Development of Fundamental Theory of Thin Impedance Vibrators

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Abstract—In the paper, we prove two theorems relating to the theory of thin impedance vibrator radiators excited by a lumped voltage generator under rather general conditions. The first theorem proves that influence of external electrodynamic media on the vibrator current distribution is limited and can be estimated using a small natural parameter. The second theorem ascertains that there exists principal possibility to compensate influence of spatial boundaries upon current distributions on a perfectly conductive vibrator by applying to its surface complex impedance with predetermined variation along the vibrator length. Several corollaries disclose a range of the theorems application and their fundamental importance.

1. INTRODUCTION

The theory of thin vibrators is now considered as classic both for perfectly conducting [1, 2] and impedance vibrators [3–10]. The theory was outlined in a large number of well-known articles and monographs (see, e.g., references in [3]). However, this problem is still of great interest, since the vibrator structures are widely used in various devices and systems to provide the required mode of excitation. Since the problem is multivariable, an experimental optimization of devices is almost impossible, and physically adequate mathematical models are needed for compound boundary value problems, non-coordinate border of spatial domains, presence of scattering irregularities, medium inhomogeneity, etc. In any case, a key stage of modeling consists in a search of current distributions on a vibrator surface. The problem solution can be greatly simplified by selection of the basic current distribution. This choice should be done taking into account the vibrator surrounding which cannot always be done relying only on analysis of available publications. Therefore, the generalization of the theoretical results concerning the influence of surrounding media upon the current distribution on the thin impedance vibrator is an actual problem.

One approach to such generalization is based on an analytical solution of an integral equation for vibrator current using small natural parameter [7]. For example, in the monograph [11], attention was drawn to the fact that the functional effect of walls of a hollow rectangular waveguide upon the current on the linear scattering vibrator located inside the waveguide contains the proportionality factor equal to the natural small parameter of the problem. An analog situation has arisen during the analytical determination of the current on a radial impedance monopole allocated on the perfectly conducting sphere [6, 9], and on the impedance vibrator over the perfectly conducting screen of finite size [10]. The boundary problem solution for these two cases requires that the total field should be represented by waves of electric and magnetic types. This article is aimed at the generalization of the influence on a boundary value problem with arbitrary boundaries that do not possess a property of mutual transformation of electric and magnetic fields. The second theorem assets that there exists principal possibility to compensate influence of spatial boundaries upon current distributions on a perfectly

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conductive radiator by covering its surface by complex impedance material with predetermined variation along the vibrator length. The second theorem states that influence of spatial boundaries upon current distributions on a perfectly conductive vibrator can be compensated by applying complex impedance with predetermined variation along the vibrator length.

2. LEMMA FORMULATION AND PROOF

To simplify the proof of the theorems, let us consider an auxiliary lemma.

Lemma. *Let a thin radiating impedance vibrator, excited by a point source is placed in an infinite homogeneous medium with material parameters (ε_1, μ_1). The vibrator is a segment of a circular cylinder, whose radius r and the length $2L$ are such that inequalities $[r/(2L)] \ll 1$ and $r\sqrt{\varepsilon_1\mu_1}/\lambda \ll 1$ (λ is the wavelength in free space) hold. Then the electric current on the vibrator can be represented by a power series $J(s) = \alpha J_1(s) + \alpha^2 J_2(s) + \ldots$ in the small parameter $\alpha = \frac{1}{2\ln[r/(2L)]}$, $|\alpha| \ll 1$, and $J_n(s)$ is the current approximation of the n-th order $(n = 1, 2 \ldots)$.*

Proof. The proof is based upon the well-known solution of the integral equation for the vibrator current obtained using a small natural parameter [7]. Consider the following equation [3]

$$\frac{1}{i\omega\varepsilon_1}\left(\text{graddiv} + k_1^2\right)\int\limits_S \hat{G}^e\left(\vec{r},\vec{r}'\right)\vec{J}\left(\vec{r}'\right)d\vec{r}' = -\vec{E}_0\left(\vec{r}\right) + z_i\left(\vec{r}\right)\vec{J}\left(\vec{r}\right), \tag{1}$$

where $z_i(\vec{r})$ is the linear intrinsic impedance ([Ohm/m]) of the vibrator, $\vec{E}_0(\vec{r})$ is the field of extraneous sources, $\hat{G}^e(\vec{r},\vec{r}')$ is the tensor Green's function of the spatial domain for the electric vector potential, $k_1 = k\sqrt{\varepsilon_1\mu_1}$, $k = \omega/c = 2\pi/\lambda$ is wave number, $c \approx 2.998 \cdot 10^{10}$ cm/s is the speed of light in vacuum. Equation (1) was obtained using boundary conditions on the vibrator surface S if time t dependence is $e^{i\omega t}$ and ω is a circular frequency of a monochromatic process.

In a thin wire approximation, the electric current induced on the vibrator surface can be represented as

$$\vec{J}(\vec{r}) = \vec{e}_s J(s)\psi(\rho,\varphi), \tag{2}$$

where $\vec{e}_s$ is the unit vector directed along the vibrator axis; s is the local coordinate along the vibrator axis; $\psi(\rho,\varphi)$ is the function of transverse ($\perp$) polar coordinates, satisfying the normalization condition $\int\limits_\perp \psi(\rho,\varphi)\rho d\rho d\varphi = 1$. If the relations

$$\int\limits_S \hat{G}^e\left(\vec{r},\vec{r}'\right)\vec{J}(\vec{r}')d\vec{r}' = \int\limits_{-L}^{L} J(s')\int\limits_{-\pi}^{\pi} \frac{e^{-ik_1\sqrt{(s-s')^2+[2r\sin(\varphi/2)]^2}}}{\sqrt{(s-s')^2+[2r\sin(\varphi/2)]^2}}\psi(r,\varphi)rd\varphi ds'$$

$$\approx \int\limits_{-L}^{L} J(s')\frac{e^{-ik_1\sqrt{(s-s')^2+r^2}}}{\sqrt{(s-s')^2+r^2}}ds' = \int\limits_{-L}^{L} J(s')\frac{e^{-ik_1 R(s,s')}}{R(s,s')}\Bigg|_{R(s,s')=\sqrt{(s-s')^2+r^2}} ds' \tag{3}$$

are valid [1, 3] and $z_i(\vec{r}) = z_i(s) \equiv \text{const}$. The surface Equation (1) can be converted to an integral equation with a quasi-one-dimensional kernel

$$\left(\frac{d^2}{ds^2} + k_1^2\right)\int\limits_{-L}^{L} J(s')\frac{e^{-ik_1 R(s,s')}}{R(s,s')}ds' = -i\omega\varepsilon_1 E_{0s}(s) + i\omega\varepsilon_1 z_i J(s), \tag{4}$$

where $E_{0s}(s)$ is projection of the extraneous source field on the vibrator axis.

Let us isolate a logarithmic singularity in Equation (4) core using the method described in [1, 7]

$$\int\limits_{-L}^{L} J(s')\frac{e^{-ik_1 R(s,s')}}{R(s,s')}ds' = \Omega(s)J(s) + \int\limits_{-L}^{L} \frac{J(s')e^{-ik_1 R(s,s')} - J(s)}{R(s,s')}ds'. \tag{5}$$

where

$$\Omega(s) = \int_{-L}^{L} \frac{ds'}{\sqrt{(s-s')^2 + r^2}} = \Omega + \gamma(s). \tag{6}$$

$\gamma(s) = \ln \frac{[(L+s)+\sqrt{(L+s)^2+r^2}]\,[(L-s)+\sqrt{(L-s)^2+r^2}]}{4L^2}$ is a function vanishing at the vibrator center and reaching maxima at the vibrator ends, where the current is zero as required by the boundary conditions $J(\pm L) = 0$ [1, 3]; $\Omega = 2\ln \frac{2L}{r}$ is a large parameter. Then, taking into account Equation (5), Equation (4) can be converted to the following integral-differential equation with a small parameter

$$\frac{d^2 J(s)}{ds^2} + k_1^2 J(s) = \alpha \left\{ i\omega\varepsilon_1 E_{0s}(s) + F[s, J(s)] - i\omega\varepsilon_1 z_i J(s) \right\}. \tag{7}$$

Here $\alpha = -\frac{1}{\Omega} = \frac{1}{2\ln[r/(2L)]}$ is the small natural parameter ($|\alpha| \ll 1$), and functional

$$
\begin{aligned}
F[s, J(s)] =\ & -\frac{dJ(s')}{ds'} \frac{e^{-ik_1 R(s,s')}}{R(s,s')} \bigg|_{-L}^{L} + \left[\frac{d^2 J(s)}{ds^2} + k_1^2 J(s) \right] \gamma(s) \\
& + \int_{-L}^{L} \frac{\left[\frac{d^2 J(s')}{ds'^2} + k_1^2 J(s') \right] e^{-ik_1 R(s,s')} - \left[\frac{d^2 J(s)}{ds^2} + k_1^2 J(s) \right]}{R(s,s')} ds'
\end{aligned} \tag{8}
$$

presents the vibrator eigenfield.

If we denote $\tilde{k} = k_1 \sqrt{1 + i\alpha\omega\varepsilon_1 z_i/k_1} = k_1\sqrt{1 + i2\alpha\bar{Z}_S/(\mu_1 kr)}$ ($\bar{Z}_S = Z_S/Z_0$ is distributed surface impedance normalized to the wave resistance $Z_0 = 120\pi$ [Ohm]), Equation (7) can be written as

$$\frac{d^2 J(s)}{ds^2} + \tilde{k}^2 J(s) = \alpha \left\{ i\omega\varepsilon_1 E_{0s}(s) + F[s, J(s)] \right\}. \tag{9}$$

Since Equation (9) is proportional to the small parameter α, its solution can be obtained by a successive approximation technique using the following algorithm

$$
\begin{cases}
\dfrac{d^2 J_1(s)}{ds^2} + \tilde{k}^2 J_1(s) = i\omega\varepsilon_1 E_{0s}(s), \\[2mm]
\dfrac{d^2 J_2(s)}{ds^2} + \tilde{k}^2 J_2(s) = F[s, J_1(s)], \\
\ . \\
\ . \\
\dfrac{d^2 J_n(s)}{ds^2} + \tilde{k}^2 J_n(s) = F[s, J_{n-1}(s)].
\end{cases} \tag{10}
$$

The solution of each differential equation can be obtained using the boundary conditions for the current $J_1(\pm L) = J_2(\pm L) = \ldots = J_n(\pm L) = 0$. Thus, we obtain the current decomposition as power series in small parameter α, i.e., $J(s) = \alpha J_1(s) + \alpha^2 J_2(s) + \ldots$, which was to be proved.

The zero approximation for the current J_0 was not included into the equation system (10), since its solution is $J_0(s) = C_1 \cos \tilde{k}s + C_2 \sin \tilde{k}s$. Taking into account losses in the medium and/or on the vibrator surface, the trigonometric functions in the solution are complex and cannot be zero for any arguments. Therefore, to satisfy the boundary conditions $J_0(\pm L) = 0$, the constants C_1 and C_2 should be zero and identities $J_0 \equiv 0$, $F[s, J_0(s)] \equiv 0$ become valid for any vibrator length.

The first approximation of the vibrator current obtained from (10) as sum of the general and partial solutions is

$$
J(s) \approx \alpha J_1(s) = -\alpha \frac{i\omega\varepsilon_1/\tilde{k}}{\sin 2\tilde{k}L} \times
\begin{cases}
\sin \tilde{k}\,(L-s) \displaystyle\int_{-L}^{s} E_{0s}\left(s'\right) \sin \tilde{k}\,(L+s')\,ds' \\[4mm]
+ \sin \tilde{k}\,(L+s) \displaystyle\int_{s}^{L} E_{0s}\left(s'\right) \sin \tilde{k}\,(L-s')\,ds',
\end{cases} \tag{11}
$$

and it does not depend on the function $F[s, J(s)]$ (8).

3. FORMULATION AND PROOF OF THE FIRST THEOREM

Theorem 1. *Let a thin radiating impedance vibrator, exited by a point source, be placed in an electrodynamic volume filled by homogeneous medium with material parameters (ε_1, μ_1). The volume boundary is an arbitrary Lyapunov surface S [12], not passing through sources of extraneous currents. The vibrator is a segment of a circular cylinder, whose radius r and the length 2L are such that inequalities $[r/(2L)] \ll 1$ and $r\sqrt{\varepsilon_1\mu_1}/\lambda \ll 1$ (λ is the wavelength in free space) hold. Then the influence of the volume boundaries upon the current distribution on the vibrator surface does not exceed an amount proportional to the small natural parameter $\alpha = \frac{1}{2\ln[r/(2L)]}$.*

Proof. The kernel $\hat{G}^e(\vec{r}, \vec{r}')$ of integral Equation (1) is the electric tensor Green's function of the closed volume. In a system of orthogonal curvilinear coordinates (q_1, q_2, q_3), this function, according to the general properties [12], satisfies the inhomogeneous Helmholtz equation

$$\Delta\hat{G}\left(\vec{q}, \vec{q}'\right) + k_1^2\hat{G}\left(\vec{q}, \vec{q}'\right) = -4\pi\hat{I}\frac{\delta\left(q_1 - q_1'\right)\delta\left(q_2 - q_2'\right)\delta\left(q_3 - q_3'\right)}{h_1 h_2 h_3}, \tag{12}$$

where $\hat{I}$ is unit tensor; (q_1', q_2', q_3') are source coordinates; $\delta(q - q')$ is Dirac delta function; h_n are Lame coefficients. Laplacian Δ applies to all tensor components. Then, the solution for the vector Hertz potential can be written as

$$\vec{\Pi}^e\left(\vec{q}\right) = \frac{1}{i\omega\varepsilon_1}\int_V \vec{J}\left(\vec{q}'\right)\hat{G}^e\left(\vec{q}, \vec{q}'\right)dv + \oint_S \left\{\text{div}\,\vec{\Pi}^e\left(\vec{q}'\right)\hat{G}^e\left(\vec{q}, \vec{q}'\right)\vec{n} - \text{div}\,\hat{G}^e\left(\vec{q}, \vec{q}'\right)\vec{\Pi}^e\left(\vec{q}'\right)\vec{n}\right.$$

$$\left. + \left[\vec{n}, \hat{G}^e\left(\vec{q}, \vec{q}'\right)\right]\text{rot}\,\vec{\Pi}^e\left(\vec{q}'\right) - \left[\vec{n}, \vec{\Pi}^e\left(\vec{q}'\right)\right]\text{rot}\,\hat{G}^e\left(\vec{q}, \vec{q}'\right)\right\}ds', \tag{13}$$

where $\vec{n}$ is the unit vector of the external normal to the surface S. The volume integral is taken over the entire volume V (dv is volume element), and the surface integral is taken over the entire surface (ds' is the area element in the primed coordinates). The expression in the curly brackets is a vector, and differentiation is performed over the primed coordinates.

Thus, the solution of the inhomogeneous Helmholtz equation is the sum of the volume and surface integrals. The surface integrals can be eliminated by building the Green's function in a special way. If the components of the Green's function $\hat{G}^e(\vec{q}, \vec{q}')$ and components of the vector potentials $\vec{\Pi}^e(\vec{q})$ satisfy the boundary conditions on the surface S, the surface integrals vanish, since integrand of surface integrals in (13) vanish. Otherwise, the solution will have the general form (13). Thus, the expression (13) allows us to use alternative forms of the Green's function.

If the Green's tensor for unbounded domain, which is the solution of Equation (12) satisfying the boundary conditions at infinity is

$$\hat{G}^e(\vec{r}, \vec{r}') = \hat{I}\frac{e^{-ik_1|\vec{r}-\vec{r}'|}}{|\vec{r} - \vec{r}'|} = \hat{I}G_0^e(\vec{r}, \vec{r}'), \tag{14}$$

the vector Hertz potential (13) can be presented as

$$\vec{\Pi}^e\left(\vec{r}\right) = \frac{1}{i\omega\varepsilon_1}\int_V \vec{J}\left(\vec{r}'\right)\hat{I}\frac{e^{-ik_1|\vec{r}-\vec{r}'|}}{|\vec{r} - \vec{r}'|}dv + \oint_S \left\{\text{div}\,\vec{\Pi}^e\left(\vec{r}'\right)\left(\hat{I}G_0^e(\vec{r}, \vec{r}')\right)\vec{n} - \text{div}\left(\hat{I}G_0^e(\vec{r}, \vec{r}')\right)\vec{\Pi}^e\left(\vec{r}'\right)\vec{n}\right.$$

$$\left. + \left[\vec{n}, \left(\hat{I}G_0^e(\vec{r}, \vec{r}')\right)\right]\text{rot}\vec{\Pi}^e\left(\vec{r}'\right) - \left[\vec{n}, \vec{\Pi}^e\left(\vec{r}'\right)\right]\text{rot}\left(\hat{I}G_0^e(\vec{r}, \vec{r}')\right)\right\}ds'. \tag{15}$$

Considering that in the general case $\vec{E}(\vec{r}) = \text{grad}\,\text{div}\vec{\Pi}^e(\vec{r}) + k^2\varepsilon_1\mu_1\vec{\Pi}^e(\vec{r})$, substituting Equation (15)

into Equation (1), we obtain

$$\frac{1}{i\omega\varepsilon_1} \left(\text{grad div} + k_1^2\right) \int_S \left(\hat{I}\frac{e^{-ik_1|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\right) \vec{J}(\vec{r}')d\vec{r}'$$

$$= -\vec{E}_0(\vec{r}) + z_i(\vec{r})\vec{J}(\vec{r}) - \left(\text{grad div} + k_1^2\right) \oint_S \left\{\text{div}\,\vec{\Pi}^e\left(\vec{r}'\right)\left(\hat{I}G_0^e(\vec{r},\vec{r}')\right)\vec{n} - \text{div}\,\left(\hat{I}G_0^e(\vec{r},\vec{r}')\right)\vec{\Pi}^e\left(\vec{r}'\right)\vec{n}\right.$$

$$\left. + \left[\vec{n}, \left(\hat{I}G_0^e(\vec{r},\vec{r}')\right)\right]\text{rot}\,\vec{\Pi}^e\left(\vec{r}'\right) - \left[\vec{n}, \vec{\Pi}^e\left(\vec{r}'\right)\right]\text{rot}\,\left(\hat{I}G_0^e(\vec{r},\vec{r}')\right)\right\}ds'. \tag{16}$$

The functional defining the field of arbitrary boundaries at the observation point $\vec{r}$ can be found, using the relation (16)

$$F_S\left(\vec{r}, \vec{J}(\vec{r})\right) = \left(\text{grad div} + k_1^2\right) \oint_S \left\{\text{div}\vec{\Pi}^e\left(\vec{r}'\right)\left(\hat{I}G_0^e(\vec{r},\vec{r}')\right)\vec{n}\right.$$

$$-\text{div}\,\left(\hat{I}G_0^e(\vec{r},\vec{r}')\right)\vec{\Pi}^e\left(\vec{r}'\right)\vec{n} + \left[\vec{n}, \left(\hat{I}G_0^e(\vec{r},\vec{r}')\right)\right]\text{rot}\,\vec{\Pi}^e\left(\vec{r}'\right)$$

$$\left. - \left[\vec{n}, \vec{\Pi}^e\left(\vec{r}'\right)\right]\text{rot}\,\left(\hat{I}G_0^e(\vec{r},\vec{r}')\right)\right\}ds', \tag{17}$$

Then, Equation (4) for the current on the vibrator surface can be written as follows

$$\left(\frac{d^2}{ds^2} + k_1^2\right) \int_{-L}^{L} J(s')\frac{e^{-ik_1R(s,s')}}{R(s,s')}ds' = -i\omega\varepsilon_1 E_{0s}(s) - i\omega\varepsilon_1 F_S\left(s, J(s)\right) + i\omega\varepsilon_1 z_i J(s), \tag{18}$$

$F_S(s, J(s))$ is the projection of the vector functional $F_S(\vec{r}, \vec{J}(\vec{r}))$ on the vibrator axis. Then, combining field functionals $f_\Sigma(\vec{r}, \vec{J}(\vec{r})) = F(\vec{r}, \vec{J}(\vec{r})) + i\omega\varepsilon_1 F_S(\vec{r}, \vec{J}(\vec{r}))$, in equation similar to (7), we can use the Lemma to proof the Theorem 1. Indeed, according to Lemma, the electric current on a thin impedance vibrator can be represented by a series in the small parameter α, therefore, its first approximation, determined by the expression (11), does not depend on the functional of the boundary influence. The influence is taken into account by successive approximations. Thus, the influence of the volume boundaries on the current distribution on the vibrator does not exceed an amount proportional to the small parameter α. The Theorem 1 is proved.

3.1. Corollaries from Theorem 1

Theorem 1 is formulated for sufficiently general conditions allowing to establish several corollaries of physical interest, which define the fundamental nature of the theorem.

Corollary 1.1. Since no restrictions were imposed on magnitude of the vibrator constant impedance, the theorem is valid for a perfectly conducting vibrator, $z_i = 0$.

Corollary 1.2. Since no restrictions were imposed on the material parameter of the medium (ε_1, μ_1) the theorem is valid both for unbounded space and hollow closed volumes with $\varepsilon_1 = \mu_1 = 1$.

Corollary 1.3. Since the formulation and proof of the theorem does not require specifying the coordinates of a feed point s_0 of the voltage δ-generator $E_{0s}(s') = V_0\delta(s' - s_0)$, the theorem is valid for arbitrary choice of the feed point. Here V_0 is the voltage amplitude, $\delta(s' - s_0)$ is one-dimensional Dirac delta function.

Commentary. Really, the expression for the first approximation of the current (11) was obtained for an arbitrary extraneous excitation field $E_{0s}(s')$. The term "arbitrary" applies only to the choice of the feed point coordinates and model describing the local excitation source. It is related to the fact that the vibrator excitation by the incident extraneous field, formed in the spatial domain with account of the borders, which influence through structure of the field $E_{0s}(s')$ will have an indirect reflection in the first approximation for the current. This certainly violates the logic of the theorem proof.

Corollary 1.4. One of the Theorem 1 conditions are the following requirements on the surface:
1) it must be a Lyapunov surface; 2) it should not cross current sources; 3) in a more general sense, it
has to be passive, i.e., without generation of extraneous fields, and 4) it does not possess the properties
of mutual transformation of spatial harmonics of the electric and magnetic fields. Since the type of
electrodynamic boundary surfaces is not defined and do not exclude the possibility of its composite
presentation, the theorem is valid both for different types of surfaces such as perfectly conducting,
impedance, partially impedance, etc. and for the different scattering inhomogeneities in the spatial
domain, whose surfaces can be interpreted as parts of the total surface.

Commentary. Parts of the boundary surface can be presented by impedance surfaces of coupling
holes between coupling volumes. Separate areas of piecewise inhomogeneous of the magneto-dielectric
filling of the electrodynamic volume can be thought as scattering irregularities.

The approach used to the proof of Theorem 1 allows us to formulate the second theorem. It
concerns the fundamental possibility to compensate the influence of the spatial boundaries upon the
current distribution on a perfectly conducting vibrator using "application" of the distributed impedance
on the surface.

4. FORMULATION AND PROOF OF THE SECOND THEOREM

In the proof of the Lemma and Theorem 1, linear impedance of the vibrator was assumed to be constant
$z_i(\vec{r}) = z_i(s) \equiv$ const. Now, let us assume that the impedance can be distributed along the vibrator
axis $z_i(\vec{r}) = z_i(s)$, or be concentrated at some points on the vibrator axis, or be superposition of these
two options.

Theorem 2. *Let a thin radiating impedance vibrator, exited by a point source, be placed in an
electrodynamic volume filled by homogeneous medium with material parameters (ε_1, μ_1). The volume
boundary is an arbitrary Lyapunov surface S [12], not passing through sources of extraneous currents.
The vibrator is a segment of a circular cylinder, whose radius r and the length 2L are such that
inequalities $[r/(2L)] \ll 1$ and $r\sqrt{\varepsilon_1\mu_1}/\lambda \ll 1$ (λ is the wavelength in free space) hold. Then, the influence
of the volume boundaries upon the current distribution on the vibrator surface can be compensated by
coating its surface with the complex impedance, varying along the vibrator axis, $z_i(s) = \frac{F_S(s, J(s))}{J(s)}$. $J(s)$
is the current distribution on the perfectly conducting vibrator, and $F_S(s, J(s))$ is the functional (17)
defining the boundary influence.*

Proof. Let us consider the equation similar to Eq. (18)

$$\left(\frac{d^2}{ds^2} + k_1^2\right) \int_{-L}^{L} J(s')\frac{e^{-ik_1 R(s,s')}}{R(s,s')}ds' = -i\omega\varepsilon_1 \left[E_{0s}(s) + F_S\left(s, J(s)\right) - z_i(s)J(s)\right], \qquad (19)$$

where $z_i(s)$ is complex surface impedance which varies along the vibrator axis. If the equality
$F_S(s, J(s)) = z_i(s)J(s)$ on the right hand side of Equation (19) holds, the vibrator current is determined
only by the fields of extraneous sources. Hence, Equation (19) can be formally represented as a system
of two equations

$$\begin{cases} \left(\dfrac{d^2}{ds^2} + k_1^2\right) \displaystyle\int_{-L}^{L} J(s')\dfrac{e^{-ik_1 R(s,s')}}{R(s,s')}ds' = -i\omega\varepsilon_1 E_{0s}(s), \\ F_S(s, J(s)) - z_i(s)J(s) = 0. \end{cases} \qquad (20)$$

The first equation is Equation (4) for a perfectly conducting vibrator, $z_i = 0$, located in an infinite
homogeneous medium. The second equation is the functional equation, which can be solved using the
current found from the first equation. These equations can be used to obtain the distribution of variable
complex impedance along the vibrator axis

$$z_i(s) = \frac{F_S(s, J(s))}{J(s)}, \qquad (21)$$

where $z_i(s)$ can be a generalized function. Thus, the influence of the boundaries can be fully compensated
by "applying" the impedance in Eq. (21) to the vibrator surface. The current distribution of such

impedance vibrator corresponds now to that of perfectly conducting vibrator, located in an infinite homogeneous medium with material parameters (ε_1, μ_1). Thus, Theorem 2 is proved.

### 4.1. Corollaries from Theorem 2

Corollary 2.1. Since the surface of the scattering irregularities, located in electrodynamic volume, can be interpreted as components of the common surface S, and the Theorem 2 is valid for this case.

Commentary. The main difficulty of the Theorem 2 application concerns determination of the functional $F_S(s, J(s))$. However, if Green's function of the electromagnetic volume is known, the problem is greatly simplified, since the fields at the boundary surface S in Eq. (17) can be found using the Green's function.

Corollary 2.2. Input resistance of the impedance vibrator with the distribution in Eq. (21), positioned in the electrodynamic volume, is equivalent to that of perfectly conductive vibrator of the same geometry located in free space.

4.2. Example of Theorem 2 Application

The application of the Theorem 2 can be demonstrated by the problem of the electro-magnetic radiation by a horizontal vibrator in a semi-infinite material medium [3]. Geometry of the structure and notation are shown in Fig. 1. Here $\{x, y, z\}$ is a Cartesian coordinate system associated with a perfectly conducting plane, a cylindrical vibrator allocated in a medium at a distance h from the plane with material parameters (ε_1, μ_1). The vibrator length is $2L$ and its radius is r.
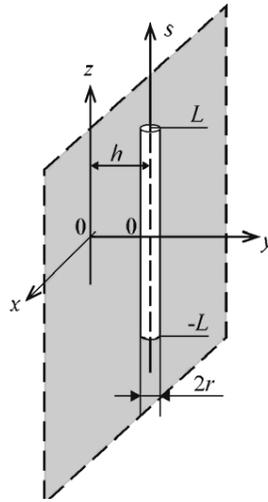
Figure 1. The geometry of the vibrator structure.

Let us substitute Green's function for the half-space

$$G_s(s, s') = \frac{e^{-ik_1\sqrt{(s-s')^2+r^2}}}{\sqrt{(s-s')^2 + r^2}} - \frac{e^{-ik_1\sqrt{(s-s')^2+(2h+r)^2}}}{\sqrt{(s-s')^2 + (2h+r)^2}}. \tag{22}$$

in Equation (4). Then the equation for the vibrator current can be written as

$$\left(\frac{d^2}{ds^2} + k_1^2\right) \int_{-L}^{L} J(s')\frac{e^{-ik_1\sqrt{(s-s')^2+r^2}}}{\sqrt{(s-s')^2 + r^2}}ds'$$

$$= -i\omega\varepsilon_1 E_{0s}(s) + \left(\frac{d^2}{ds^2} + k_1^2\right) \int_{-L}^{L} J(s')\frac{e^{-ik_1\sqrt{(s-s')^2+(2h+r)^2}}}{\sqrt{(s-s')^2 + (2h+r)^2}}ds' + i\omega\varepsilon_1 z_i(s)J(s). \tag{23}$$

Without loss of generality, we assume that the vibrator is excited at the center of a lumped voltage generator with an amplitude V_0. Let us apply the Theorem 2. First, we find the vibrator current in the infinite material medium by solving the first equation of the system (20) using the averaging method. The solution [3, 4, 7] is

$$J(s) = -\alpha V_0 \left(\frac{i\omega\varepsilon_1}{2k_1} \right) \frac{\sin k_1(L - |s|) + \alpha P_\delta^s(k_1 r, k_1 s)}{\cos k_1 L + \alpha P_L^s(k_1 r, k_1 L)}. \tag{24}$$

Here $P_\delta^s(k_1 r, k_1 s) = P^s[k_1 r, k_1(L+s)] - (\sin k_1 s + \sin k_1 |s|) P_L^s(k_1 r, k_1 L)$, the constants $P^s[k_1 r, k_1(L+s)]$ and $P_L^s(k_1 r, k_1 L)$ are defined by the analytical formulas [3, 4, 7].

Then, substituting the current $J(s)$ from (24) into the formula (21), we obtain

$$z_i(s) = \frac{\left(\dfrac{d^2}{ds^2} + k_1^2 \right) \displaystyle\int_{-L}^{L} \left(\sin k_1(L - |s'|) + \alpha P_\delta^s(k_1 r, k_1 s') \right) \dfrac{e^{-ik_1\sqrt{(s-s')^2 + (2h+r)^2}}}{\sqrt{(s - s')^2 + (2h + r)^2}} ds'}{\sin k_1(L - |s|) + \alpha P_\delta^s(k_1 r, k_1 s)}. \tag{25}$$

Thus, the functional $F_S(\vec{r}, \vec{J}(\vec{r}))$ in (17), wherein the integration is done over an infinite boundary surface, is replaced by the functional, wherein integration is performed only over the surface of the mirror vibrator image. Since the kernel of the integral operator is a smooth function, the differentiation operator can be moved under the integral sign. The final expression for the impedance distribution function can be written as

$$z_i(s) = \frac{1}{Fi(s)} \int_{-L}^{L} Fi\left(s' \right) Fe\left(s, s' \right) ds', \tag{26}$$

were

$$Fi(s) = \sin k_1(L - |s|) + \alpha P_\delta^s(k_1 r, k_1 s), \quad R_1\left(s, s' \right) = \sqrt{(s - s')^2 + (2h + r)^2},$$

$$Fe\left(s, s' \right) = \frac{e^{-ik_1 R_1(s,s')}}{(R_1\left(s, s' \right))^4} \left[\begin{array}{l} (s - s')^2 \left(3ik_1 - \dfrac{k_1^2 - 3}{R_1\left(s, s' \right)} \right) \\ -R_1\left(s, s' \right) - ik_1 \left(R_1\left(s, s' \right) \right)^2 + k_1^2 \left(R_1\left(s, s' \right) \right)^3 \end{array} \right].$$

As might be expected, if $h \to \infty$ in expression (26), $|z_i(s)| \to 0$ for the any point on the vibrator surface.

5. CONCLUSION

The paper presents a generalization of the theory of thin impedance vibrators which can be found in a number of publications devoted to vibrator radiators with a lumped excitation in free space or in electrodynamic volumes with coordinate boundaries. Taking into account methodological aspects of the problem, the authors decided to make such a generalization in the form of two theorems. On the one hand, the approach allows systematically assessing already known results, and, on the other hand, extend the methodology to the solutions of new boundary value problems with compound surface boundaries. Requirements for the boundary surface are as follows: it must be a Lyapunov surface, which does not cross current sources, or in a more general sense, it has to be passive and does not possess the properties of mutual transformation of spatial harmonics of the electric and magnetic fields.

The first theorem proves the limited influence of the external electrodynamic medium upon the current distribution on the radiating vibrator. Quantification of this effect was related to the natural small parameter $\alpha = \frac{1}{2\ln[r/(2L)]}$; therefore, the result would not exceed the small parameters α. The theorem proof is based on the lemma, which has independent significance. Four corollaries disclose more fully the theorems' scope. The theorem can be used for selection of the current distribution in the vibrator for the solution of complicated boundary value problems and assessing the accuracy of this approximation.

The second theorem states that the influence of the spatial boundaries, including borders of scattering irregularities on the current distribution in a perfectly conducting radiator can be

compensated by "applying" an extended complex impedance having the required distribution to the vibrator surface. The theorems' application was demonstrated by solving the problem for the horizontal vibrator located above a perfectly conducting plane.

The results presented in the paper can be useful both for electromagnetic theory of thin impedance vibrators and for solution of boundary value problems with vibrator excitations including the questions related physical interpretation of mathematical modeling results.

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