*Progress In Electromagnetics Research C, Vol. 63, 75–83, 2016*

# A Hybrid Algorithm for Synthesizing Linear Sparse Arrays

**Xiaowen Zhao[1, 2, 3, *], Yunhua Zhang[1, 2], and Qingshan Yang[1, 2]**

**Abstract**—A hybrid algorithm based on the invasive weed optimization (IWO) and the convex optimization (CVX) is proposed for minimizing the peak sidelobe level (PSLL) of linear array with focused and/or shaped beam pattern. In this approach, IWO is adopted to produce the array (described by element positions), and CVX is used to determine the excitations for each produced array. Then the corresponding PSLL acts as the fitness function of IWO to find the optimal positions which lead to the minimum PSLL. Numerical experiments are conducted to validate the effectiveness and robustness of the proposed hybrid approach. Compared with other techniques, a lower PSLL can be achieved with a fixed main beam width or with a shaped main beam using this hybrid algorithm. Moreover, this method can easily cope with some constraints on the aperture, such as the minimum element spacing and the total number of elements.

## 1. INTRODUCTION

Nowadays, the synthesis of aperiodic arrays has attracted growing attentions as a challenging and meaningful problem. Compared with uniform arrays, aperiodic arrays with unequally-spaced elements have many outstanding advantages, such as easy to get rid of grating-lobes while achieving high directivity, the number of elements can be saved, the effect of mutual coupling reduced and the system cost well controlled [1]. As a result, aperiodic arrays have been widely used in the fields of satellite communication, tracking radars, radio astronomy and microwave remote sensing. In the synthesis of aperiodic arrays, the minimization of the peak sidelobe level (PSLL) is usually of major concern to the designer. Generally speaking, aperiodic arrays can be classified into two categories: sparse arrays with optimally spaced elements and thinned arrays that are derived by turning off some elements in an initial equally spaced array [2]. The inter-element spacings of a thinned array is an integer times of the uniform spacing of an initial equally spaced array, thus antenna elements are on originally uniform grids in some sense. However, the elements in sparse array are randomly distributed in the span of array aperture. Therefore, sparse arrays have more freedom degrees of optimization to achieve lower PSLL than thin arrays have. Recently, several approaches, such as analytical methods [3–5], modified genetic algorithm (MGA) [6], differential evolution algorithm (DEA) [7], vector mapping and simultaneous perturbation stochastic approximation [8], and invasive weed optimization (IWO) [9] have been developed for synthesizing sparse arrays. The performances of those techniques on achieving low PSLL are quite limited as a result of position-only optimization. To improve the performance of PSLL, the improved genetic algorithm (IGA) [10] and improved genetic algorithm-extremum disturbed simple particle swarm optimization (IGA-edsPSO) [11] method have been proposed for the synthesis of sparse arrays by simultaneously adjusting element positions and excitations. More recently, the convex optimization has been adopted to solve the nonlinear nonconvex problem of sparse array synthesis [12, 13]. In the approach, the problem was formulated by first finely discretizing an array aperture to get the candidate positions for antenna elements, and then transforming the synthesis

*Received 3 March 2016, Accepted 31 March 2016, Scheduled 16 April 2016*
* Corresponding author: Xiaowen Zhao (xiaowenzhao923@gmail.com).
[1] Key Laboratory of Microwave Remote Sensing, Chinese Academy of Sciences, Beijing, China. [2] National Space Science Center, Chinese Academy of Sciences, Beijing, China. [3] University of Chinese Academy of Sciences, Beijing, China.

problem into a convex optimization problem with respect to excitation variables and removing these elements with small excitation coefficients from the supposed positions. This method leads to a solution to the synthesis problem with the active elements positioned on the grids. However, the array may not be sparsely arranged on the aperture oftentimes because the elements may not fall just onto the finite grids defined by the discretization. And, fine discretization with more grids will definitely increase the memory requirement during the calculation process and lead to numerical instability issues as well. In addition, we should point out that these previous research efforts have focused mostly on synthesis of sparse arrays with focused beam patterns, and the methods for shaped beam patterns in [14] and [15] have been developed only for equally spaced array. In view of the drawbacks of the above works and aim to overcome them, a hybrid approach based on the stochastic optimization and the convex optimization is proposed in this paper, which tries to keep a balance between the search capabilities of these two algorithms, at the same time, to achieve a much lower PSLL for any shaped beam pattern with respect to the previous techniques. Compared with other stochastic optimization methods, e.g., GA, particle swarm optimization (PSO), and DEA, IWO is simple, efficient and robust [9]. Here, the IWO is adopted to optimize element positions with the inter-element spacing controllable in the proposed hybrid approach. For each predefined set of locations, the array synthesis is a convex problem related to excitations, which can be solved by a convex solver CVX. At the same time, the corresponding PSLL with both positions and excitations determined is treated as a fitness function of IWO to guide the hybrid method to find the optimal positions and excitations. The hybrid method is named as IWO-CVX algorithm, which can be used for synthesising sparse arrays by simultaneously adjusting the antenna elements' positions and excitations to minimize the PSLL of the focused and/or shaped beam pattern under the constraints of fixed number of elements, array aperture size and the control of inter-element spacing.

The remainder of this paper is organized as follows. The configuration of the synthesis problem is briefly described in Section 2. The hybrid IWO-CVX method is proposed and formulated to synthesize sparse arrays in Section 3. Then, Section 4 conducts numerical experiments to validate the performance of the proposed algorithm, and finally conclusion is drawn in Section 5.

## 2. FORMULATION FOR SYNTHESIS PROBLEM

Let us consider an arbitrary $N$-element linear array positioned in the span of aperture $L$, the array factor can be characterized as

$$AF(\theta) = \sum_{n=1}^{N} w_n e^{jkx_n \sin(\theta)} \tag{1}$$

where $k = 2\pi/\lambda$, $\lambda$ is the wavelength and $w_n$ the excitation coefficient of the $n$-th element located at position $x_n$. Let $x_1 = 0$ and $x_N = L$ in order that the aperture is always $L$, and $\theta$ is the steering angle in respect to the broadside. The object is to minimize the PSLL by optimizing both excitation weights and element positions in consideration of the constraints of element number, aperture size as well as minimum element spacing. The synthesis problem is expressed mathematically as

$$\begin{cases} find\ \mathbf{x} = [x_1, \ldots, x_N]^T \ \ and\ \mathbf{w} = [w_1, \ldots, w_N]^T \\ \min\{\mathrm{PSLL}(\mathbf{x}, \mathbf{w})\} \\ s.t.\ x_{i+1} - x_i \geq d_c > 0 \\ i \in \mathbb{Z}, 1 \leq i \leq N-1 \\ x_1 = 0,\ x_N = L \end{cases} \tag{2}$$

where $d_c$ is the minimum allowable spacing, and the PSLL is defined as

$$PSLL(\mathbf{x}, \mathbf{w}) = \max \left| \frac{\sum_{n=1}^{N} w_n e^{jkx_n \sin(\theta_{\mathrm{sl}})}}{AF(\theta_{\max})} \right| \tag{3}$$

$\theta_{\max}$ is the mainlobe direction and $\theta_{\mathrm{sl}}$ the observation angles over the sidelobe outside the given mainlobe region.

## 3. THE HYBRID METHOD AND SYNTHESIS PROCEDURE

The hybrid IWO-CVX method is proposed to solve the synthesis problem in Section 2. The main steps of the procedure are summarized below.

### 3.1. Initialization of Element Positions

The initialization strategy for element positions in [6] is adopted here for initializing the solution. Let us take the position variable vector $\mathbf{x} = [x_1, \ldots, x_N]^T$ as the individual. In order to meet the constraints of minimum element spacing $d_c$ and aperture dimension $L$, the position vector $\mathbf{x}$ is set as follows

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ L \end{bmatrix} + \begin{bmatrix} 0 \\ d_c \\ 2d_c \\ \vdots \\ (N-2)\, d_c \\ 0 \end{bmatrix} \tag{4}$$

The sub-vector $\mathbf{a} = [a_1, \ldots, a_{N-2}]^T$ contains $N - 2$ random real numbers within the range of $[0, L - (N-1)d_c]$, where $a_1 \leq a_2 \leq \ldots \leq a_{N-2}$. Obviously, the position vector $\mathbf{x}$ can be determined as long as the corresponding sub-vector $\mathbf{a}$ is generated randomly to meet the requirements. We take $\mathbf{a}$ as a seed for the IWO. By producing sub-vector $\mathbf{a}$ $M$ times independently, a starting population containing $M$ seeds is initialized, thus $M$ sets of position vector are initialized too.

### 3.2. Excitation Optimization and Fitness Determination

The synthesis problem is convex with respect to the excitation variables as long as the corresponding position variables are predetermined. Convex optimization solver CVX can be used to optimize the excitation variables in order to minimize the PSLL of the array pattern. The mathematical expressions of optimization are described as

$$\begin{cases} find\ \ (w_1, \ldots, w_N) \\ \min \{\text{PSLL}\,(w_1, \ldots, w_N)\} \\ s.t.\ \ |AF\,(\theta_{\text{main}}) - F_{red}\,(\theta_{\text{main}})| \leq \varepsilon \end{cases} \tag{5}$$

where $\theta_{\text{main}}$ is the direction in the main beam region, and $F_{red}(\theta_{\text{main}})$ represents the main beam with any desired shape. Meanwhile, the obtained PSLL of each array is treated as the fitness value of the corresponding seed in the population, and then each initial seed grows to a weed after evaluating the fitness.

### 3.3. Reproduction

The reproductive capability of each weed depends on the fitness value. The number of reproduced seeds from each weed varies linearly with its own fitness value. For the synthesis problem concerned, the weeds with smaller fitness values are more likely reserved in the population, and consequently produce more seeds. The number of seeds produced by the $m$-th weed is determined as

$$S_m = \frac{S_{\max} - S_{\min}}{f_{\max} - f_{\min}} \left(f_{\max} - f_m\right) + S_{\min} \tag{6}$$

where $f_{\max}$ and $f_{\min}$ are the maximum and minimum PSLLs in the current population, respectively; $S_{\max}$ and $S_{\min}$ are the maximum and minimum allowable seeds, respectively; $f_m$ is the fitness value of the $m$-th weed. An important advantage of the IWO introduced in this step is that it allows all weeds to participate in the reproduction process, which means that the IWO gives the worse weeds a chance to reproduce and survive. This characteristic is very advantageous since some infeasible weeds probably carry more useful information than the feasible ones during the evolution process.

### 3.4. Spatial Dispersal

The produced seeds dispread randomly over the searching space and the process obeys to the normal distribution with the mean referring to the parent weed's location. The standard deviation $\sigma$ will be reduced after several iterations from the predefined initial value $\sigma_{\text{initial}}$ to the final value $\sigma_{\text{final}}$. The current $\sigma$ of the *iter*-th iteration is given as

$$\sigma = \frac{(iter_{\max} - iter)^n}{(iter_{\max})^n}\left(\sigma_{\text{initial}} - \sigma_{finial}\right) + \sigma_{finial} \tag{7}$$

where $iter_{\max}$ is the maximum number of iterations and $n$ the nonlinear modulation index. Then the $k$-th seed produced by the $m$-th weed can be expressed as

$$\mathbf{a}_{n,k} = \mathbf{a}_n + N\left(0, \sigma\right) \tag{8}$$

Meanwhile, the elements of each $\mathbf{a}$ (seed) are limited in the range of $[0, L-(N-1)d_c]$ and are reordered in an increasing sequence, i.e., $a_1 \leq a_2 \leq \ldots \leq a_{N-2}$. Using Eq. (4), the corresponding position vector $\mathbf{x} = [x_1, \ldots, x_N]^T$ is subsequently generated, then CVX is used to optimize the excitation vector to minimize the PSLL of the array pattern according to the given position vector. By this way, the fitness of all of the produced seeds and parent weeds can be evaluated.

### 3.5. Competitive Exclusion

The new seeds will grow to weeds and are later ranked together with their parents based on their fitness values. There should be some kind of competition between weeds for limiting the maximum number of weeds in the colony. Here, the weeds with poor fitness are eliminated from the current colony when the maximum number of weeds $P_{\max}$ is reached after some iterations. On the contrary, the ones with better fitness should survive and are allowed to reproduce their next generations. The iteration process is repeated from step 3.3 until the termination criterion is met, that is, either the maximum number of iterations $iter_{\max}$ is reached or the improvement of the best fitness after successive iterations is smaller than an acceptable level.

## 4. SIMULATION EXPERIMENTS

In this section, numerical applications of sparse array synthesis with focused or/and shaped beam patterns are considered to evaluate the effectiveness of the proposed hybrid method. The three parameters of the algorithm including the initial standard deviation $\sigma_{\text{initial}}$, the finial standard deviation $\sigma_{\text{final}}$ and the nonlinear modulation index $n$ should be carefully chosen so as to obtain the proper $\sigma$ for each iteration since they are the major factors affecting the convergence of the algorithm. In the following simulation experiments, $\sigma_{\text{initial}}$ is set around 5% of the dimension of $L-(N-1)d_c$, $\sigma_{\text{final}}$ is selected as small as possible, e.g., 0.0005, and $n = 3$ is used (as we know, the best choice of $n$ is suggested to be 3 for IWO in open literatures). In addition, other parameters are selected as $M = 20$, $S_{\min} = 0$, $S_{\max} = 5$, $P_{\max} = 50$, $iter_{\max} = 100$.

### 4.1. Focused Beam Synthesis Addressed in [12]

As the first experiment, the proposed hybrid method is now applied to solving the same synthesis problem presented in [12], where a 25-element sparse array with PSLL equal to $-20\,\text{dB}$ and main beam confined in $|u| \leq 0.04\,(u = \sin(\theta))$ has been synthesized by sequential convex optimization. Let the proposed algorithm run 20 times independently to verify its robustness. As can be seen, all experiment results obtained with PSLL less than $-20\,\text{dB}$ are better than that of literature [12]. Specifically, the lowest PSLL in 20 independent runs is $-20.56\,\text{dB}$ as seen from Fig. 1. Meanwhile, Table 1 lists the positions and excitations of the best array. Fig. 2 shows the synthesized pattern by our method along with that by the method in [12], from which one can clearly see that the proposed IWO-CVX method can achieve a lower PSLL with the same main beam. Compared to the method in [12], the proposed method can get optimal element positions since it never restricts the antenna elements to the fixed grids. Moreover, during the calculation process, the memory requirement for our method is only proportional

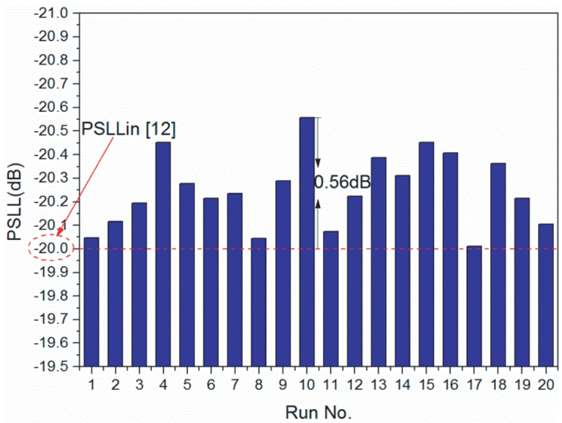



**Figure 1.** PSLLs obtained by the proposed method for 20 independent runs.

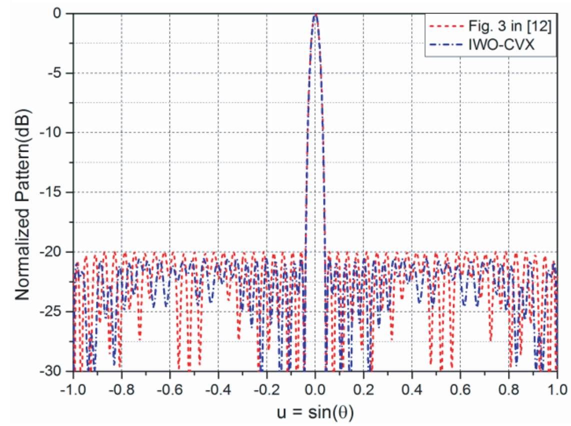



**Figure 2.** Optimal pattern by the proposed method (blue dashed dot line: PSLL = $-20.56$ dB) along with the synthesized pattern by the sequential convex optimization in [12] (red dashed line: PSLL = $-20$ dB).

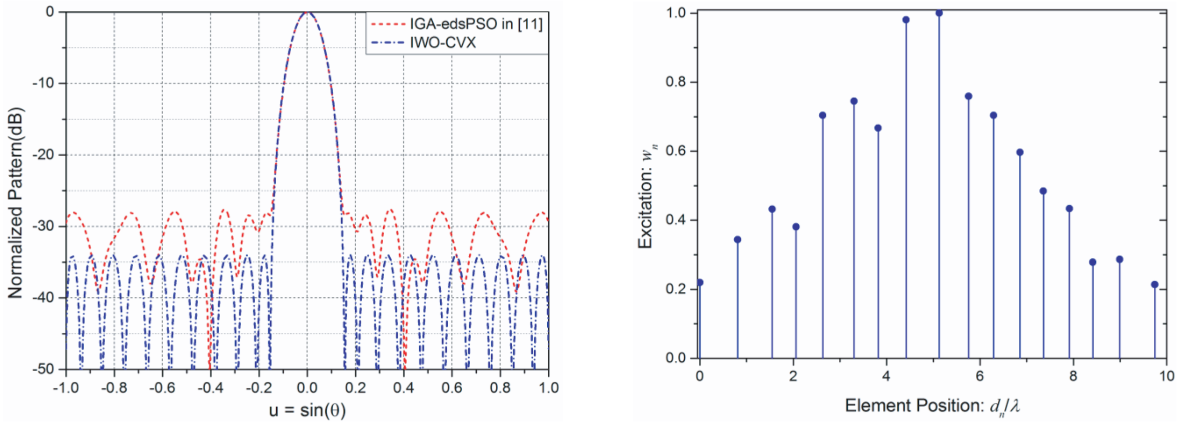
**Table 1.** Positions and excitations of the optimal array synthesised by the proposed method.

| No. | $d_n/\lambda$ | $w_n$ | No. | $d_n/\lambda$ | $w_n$ |
|---|---|---|---|---|---|
| 1 | 0 | 0.7141 | 14 | 15.2375 | 0.8597 |
| 2 | 0.9000 | 0.6180 | 15 | 16.1980 | 0.8258 |
| 3 | 3.8083 | 1 | 16 | 17.1424 | 0.7793 |
| 4 | 5.6661 | 0.9544 | 17 | 18.0941 | 0.6922 |
| 5 | 6.6157 | 0.6488 | 18 | 19.0616 | 0.6634 |
| 6 | 7.6439 | 0.7108 | 19 | 20.0101 | 0.6480 |
| 7 | 8.5839 | 0.8022 | 20 | 21.0074 | 0.7427 |
| 8 | 9.5290 | 0.6978 | 21 | 21.9703 | 0.6264 |
| 9 | 10.4676 | 0.9136 | 22 | 22.9307 | 0.6156 |
| 10 | 11.4233 | 0.9034 | 23 | 23.8524 | 0.6427 |
| 11 | 12.4074 | 0.9028 | 24 | 24.7675 | 0.6249 |
| 12 | 13.3663 | 0.7991 | 25 | 25.6821 | 0.4896 |
| 13 | 14.3179 | 0.7512 | | | |

to the number of active elements (here is 25), however, for the method in [12], it is proportional to the number of the discretized grids (e.g., $24\lambda/(\lambda/200) = 4800$, $\lambda/200$ is the discretization step), which is far larger than the number of active elements. We should emphasize that with our method, one can conveniently add some other constraints on the linear geometry so as to make the synthesized sparse array well adapted to the requirements of the designer since the inter-element spacing can be controlled flexibly.

### 4.2. Focused Beam Synthesis Addressed in [11]

In [11], IGA-edsPSO was proposed to synthesize a 17-element linear sparse array with an aperture of $9.744\lambda$ by optimizing the antenna positions and excitations. The achieved radiation focused beam pattern exhibits a PSLL of $-27.67$ dB with the main beam confined in $|u| \leq 0.156$, i.e., $|\theta| \leq 9°$. Now,




(a) (b)

**Figure 3.** Synthesized array by the proposed method. (a) Beam pattern. (b) Element positions and excitations.

**Table 2.** PSLL values of 10 independent runs with 17 antenna elements.

| No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| PSLL (dB) | −33.93 | -33.92 | −33.93 | −33.95 | −33.90 | −33.99 | −33.95 | −33.91 | −33.97 | −33.96 |

we use the proposed IWO-CVX method to solve the same synthesis problem for further reduction on PSLL. An improved PSLL of $-33.99$ dB is achieved with the same main beamwidth. The synthesized pattern, as compared to the one in [11], is shown in Fig. 3(a). It can be seen that the main beamwidth is not broadened along with the PSLL is decreased, that is just the goal of optimization. In Fig. 3(b), the layout of the corresponding sparse array is also presented. In addition, we run the simulation 10 times independently to verify the robustness of the proposed method. The PSLLs obtained in each run are listed in Table 2, from which one can see that the proposed method is very stable since the PSLLs in those independent runs are almost unchanged and are nearly $-33.9$ dB.

## 4.3. Flat-Top Beam Synthesis Addressed in [14]

This example will show the capability of our method to optimize amplitudes, phases and positions simultaneously in order to further reduce the number of elements and the PSLL of a flat-top beam pattern compared with the case of uniformly spaced array. In [14], a 15-element uniformly spaced array with an aperture of $7\lambda$ has been synthesized to obtain a flat-top beam pattern with PSLL $= -35.45$ dB. Using the proposed IWO-CVX method, a sparse array of 12 elements is synthesized with $-38.33$ dB PSLL for the same shaped main beam of [14]. This means that 3 elements are saved as well as the PSLL is reduced about 3 dB compared with the uniformly spaced array synthesized by [14]. The complex weightings and element positions are given in Table 3. The comparison of the synthesized patterns is shown in Fig. 4.

## 4.4. Cosecant Beam Synthesis Addressed in [15]

The synthesis of a sparse linear array with $9.5\lambda$ aperture radiating a cosecant beam pattern with PSLL equal to $-25$ dB (Fig. 3 in [15]) is considered. This pattern was originally synthesized by the touring ant colony optimization (TACO) algorithm with 20 equal spacing elements. For this problem, the FBMPM-based synthesis method has been proposed in [16], where only 15 elements were used to obtain the same cosecant main beam and the same PSLL of [15]. Now, using the proposed IWO-CVX method, an improved PSLL of $-26$ dB with the same cosecant main beam has been achieved compared

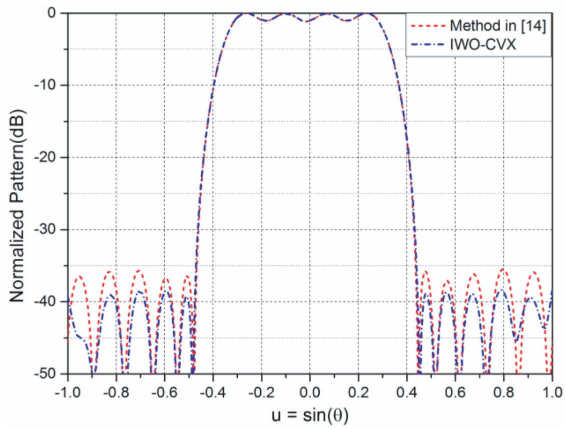



**Figure 4.** Synthesized flat-top patterns by the proposed method (blue dashed dot line: a 12-element sparse array with PSLL = $-38.33$ dB) and the method in [14] (red dashed line: a 15-element uniformly spaced array with PSLL = $-35.45$ dB).

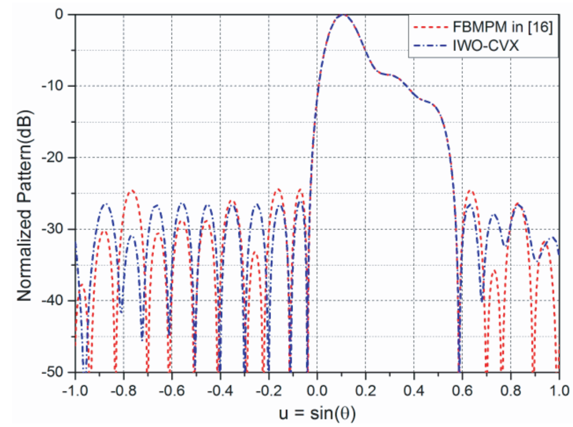



**Figure 5.** Synthesized cosecant beam patterns by the proposed IWO-CVX method (blue dashed dot line: PSLL = $-26$ dB) and the FBMPM method in [16] (red dashed line: PSLL = $-25$ dB).

**Table 3.** Element positions and excitations synthesised by the proposed method.

| No. | $d_n/\lambda$ | $\|w_n\|$ | $\angle w_n$ (rad) | No. | $d_n/\lambda$ | $\|w_n\|$ | $\angle w_n$ (rad) |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.1769 | 0.7873 | 7 | 3.805 | 0.46 | 0.6310 |
| 2 | 0.612 | 0.5336 | 0.7187 | 8 | 4.454 | 0.7758 | 1.5634 |
| 3 | 1.240 | 0.7526 | 0.4785 | 9 | 5.103 | 0.5061 | 2.1154 |
| 4 | 1.873 | 0.7661 | −0.1957 | 10 | 5.752 | 0.2971 | −2.5885 |
| 5 | 2.512 | 1 | −0.7925 | 11 | 6.386 | 0.3256 | −1.8901 |
| 6 | 3.157 | 0.8178 | −0.7473 | 12 | 7 | 0.1406 | −1.6653 |

**Table 4.** Element positions and excitations synthesised by the proposed method.

| No. | $d_n/\lambda$ | $\|w_n\|$ | $\angle w_n$ (rad) | No. | $d_n/\lambda$ | $\|w_n\|$ | $\angle w_n$ (rad) |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.1793 | −0.6645 | 9 | 5.236 | 0.9642 | 2.5299 |
| 2 | 0.679 | 0.1659 | −0.6722 | 10 | 5.928 | 0.6646 | 1.8142 |
| 3 | 1.466 | 0.2693 | −0.9390 | 11 | 6.729 | 0.4030 | 1.6570 |
| 4 | 2.159 | 0.3533 | −1.5022 | 12 | 7.472 | 0.3190 | 1.4393 |
| 5 | 2.865 | 0.3160 | −1.7277 | 13 | 8.096 | 0.2418 | 0.8942 |
| 6 | 3.379 | 0.3658 | −1.6106 | 14 | 8.853 | 0.1535 | 0.6583 |
| 7 | 3.944 | 0.7790 | −2.1987 | 15 | 9.5 | 0.1734 | 0.6654 |
| 8 | 4.589 | 1 | −2.9221 | | | | |

to the FBMPM method in [16]. Fig. 5 shows the comparison of the synthesized shaped beam patterns. Table 4 shows positions and excitations of the elements obtained by the IWO-CVX method.

We should point out that in all of the above simulation experiments, the improved performances on PSLLs with unchanged main beam achieved by the proposed IWO-CVX method compared with other methods have been demonstrated. For better comparison and showing the improvements of the proposed method, Table 5 specially lists the PSLL results obtained by different methods for each simulation.

**Table 5.** Comparison of PSLLs achieved by the proposed method and other methods for the above simulation experiments.

| | Simulation 1 | | Simulation 2 | |
|---|---|---|---|---|
| Method | IWO-CVX | Method in [12] | IWO-CVX | IGA-edsPSO in [11] |
| PSLL | $-20.56$ dB | $-20$ dB | $-33.99$ dB | $-27.67$ dB |
| | Simulation 3 | | Simulation 4 | |
| Method | IWO-CVX | Method in [14] | IWO-CVX | FBMPM in [16] |
| PSLL | $-38.33$ dB | $-35.45$ dB | $-26$ dB | $-25$ dB |

## 5. CONCLUSION

This paper provides a hybrid IWO-CVX approach for synthesis of linear sparse arrays with some constraints imposed. The proposed method aims to minimize the PSLL of a sparse array by optimizing both the positions and the excitations. Simulation experiments are carried out to demonstrate the effectiveness and feasibility of the proposed algorithm. Compared with other methods, the proposed method can achieve lower PSLL while keep the main beam unchanged. In addition, the proposed method can handle the synthesis problem for any shaped beam patterns without any restrictions. Besides, the proposed method is robust and stable. Moreover, the proposed method can easily cope with some constraints on the aperture, e.g., the minimum inter-element spacing and the total number of elements, etc. Therefore, it is more practically applicable to real situations for engineering applications. As expected, the proposed method can guarantee that the minimum inter-element spacing is always larger than $\lambda/2$, which means that the mutual coupling of the synthesized sparse array can be controlled to a low level and even can be neglected. Finally, we should point out that the proposed method is also extendable to the synthesis of planar arrays.

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