n_2} a^2 X^{\frac{-1}{2}} \cos \alpha + a^2 X^{\frac{-1}{2}} \cos \alpha \cos(\alpha + \alpha_2) + b^2 X^{\frac{-1}{2}} \sin \alpha \sin(\alpha + \alpha_2) - x \cos \gamma \sin(\alpha + \alpha_2) - y \sin \gamma \sin(\alpha + \alpha_2) - z \cos(\alpha + \alpha_2) \right\} \right] d\alpha d\gamma \quad (57)$$

where H is the angle subtended by the 3D elliptical PEC reflector with the aperture.

3. RESULTS AND DISCUSSION

The equations which describe the high frequency fields around the focal points of a 3D elliptical reflector with chiral background are derived in the previous sections using GO and Maslov's method. Equations (54)–(57) are solved numerically, and the results are obtained accordingly. The values of some of the parameters are taken to be $ka = 250$, $kb = 200$ and $H = \pi/4$. The field patterns are plotted along z -axis by keeping $kx = 0$. Figure 2 and Figure 3 show the high frequency field intensity for LL and RR waves represented by u_{LL} and u_{RR} . The focal point positions of the 3D elliptical reflector are similar to the case when the background material is achiral. Therefore, focal points LL and RR rays

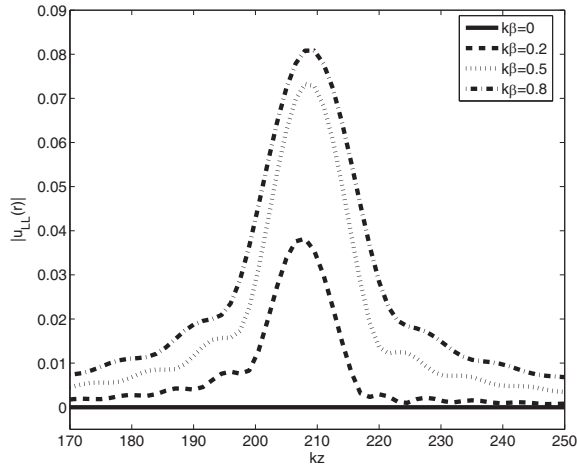


Figure 2. Plot for $|u_{LL}|$ around the focal points with $ka = 250$, $kb = 200$ and different values of chirality parameter $k\beta = 0$, $k\beta = 0.2$, $k\beta = 0.5$ and $k\beta = 0.8$.

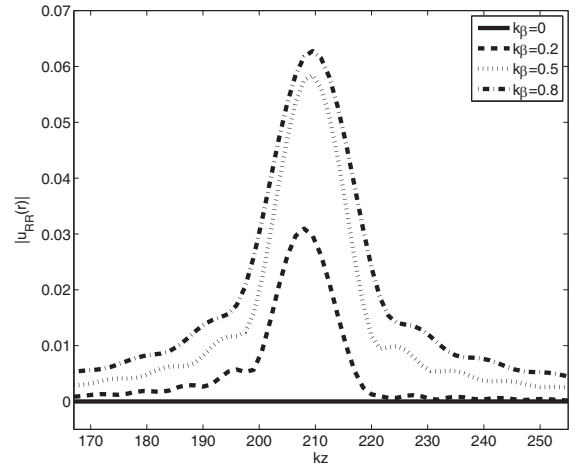


Figure 3. Plot for $|u_{RR}|$ around the focal points with $ka = 250$, $kb = 200$ and different values of chirality parameter $k\beta = 0$, $k\beta = 0.2$, $k\beta = 0.5$ and $k\beta = 0.8$.

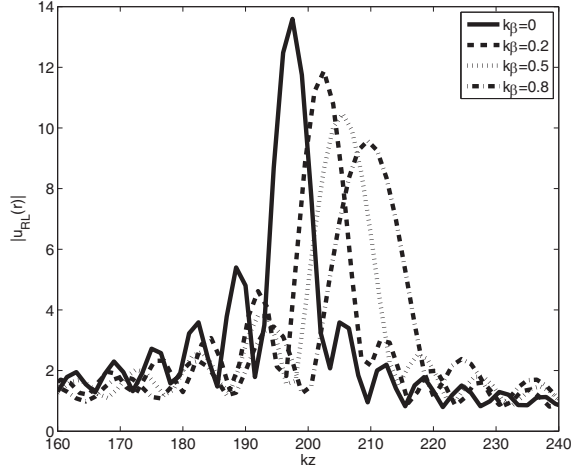


Figure 4. Plot for $|u_{RL}|$ around the focal points with $ka = 250$, $kb = 200$ and different values of chirality parameter $k\beta = 0$, $k\beta = 0.2$, $k\beta = 0.5$ and $k\beta = 0.8$.

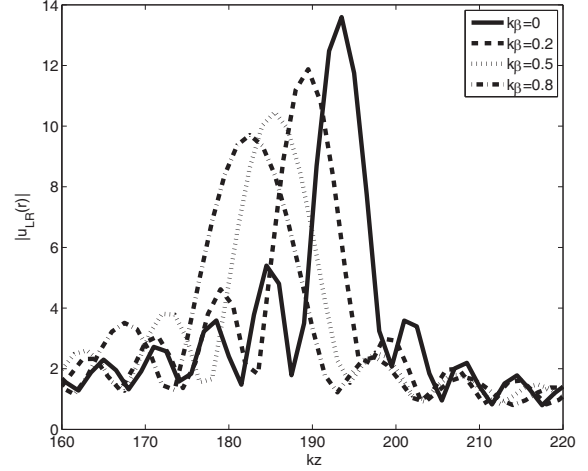


Figure 5. Plot for $|u_{LR}|$ around the focal points with $ka = 250$, $kb = 200$ and different values of chirality parameter $k\beta = 0$, $k\beta = 0.2$, $k\beta = 0.5$ and $k\beta = 0.8$.

overlap for all values of $k\beta$. If we want to discuss the special case of conventional medium by putting the value of chirality parameter $k\beta = 0$, then $n_1 = n_2 = 1$ and the field vanishes as given by

$$u_{LL} = u_{RR} = 0 \tag{58}$$

The magnitude values of u_{LL} and u_{RR} around the focal point increase with the increase in the chirality parameter $k\beta$ as depicted in Figure 2 and Figure 3, respectively. Figure 4 and Figure 5 illustrate that as the value of chirality parameter $k\beta$ increases, the focal point for RL is shifted towards left and focal point for LR wave shifted towards right. With the increase in value of chirality parameter $k\beta$, the gap between the focal points of RL and LR rays increases. The variation in field pattern for different values of the chirality parameter $k\beta$ is indicated. If $k\beta = 0$, then $n_1 = n_2 = 1$ and the field pattern decreases to ordinary medium. The results derived in this paper represent a case for spherical reflector placed in chiral medium [18]. Furthermore, the results obtained can be converted to that obtained for spherical reflector in conventional medium as discussed in [10] by putting the major axis length to the minor axis, i.e., $ka = kb$ and the chirality parameter value $k\beta = 0$. All the expressions derived can be converted to the expressions derived in the previous results which also indicate the validity of the derived equations in this article. The 3D elliptical reflector gives us an extra degree of freedom in the design of reflector antenna for high frequency applications compared to spherical reflector antenna. The tunability due to chirality and also more tunable variables in the reflector design make it suitable for advanced applications.

4. CONCLUSIONS

The high frequency fields of the 3D elliptical reflector with chiral background is derived using GO approximation. The chiral background having birefringence nature supports both LCP and RCP waves when a linearly polarized wave propagates through. Thus, four waves are reflected by the 3D elliptical reflector which are represented by LL, RR, RL and LR in this paper. Focal points for LL and RR rays are located at the same position, and focal points for RL and LR are located on the opposite side of the focal point for RR and LL rays. Chirality parameter $k\beta > 0$ LCP wave moves slower than RCP and is focused near the reflector, and RCP wave is focused away from the elliptical reflector. The roles of LCP and RCP waves get reversed by considering the chirality factor $k\beta < 0$. If the value of chirality parameter increases, the gap between the focal points increases, and if the chirality parameter $k\beta = 0$ is zero, the high frequency field for LL and RR becomes zero, and that for RL and LR is reduced to the case of conventional medium.

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