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Generalized Optical Theorem in the Time Domain

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Abstract—The optical theorem is a fundamental result that describes the energy budget of wave scattering phenomena. Most past formulations have been derived in the frequency domain and thus apply only to linear time-invariant (LTI) scatterers and background media. In this paper we develop a new theory of the electromagnetic form of the optical theorem directly in the time domain. The derived formulation covers not only the ordinary optical theorem but also the most general form of this result, known as the generalized optical theorem. The developed formulation provides a very general description of the optical theorem for arbitrary probing fields and general scatterers that can be electromagnetically nonlinear, time-varying, and lossy. In the derived formalism, both the scatterer and the background medium can be nonhomogeneous and anisotropic, but the background is assumed to be LTI and lossless. The derived results are illustrated with a computer simulation study of scattering in the presence of a corner reflector which acts as the background. Connections to prior work on the time-domain optical theorem under plane wave excitation in free space are also discussed.

1. INTRODUCTION

The optical theorem is a fundamental relation that characterizes the energy budget of scattering phenomena (see [1], sec. 13.3 and 13.6, and [2], sec. 1.3.9, for an overview). This fundamental result of wave theory has been the topic of a large body of literature in the past 15 years, which has addressed, among other areas, generalizations to arbitrary probing fields and media [3–5], detailed accounts for nonhomogeneous media [6, 7], descriptions from the point of view of Green's function extraction from field correlations [8, 9], and Green-operator-based optical theorem formulations [10]. Other contributions include formulations of the optical theorem for specialized sensing geometries and coordinate systems [11–13], the optical theorem for active scatterers [14], as well as applications to periodic structures [15], sensors [16], lasers [17], and computational methods [18]. The vast majority of the past efforts in this area have focused on the frequency domain formulation relevant to linear time-invariant (LTI) media. On the other hand, a few papers [19–21] have addressed the time-domain generalization, including a recent paper [22] which considers the application to antenna scattering.

In this contribution, we further expand the theoretical repertoire on the optical theorem in the time domain, addressing pending aspects of the theory not covered in the previous key papers [19–22]. In particular, we consider not only the ordinary optical theorem which is the focus of these papers but also the most general form of the optical theorem known as the generalized optical theorem (see [1], p. 723, and [5] for an overview of the frequency domain version). To the best of our knowledge, this is the first formulation of the generalized optical theorem in the time domain. In addition, it has been shown that in addition to the conventional "real power" optical theorem there is also a less known reactive optical theorem [5]. Both forms of the optical theorem have been found to be quite useful in the design of novel change detection algorithms for random and complex media [14, 23]. In this work we provide the first time domain formulation of the reactive optical theorem. Furthermore, the focus of past time-domain formulations has been on homogeneous backgrounds such as free space, as well as

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probing of the scatterer by homogeneous plane waves. We generalize the time-domain optical theorem to more general nonhomogeneous background media that are LTI and lossless, as well as to arbitrary probing fields, including near fields which are relevant, e.g., in super-resolution imaging. Moreover, the background can be anisotropic and nonreciprocal. Importantly, as in the previous key paper [22], the scatterer itself can be very general. For instance, it can be electromagnetically nonlinear, time-varying, and lossy. In fact, perhaps the main appeal of the time-domain optical theorem is its applicability to arbitrary targets that can be time-varying and nonlinear, for which the frequency domain theory of most previous work (relevant only to LTI scatterers) is not applicable.

It is worth emphasizing that even though the scatterer itself is entirely arbitrary, the background medium in which the scatterer is embedded must be lossless. This is required to be able to measure electromagnetic interaction or power associated to the source induced in the scatterer in the form of a reaction. In addition, the LTI nature of the background is also required to ensure that the reciprocity theorem of the convolution type holds, see, e.g., [24]. This in turn enables the remote sensing of scattering-linked power in the form of a force or field (with an antenna) which is, in essence, the practical implementation and meaning of the optical theorem.

We conclude this introduction with remarks about the notation adopted in the following. We denote time and position vector in three-dimensional space as $t \in \mathbb{R}$ and $\mathbf{r} \in \mathbb{R}^3$, respectively. For simplicity, we do not use the familiar boldface font for vector fields, for example, the electric field is denoted simply as $E(\mathbf{r}, t)$. In addition, we conveniently introduce the convolution inner product $\odot$ defined for any vector fields F and G as

$$(F \odot G)(\mathbf{r}, t) \equiv \int_{-\infty}^{\infty} d\tau F(\mathbf{r}, \tau) \cdot G(\mathbf{r}, t - \tau). \tag{1}$$

It follows from a well-known convolution property that

$$F \odot \frac{\partial}{\partial t} G = \frac{\partial}{\partial t} F \odot G = \frac{\partial}{\partial t}(F \odot G), \tag{2}$$

a result to be recalled later. We also introduce the convolution cross product $\otimes$ defined as

$$(F \otimes G)(\mathbf{r}, t) \equiv \int_{-\infty}^{\infty} d\tau F(\mathbf{r}, \tau) \times G(\mathbf{r}, t - \tau). \tag{3}$$

Similarly, we consider the time reversal operation $\bar{}$ defined as

$$\bar{F}(\mathbf{r}, t) \equiv F(\mathbf{r}, -t) \tag{4}$$

and introduce the correlation inner product $\diamond$ defined as

$$(F \diamond G)(\mathbf{r}, t) \equiv (\bar{F} \odot G)(\mathbf{r}, t) = \int_{-\infty}^{\infty} d\tau F(\mathbf{r}, \tau) \cdot G(\mathbf{r}, t + \tau) \tag{5}$$

as well as the correlation cross product $\star$ defined as

$$(F \star G)(\mathbf{r}, t) \equiv (\bar{F} \otimes G)(\mathbf{r}, t) = \int_{-\infty}^{\infty} d\tau F(\mathbf{r}, \tau) \times G(\mathbf{r}, t + \tau). \tag{6}$$

Note that $F \odot G = G \odot F$ and $F \otimes G = -G \otimes F$ while $(F \diamond G)(t) = (G \diamond F)(-t)$ and $(F \star G)(t) = -(G \star F)(-t)$.

2. PROBLEM FORMULATION, RECIPROCITY, AND INTERACTION RELATIONS IN THE TIME DOMAIN

We consider an arbitrary scatterer embedded in a general locally-reacting, LTI, lossless, anisotropic background medium. The background is in general nonhomogeneous and can be nonreciprocal. It is assumed that the background medium is made of material of finite spatial extent so that the background's constitutive properties are equal to those of free space at infinity. The fields generated by sources and scatterers behave as outgoing waves at infinity. The scatterer is assumed to be of compact spatial support V_0 but is otherwise quite general. For example, the medium constituting the scatterer can be nonlinear, time-varying, and lossy.

The starting point is provided by Maxwell's equations in the background medium, in particular,

$$\begin{aligned} \nabla \times E(\mathbf{r}, t) &= -M(\mathbf{r}, t) - \frac{\partial}{\partial t} B(\mathbf{r}, t) \\ \nabla \times H(\mathbf{r}, t) &= J(\mathbf{r}, t) + \frac{\partial}{\partial t} D(\mathbf{r}, t) \end{aligned} \tag{7}$$

where E, H, D, B, J, and M denote, respectively, the electric field, the magnetic field, the electric flux density, the magnetic flux density, the impressed electric current density, and the impressed magnetic current density. In the background medium, the electric flux density D is given by

$$D(\mathbf{r}, t) = \underline{\epsilon}(\mathbf{r}, t) \odot E(\mathbf{r}, t) \tag{8}$$

where $\underline{\epsilon}$ is the background medium's permittivity tensor. The magnetic flux density B is given by

$$B(\mathbf{r}, t) = \underline{\mu}(\mathbf{r}, t) \odot H(\mathbf{r}, t) \tag{9}$$

where $\underline{\mu}$ is the permeability tensor of the background.

To develop the optical theorems in a framework applicable to both reciprocal and nonreciprocal media, we consider the complementary medium in which the permittivity and permeability are given by

$$\begin{aligned} \underline{\epsilon}^C(\mathbf{r}, t) &= \underline{\epsilon}^T(\mathbf{r}, t) \\ \underline{\mu}^C(\mathbf{r}, t) &= \underline{\mu}^T(\mathbf{r}, t) \end{aligned} \tag{10}$$

where T denotes the transpose. If $\underline{\epsilon}^T = \underline{\epsilon}$ and $\underline{\mu}^T = \underline{\mu}$ the medium is reciprocal. If these conditions do not hold the medium is nonreciprocal.

Consider any two pairs of sources J_1, M_1 and J_2, M_2 having respective supports V_1 and V_2. Sources "1" generate fields E_1, H_1 in the background medium. The same sources generate fields E_1^C, H_1^C in the complementary medium. Sources "2" generate fields E_2, H_2 in the background. The same sources generate fields E_2^C, H_2^C in the complementary medium. We develop next the fundamental reciprocity relations describing the reaction and interaction of these two sets of sources and their fields.

It is well known (see [25], Eq. (24), and [26]) that the following modified reciprocity theorem holds:

$$\int_{V_1} dV \left[\left(E_2^C \odot J_1 \right)(\mathbf{r}, t) - \left(H_2^C \odot M_1 \right)(\mathbf{r}, t) \right] = \int_{V_2} dV \left[(E_1 \odot J_2)(\mathbf{r}, t) - (H_1 \odot M_2)(\mathbf{r}, t) \right]. \tag{11}$$

In the special case in which the background is reciprocal then $E_2^C = E_2$, $H_2^C = H_2$ so that this reduces to the familiar Lorentz reciprocity theorem.

We develop next two correlation-type reciprocity relations which define the signal processing associated to the optical theorems of this work. The starting point is provided by the background medium Maxwell equations corresponding to sources and fields "1", which we write as

$$\begin{aligned} \nabla \times E_1(\mathbf{r}, \tau) &= -M_1(\mathbf{r}, \tau) - \frac{\partial}{\partial \tau} B_1(\mathbf{r}, \tau) \\ \nabla \times H_1(\mathbf{r}, \tau) &= J_1(\mathbf{r}, \tau) + \frac{\partial}{\partial \tau} D_1(\mathbf{r}, \tau). \end{aligned} \tag{12}$$

The corresponding equations for sources and fields "2" are

$$\begin{aligned} \nabla \times E_2(\mathbf{r}, \tau) &= -M_2(\mathbf{r}, \tau) - \frac{\partial}{\partial \tau} B_2(\mathbf{r}, \tau) \\ \nabla \times H_2(\mathbf{r}, \tau) &= J_2(\mathbf{r}, \tau) + \frac{\partial}{\partial \tau} D_2(\mathbf{r}, \tau). \end{aligned} \tag{13}$$

Substituting τ by $t + \tau$ in the above result we obtain

$$\begin{aligned} \nabla \times E_2(\mathbf{r}, t+\tau) &= -M_2(\mathbf{r}, t+\tau) - \frac{\partial}{\partial \tau} B_2(\mathbf{r}, t+\tau) \\ \nabla \times H_2(\mathbf{r}, t+\tau) &= J_2(\mathbf{r}, t+\tau) + \frac{\partial}{\partial \tau} D_2(\mathbf{r}, t+\tau). \end{aligned} \tag{14}$$

Multiplying the first of Eq. (12) by $H_2(\mathbf{r}, t+\tau)$ (i.e., applying $H_2(\mathbf{r}, t+\tau)\cdot$) one obtains

$$H_2(\mathbf{r}, t+\tau) \cdot [\nabla \times E_1(\mathbf{r}, \tau)] = -H_2(\mathbf{r}, t+\tau) \cdot M_1(\mathbf{r}, \tau) - H_2(\mathbf{r}, t+\tau) \cdot \frac{\partial}{\partial \tau} B_1(\mathbf{r}, \tau). \tag{15}$$

Multiplying the last of Eq. (14) by $E_1(\mathbf{r}, \tau)$ we get

$$E_1(\mathbf{r}, \tau) \cdot [\nabla \times H_2(\mathbf{r}, t+\tau)] = E_1(\mathbf{r}, \tau) \cdot J_2(\mathbf{r}, t+\tau) + E_1(\mathbf{r}, \tau) \cdot \frac{\partial}{\partial \tau} D_2(\mathbf{r}, t+\tau). \tag{16}$$

It follows from Eqs. (15), (16) and the vector identity

$$\nabla \cdot (A \times B) = B \cdot \nabla \times A - A \cdot \nabla \times B \tag{17}$$

that

$$\begin{aligned} \nabla \cdot [E_1(\mathbf{r}, \tau) \times H_2(\mathbf{r}, t+\tau)] \;=\; & -H_2(\mathbf{r}, t+\tau) \cdot M_1(\mathbf{r}, \tau) - E_1(\mathbf{r}, \tau) \cdot J_2(\mathbf{r}, t+\tau) \\ & -H_2(\mathbf{r}, t+\tau) \cdot \frac{\partial}{\partial \tau} B_1(\mathbf{r}, \tau) - E_1(\mathbf{r}, \tau) \cdot \frac{\partial}{\partial \tau} D_2(\mathbf{r}, t+\tau). \end{aligned} \tag{18}$$

By means of a similar procedure involving the last of Eq. (12) and the first of Eq. (14) one also obtains

$$\begin{aligned} \nabla \cdot [E_2(\mathbf{r}, t+\tau) \times H_1(\mathbf{r}, \tau)] \;=\; & -H_1(\mathbf{r}, \tau) \cdot M_2(\mathbf{r}, t+\tau) - E_2(\mathbf{r}, t+\tau) \cdot J_1(\mathbf{r}, \tau) \\ & -H_1(\mathbf{r}, \tau) \cdot \frac{\partial}{\partial \tau} B_2(\mathbf{r}, t+\tau) - E_2(\mathbf{r}, +\tau) \cdot \frac{\partial}{\partial \tau} D_1(\mathbf{r}, \tau). \end{aligned} \tag{19}$$

For the special case in which sources and fields "1" are identical to sources and fields "2", and for $t = 0$, the above results, Eqs. (18), (19), reduce to the differential form of Poynting's theorem:

$$\begin{aligned} \nabla \cdot [E_1(\mathbf{r}, \tau) \times H_1(\mathbf{r}, \tau)] \;=\; & -H_1(\mathbf{r}, \tau) \cdot M_1(\mathbf{r}, \tau) - E_1(\mathbf{r}, \tau) \cdot J_1(\mathbf{r}, \tau) \\ & -H_1(\mathbf{r}, \tau) \cdot \frac{\partial}{\partial \tau} B_1(\mathbf{r}, \tau) - E_1(\mathbf{r}, \tau) \cdot \frac{\partial}{\partial \tau} D_1(\mathbf{r}, \tau). \end{aligned} \tag{20}$$

This connection is important. It reveals that the correlation reciprocity relations to be developed next can be interpreted as a generalization of the familiar electromagnetic energy conservation relation based on Poynting's theorem. The left side term in Eq. (20) is related to the flux density of power exiting the differential volume. The first two source-field interaction terms in the right side of Eq. (20) define the density of power put by the sources. Finally, the last two terms in the right side of the same equation define the time rate at which energy is stored in the differential volume, which is usually termed the reactive power density.

Adding Eqs. (18) and (19) and integrating the resulting expression over τ we get

$$\begin{aligned} \nabla \cdot [E_1 \star H_2 - H_1 \star E_2] \;=\; & -M_1 \diamond H_2 - H_1 \diamond M_2 - E_1 \diamond J_2 - J_1 \diamond E_2 \\ & -\frac{\partial}{\partial t} B_1 \diamond H_2 - H_1 \diamond \frac{\partial}{\partial t} B_2 - E_1 \diamond \frac{\partial}{\partial t} D_2 - \frac{\partial}{\partial t} D_1 \diamond E_2 \end{aligned} \tag{21}$$

where we conveniently suppress the space-time dependence $(\mathbf{r}, t)$ with the implicit understanding that all the quantities appearing in the formulation depend on time and position. Also, subtracting Eqs. (18) and (19) and integrating over τ we obtain

$$\begin{aligned} \nabla \cdot [E_1 \star H_2 + H_1 \star E_2] \;=\; & -M_1 \diamond H_2 + H_1 \diamond M_2 - E_1 \diamond J_2 + J_1 \diamond E_2 \\ & -\frac{\partial}{\partial t} B_1 \diamond H_2 + H_1 \diamond \frac{\partial}{\partial t} B_2 - E_1 \diamond \frac{\partial}{\partial t} D_2 + \frac{\partial}{\partial t} D_1 \diamond E_2. \end{aligned} \tag{22}$$

In view of Eqs. (2), (5), (8), (9), the result in Eq. (21) can be written in the following form which is consistent with a result derived earlier in [25], Eq. (29):

$$\begin{aligned} \nabla \cdot [E_1 \star H_2 - H_1 \star E_2] \;=\; & -M_1 \diamond H_2 - H_1 \diamond M_2 - E_1 \diamond J_2 - J_1 \diamond E_2 \\ & +\frac{\partial}{\partial t} \left\{ \bar{H}_1 \odot \left[\left(\underline{\bar{\mu}}^T - \underline{\mu} \right) \odot H_2 \right] \right\} + \frac{\partial}{\partial t} \left\{ \bar{E}_1 \odot \left[\left(\underline{\bar{\epsilon}}^T - \underline{\epsilon} \right) \odot E_2 \right] \right\} \end{aligned} \tag{23}$$

which conveniently reduces to

$$\nabla \cdot [E_1 \star H_2 - H_1 \star E_2] = -M_1 \diamond H_2 - H_1 \diamond M_2 - E_1 \diamond J_2 - J_1 \diamond E_2 \tag{24}$$

if the condition

$$\begin{aligned} \underline{\epsilon}^T(\mathbf{r},-t) &= \underline{\epsilon}(\mathbf{r},t) \\ \underline{\mu}^T(\mathbf{r},-t) &= \underline{\mu}(\mathbf{r},t) \end{aligned} \tag{25}$$

holds. In the rest of the paper we assume this condition corresponding to a lossless background. For the special case of a nondispersive medium for which the permittivity and permeability are of the form

$$\begin{aligned} \underline{\epsilon}(\mathbf{r},t) &= \underline{\epsilon}_0(\mathbf{r})\delta(t) \\ \underline{\mu}(\mathbf{r},t) &= \underline{\mu}_0(\mathbf{r})\delta(t) \end{aligned} \tag{26}$$

where $\delta(\cdot)$ is Dirac's delta function and where $\underline{\epsilon_0}$ and $\underline{\mu_0}$ are space-dependent dyadics, this further implies that the medium is reciprocal, in particular, $\underline{\epsilon}_0^T = \underline{\epsilon}_0$ and $\underline{\mu}_0^T = \underline{\mu}_0$.

Similarly, in view of Eqs. (2), (5), (8), (9) the result in Eq. (22) reduces to

$$\begin{aligned} \nabla \cdot [E_1 \star H_2 + H_1 \star E_2] \;=\; & -M_1 \diamond H_2 + H_1 \diamond M_2 - E_1 \diamond J_2 + J_1 \diamond E_2 \\ & +2\frac{\partial}{\partial t}[\bar{H}_1 \odot (\underline{\mu} \odot H_2)] + 2\frac{\partial}{\partial t}[\bar{E}_1 \odot (\underline{\epsilon} \odot E_2)]. \end{aligned} \tag{27}$$

The correlation reciprocity relation in Eq. (27) has linkages to the reactive energy and appears to be new.

The integral form of (24) follows from the divergence theorem and can be written as

$$\int_{\partial V} dS\hat{\mathbf{n}} \cdot (E_1 \star H_2 - H_1 \star E_2) = -\int_{V_1} dV(M_1 \diamond H_2 + J_1 \diamond E_2) - \int_{V_2} dV(H_1 \diamond M_2 + E_1 \diamond J_2) \tag{28}$$

where V is a volume containing V_1 and V_2 ($V_1 \subseteq V$ and $V_2 \subseteq V$), ∂V is the boundary of V, dS is surface differential element, and $\hat{\mathbf{n}}$ is the unit vector in the outward-normal direction associated to the differential element. Similarly, the integral form of Eq. (27) is

$$\begin{aligned} \int_{\partial V} dS\hat{\mathbf{n}} \cdot (E_1 \star H_2 + H_1 \star E_2) \;=\; & \int_{V_1} dV(-M_1 \diamond H_2 + J_1 \diamond E_2) + \int_{V_2} dV(H_1 \diamond M_2 - E_1 \diamond J_2) \\ & +2\frac{\partial}{\partial t}\int_V dV \left\{\bar{H}_1 \odot \left[\underline{\mu} \odot H_2\right] + \bar{E}_1 \odot \left[\underline{\epsilon} \odot E_2\right]\right\}. \end{aligned} \tag{29}$$

Two classes of time-domain optical theorems follow from the key results in Eqs. (28), (29). The first class is the most general and will be referred to in the rest of the paper as "cross-correlation-type optical theorems" or simply "correlation-type optical theorems". The second class is an important special case of the former, corresponding to "autocorrelations" as opposed to the more general "cross-correlations" of the general theory. As we shall see next the autocorrelations in question correspond to the physical electromagnetic energies. We shall term this special class "autocorrelation-type optical theorems" or simply "energy-type optical theorems".

Furthermore, within each class, "correlation-type" and "autocorrelation" or energy-type, we shall develop both general and special forms of the optical theorem. The key result in Eq. (28) will lead to the new correlation-type and energy-type generalized optical theorems. The special form of Eq. (28) corresponding to identical sources and fields "1" and "2" will lead to the new correlation-type and energy-type ordinary optical theorems. On the other hand, the general expression (29) will lead to the new correlation-type reactive optical theorem and an associated correlation-type generalized optical theorem related to the reactive power. The latter gives, as a special case, an energy-type generalized optical theorem related to the reactive power.

We conclude this section with the associated precursor relations for the ordinary and reactive optical theorems. In particular, for the special case in which the sources and fields "1" and "2" are identical the key result in Eq. (28) takes the form

$$\int_{\partial V} dS\hat{\mathbf{n}} \cdot (E_1 \star H_1 - H_1 \star E_1) = -\int_{V_1} dV(M_1 \diamond H_1 + J_1 \diamond E_1 + H_1 \diamond M_1 + E_1 \diamond J_1). \tag{30}$$

This result is the precursor of the ordinary optical theorems derived in this work. The correlation-type ordinary optical theorem follows from the general expression (30). The more specialized energy-type version is derived by evaluating Eq. (30) for $t = 0$, which gives

$$\int_{-\infty}^{\infty} d\tau \int_{\partial V} dS\hat{\mathbf{n}} \cdot [E_1(\mathbf{r},\tau) \times H_1(\mathbf{r},\tau)] = -\int_{-\infty}^{\infty} d\tau \int_{V_1} dV[M_1(\mathbf{r},\tau) \cdot H_1(\mathbf{r},\tau) + J_1(\mathbf{r},\tau) \cdot E_1(\mathbf{r},\tau)]. \quad (31)$$

The term in the left side of this equation is the total radiated energy. It is equal to the time-integrated sum of the source-field interaction integrals in the right side of the equation. Appendix A provides further interpretative results pertinent to Eqs. (30), (31).

Similarly, for the special case in which the sources and fields "1" and "2" are identical, the key result in Eq. (29) becomes

$$\begin{aligned}\int_{\partial V} dS\hat{\mathbf{n}} \cdot (E_1 \star H_1 + H_1 \star E_1) &= \int_{V_1} dV(-M_1 \diamond H_1 + J_1 \diamond E_1 + H_1 \diamond M_1 - E_1 \diamond J_1) \\ &\quad + 2\frac{\partial}{\partial t}\int_V dV \left[\bar{H}_1 \odot (\underline{\mu} \odot H_1) + \bar{E}_1 \odot (\underline{\epsilon} \odot E_1)\right]. \end{aligned} \quad (32)$$

Evaluating this for $t = 0$ we get that the time-integrated sum of the electric and magnetic reactive powers is equal to zero, i.e.,

$$\int_{-\infty}^{\infty} d\tau \int_V dV \left[H_1(\mathbf{r},\tau) \cdot \frac{\partial}{\partial \tau}B_1(\mathbf{r},\tau) + E_1(\mathbf{r},\tau) \cdot \frac{\partial}{\partial \tau}D_1(\mathbf{r},\tau)\right] = 0. \quad (33)$$

For $t \neq 0$ the general expression (32) can be exploited to gain information about the reactive energy, near field dynamics, which motivates the new correlation-type reactive optical theorem.

3. SCATTERING FORMULATION IN THE TIME DOMAIN

To develop the optical theorems we formulate electromagnetic scattering in a general framework that is applicable to scatterers composed of any kind of material, which can be electromagnetically nonlinear and time-varying. On the other hand, first we develop the basic ideas in the familiar context of LTI media. Later we establish the extensions to arbitrary scatterers and adopt them in the remainder of the paper.

Let V_s be the surveillance or scattering region where the scatterer of support $V_0 \subseteq V_s$ is contained. The scatterer is embedded in the background medium having permittivity and permeability dyadics $\underline{\epsilon}$ and $\underline{\mu}$, respectively. It follows from (7) that when the scatterer is interrogated by quite arbitrary sources labelled "n" which are located outside the volume V_s the corresponding probing fields $E_i^{(n)}$, $H_i^{(n)}$ due to those sources obey in V_s

$$\begin{aligned}\nabla \times E_i^{(n)}(\mathbf{r},t) &= -\frac{\partial}{\partial t}B_i^{(n)}(\mathbf{r},t) \\ \nabla \times H_i^{(n)}(\mathbf{r},t) &= \frac{\partial}{\partial t}D_i^{(n)}(\mathbf{r},t)\end{aligned} \quad (34)$$

where $D_i^{(n)}$ and $B_i^{(n)}$ are the electric and magnetic flux densities associated to the electric and magnetic fields $E_i^{(n)}$ and $H_i^{(n)}$. In the background Eq. (34) can be written as

$$\begin{aligned}\nabla \times E_i^{(n)}(\mathbf{r},t) &= -\frac{\partial}{\partial t}(\underline{\mu} \odot H_i^{(n)})(\mathbf{r},t) \\ \nabla \times H_i^{(n)}(\mathbf{r},t) &= \frac{\partial}{\partial t}(\underline{\epsilon} \odot E_i^{(n)})(\mathbf{r},t).\end{aligned} \quad (35)$$

Let $\underline{\epsilon}_t$ and $\underline{\mu}_t$ be, respectively, the permittivity and permeability dyadics of the total medium composed by the background plus the scatterer. It follows that, relative to the background, the scatterer's constitutive properties are $\underline{\delta\epsilon} = \underline{\epsilon}_t - \underline{\epsilon}$ and $\underline{\delta\mu} = \underline{\mu}_t - \underline{\mu}$. The total fields $E_t^{(n)}$, $H_t^{(n)}$ in the total medium composed by the background plus the scatterer upon the excitation by sources "n" located outside V_s obey in V_s

$$\begin{aligned} \nabla \times E_t^{(n)}(\mathbf{r},t) &= -\frac{\partial}{\partial t} B_t^{(n)}(\mathbf{r},t) \\ \nabla \times H_t^{(n)}(\mathbf{r},t) &= \frac{\partial}{\partial t} D_t^{(n)}(\mathbf{r},t). \end{aligned} \tag{36}$$

This can be written in terms of the fields as

$$\begin{aligned} \nabla \times E_t^{(n)}(\mathbf{r},t) &= -\frac{\partial}{\partial t}(\underline{\delta\mu} \odot H_t^{(n)})(\mathbf{r},t) - \frac{\partial}{\partial t}(\underline{\mu} \odot H_t^{(n)})(\mathbf{r},t) \\ \nabla \times H_t^{(n)}(\mathbf{r},t) &= \frac{\partial}{\partial t}(\underline{\delta\epsilon} \odot E_t^{(n)})(\mathbf{r},t) + \frac{\partial}{\partial t}(\underline{\epsilon} \odot E_t^{(n)})(\mathbf{r},t). \end{aligned} \tag{37}$$

It follows from Eqs. (35), (37) that the scattered fields

$$\begin{aligned} E_s^{(n)}(\mathbf{r},t) &= E_t^{(n)}(\mathbf{r},t) - E_i^{(n)}(\mathbf{r},t) \\ H_s^{(n)}(\mathbf{r},t) &= H_t^{(n)}(\mathbf{r},t) - H_i^{(n)}(\mathbf{r},t) \end{aligned} \tag{38}$$

obey

$$\begin{aligned} \nabla \times E_s^{(n)}(\mathbf{r},t) &= -M_s^{(n)}(\mathbf{r},t) - \frac{\partial}{\partial t}(\underline{\mu} \odot H_s^{(n)})(\mathbf{r},t) \\ \nabla \times H_s^{(n)}(\mathbf{r},t) &= J_s^{(n)}(\mathbf{r},t) + \frac{\partial}{\partial t}(\underline{\epsilon} \odot E_s^{(n)})(\mathbf{r},t) \end{aligned} \tag{39}$$

where we have conveniently introduced the electric and magnetic sources, $J_s^{(n)}$ and $M_s^{(n)}$, respectively, that are induced in the scatterer upon excitation by the sources labeled "n" located outside the scattering region V_s. They are given by

$$\begin{aligned} J_s^{(n)}(\mathbf{r},t) &= \frac{\partial}{\partial t}(\underline{\delta\epsilon} \odot E_t^{(n)})(\mathbf{r},t) \\ M_s^{(n)}(\mathbf{r},t) &= \frac{\partial}{\partial t}(\underline{\delta\mu} \odot H_t^{(n)})(\mathbf{r},t). \end{aligned} \tag{40}$$

The scattered fields $E_s^{(n)}$, $H_s^{(n)}$ are uniquely defined by Eq. (39) subject to the requirement that they behave as outgoing waves at infinity.

Expressions (35), (36) hold for arbitrary scatterers. The particular expressions (40) for the induced sources $J_s^{(n)}$, $M_s^{(n)}$ hold only for LTI scatterers. On the other hand, expression (39) and the associated outgoing wave radiation condition for the scattered fields hold for any scatterer including those for which the map from the total fields $E_t^{(n)}$, $H_t^{(n)}$ to the sources induced in the scatterer $J_s^{(n)}$, $M_s^{(n)}$ is nonlinear or time-varying.

3.1. Field Measurements

To formulate the optical theorems, we need to define how to implement scattered field measurements. It follows from antenna theory that the most general measurement of the scattered fields $E_s^{(n)}$, $H_s^{(n)}$ is of the form

$$V^{(m,n)}(t) = \int_{V_r} dV \left[\left(E_s^{(n)} \odot h_e^{(m)} \right)(\mathbf{r},t) - \left(H_s^{(n)} \odot h_m^{(m)} \right)(\mathbf{r},t) \right] \tag{41}$$

where V_r represents the probe region of localization, $V^{(m,n)}$ represents the output voltage due to the sensing of fields $E_s^{(n)}$, $H_s^{(n)}$ with electric and magnetic probe modes labeled "m" and characterized by local impulse responses $h_e^{(m)}$ and $h_m^{(m)}$, respectively. In view of the reciprocity theorem Eq. (11) the value of the voltage $V^{(m,n)}$ is given in terms of the sources of the scattered field, i.e., the induced sources $J_s^{(n)}$, $M_s^{(n)}$, via

$$V^{(m,n)}(t) = \int_{V_s} dV \left[\left(E_h^{C(m)} \odot J_s^{(n)} \right)(\mathbf{r},t) - \left(H_h^{C(m)} \odot M_s^{(n)} \right)(\mathbf{r},t) \right] \tag{42}$$

where $E_h^{C(m)}$, $H_h^{C(m)}$ are the electric and magnetic fields generated by electric and magnetic sources $h_e^{(m)}$, $h_m^{(m)}$ which obey Maxwell's equations in the complementary medium defined in Eq. (10), namely,

$$\begin{aligned} \nabla \times E_h^{C(m)}(\mathbf{r},t) &= -h_m^{(m)}(\mathbf{r},t) - \frac{\partial}{\partial t}(\underline{\mu}^C \odot H_h^{C(m)})(\mathbf{r},t) \\ \nabla \times H_h^{C(m)}(\mathbf{r},t) &= h_e^{(m)}(\mathbf{r},t) + \frac{\partial}{\partial t}(\underline{\epsilon}^C \odot E_h^{C(m)})(\mathbf{r},t) \end{aligned} \tag{43}$$

plus the radiation condition at infinity.

4. THE ORDINARY OPTICAL THEOREM IN THE TIME DOMAIN

The correlation-type form of the ordinary optical theorem follows from Eq. (31) with the substitutions of E_1, H_1 for the scattered fields $E_s^{(n)}$, $H_s^{(n)}$, and of J_1, M_1 for the induced sources $J_s^{(n)}$, $M_s^{(n)}$. In addition, we substitute V_1 for V_s and ∂V for the boundary ∂V_s of V_s. We also borrow from (38). We obtain the following correlation-type optical theorem:

$$U^{(n)+}(t) = S_e^{(n,n)+}(t) + S_d^{(n,n)+}(t) \tag{44}$$

where

$$U^{(n)+}(t) = \frac{1}{2}\left[U^{(n)}(t) + U^{(n)}(-t)\right], \tag{45}$$

where $U^{(n)}$ is the quantity defined by

$$U^{(n)} = \int_{V_s} dV \left(H_i^{(n)} \diamond M_s^{(n)} + E_i^{(n)} \diamond J_s^{(n)}\right) = \int_{V_s} dV \left(\bar{H}_i^{(n)} \odot M_s^{(n)} + \bar{E}_i^{(n)} \odot J_s^{(n)}\right), \tag{46}$$

and where

$$S_e^{(n,n)+}(t) = \frac{1}{2}\left[S_e^{(n,n)}(t) + S_e^{(n,n)}(-t)\right] \tag{47}$$

where

$$S_e^{(n,n)} = \int_{\partial V_s} dS\hat{\mathbf{n}} \cdot \left(E_s^{(n)} \star H_s^{(n)}\right), \tag{48}$$

while

$$S_d^{(n,n)+}(t) = \frac{1}{2}\left[S_d^{(n,n)}(t) + S_d^{(n,n)}(-t)\right] \tag{49}$$

where

$$S_d^{(n,n)} = \int_{V_s} dV \left(H_t^{(n)} \diamond M_s^{(n)} + E_t^{(n)} \diamond J_s^{(n)}\right). \tag{50}$$

To complete the statement of the optical theorem, it is necessary to specify how to implement the key measurement in Eq. (46) using probes located outside the scattering region V_s. It follows from Eqs. (41), (42), (46) that $U^{(n)}(t)$ can be measured in the form of a scattered field measurement $V^{(n,n)}(t)$ as defined in Eq. (41), i.e., $U^{(n)}(t) = V^{(n,n)}(t)$, where according to (42) the required electric and magnetic field probes characterized by the local impulse response functions $h_e^{(n)}$, $h_m^{(n)}$ are such that they generate, when operating as transmitters, the time-reversed incident fields $\bar{E}_i^{(n)}$, $-\bar{H}_i^{(n)}$ for $\mathbf{r} \in V_s$ in the complementary medium defined in (10), in particular,

$$\begin{aligned} E_h^{C(n)}(\mathbf{r},t) &= E_i^{(n)}(\mathbf{r},-t) \quad \mathbf{r} \in V_s \\ H_h^{C(n)}(\mathbf{r},t) &= -H_i^{(n)}(\mathbf{r},-t) \quad \mathbf{r} \in V_s. \end{aligned} \tag{51}$$

Furthermore, for the important case of an incident electromagnetic pulse passing by the scattering region V_s in a finite, time interval, say, $[0, T]$, it suffices that (51) be obeyed within that interval. It is not hard to show from Maxwell's equations for the complementary medium (and manipulations involving the vector analogues of Green's theorem and Kirchhoff's integral formula [27], sec. 10.6) that these fields are realizable using sources located outside V_s, thus the optical theorem measurement is realizable with probes located outside V_s. Thus for probes $(h_e^{(n)}, h_m^{(n)})$ having these characteristics, the optical theorem expression (44) can be written in terms of the scattering measurement $V^{(n,n)}(t)$ as

$$V^{(n,n)+}(t) = U^{(n)+}(t) = S_e^{(n,n)+}(t) + S_d^{(n,n)+}(t) \tag{52}$$

where

$$V^{(n,n)+}(t) = \frac{1}{2}\left[V^{(n,n)}(t) + V^{(n,n)}(-t)\right]. \tag{53}$$

Here it is very important to point out that the actual probes $(h_e^{(n)}, h_m^{(n)})$ used to implement in practice the optical theorem in Eq. (52) are nonunique, since there is an infinite number of sources outside V_s that generate the fields in Eq. (51). This fundamental nonuniqueness of optical theorem detectors has been elaborated for scalar fields in free space in [28].

The energy-type form of the ordinary optical theorem follows from Eq. (31) via the same substitutions or, equivalently, by evaluating the expression for the correlation-type optical theorem in Eq. (52) for $t = 0$. In particular, the extinct energy, taken away from the probing beam by the scatterer, is equal to $U^{(n)+}(0) = U^{(n)}(0)$, which is given from Eqs. (45), (46) by

$$U^{(n)+}(0) = \int_{-\infty}^{\infty} d\tau \int_{V_s} dV \left[H_i^{(n)}(\mathbf{r},\tau)\cdot M_s^{(n)}(\mathbf{r},\tau) + E_i^{(n)}(\mathbf{r},\tau)\cdot J_s^{(n)}(\mathbf{r},\tau)\right]. \tag{54}$$

As expected, it is given by the integral of the interaction of the incident field $(E_i^{(n)}, H_i^{(n)})$ with the source $(J_s^{(n)}, M_s^{(n)})$ that is induced in the scatterer. Note that if the incident electromagnetic pulse passes by the scattering region V_s in a finite time window, say $[0, T]$, then the only relevant value of the induced source is the one corresponding to that particular interval. Furthermore, it follows from Eq. (41) that the extinct energy can be measured with probes $h_e^{(n)}$, $h_m^{(n)}$ located outside the scattering region V_s via an optical theorem measurement of the form

$$V^{(n,n)+}(0) = \int_{-\infty}^{\infty} d\tau \int_{V_r} dV \left[E_s^{(n)}(\mathbf{r},\tau)\cdot h_e^{(n)}(\mathbf{r},-\tau) - H_s^{(n)}(\mathbf{r},\tau)\cdot h_m^{(n)}(\mathbf{r},-\tau)\right] \tag{55}$$

where the probes $(h_e^{(n)}(\mathbf{r},t), h_m^{(n)}(\mathbf{r},t))$ obey the required conditions in Eq. (51). In addition, the source-field correlation function $S_e^{(n,n)+}(t)$ reduces for $t = 0$ to the total (over all time) scattered energy,

$$S_e^{(n,n)+}(0) = \text{Energy}_s^{(n)} = \int_{-\infty}^{\infty} d\tau \int_{\partial V_s} dS \hat{\mathbf{n}} \cdot \left[E_s^{(n)}(\mathbf{r},\tau) \times H_s^{(n)}(\mathbf{r},\tau)\right]. \tag{56}$$

Also, the source-field correlation function $S_d^{(n,n)+}(t)$ reduces for $t = 0$ to the total energy that is dissipated in the scatterer,

$$S_d^{(n,n)+}(0) = \text{Energy}_d^{(n)} = \int_{-\infty}^{\infty} d\tau \int_{V_s} dV \left[M_s^{(n)}(\mathbf{r},\tau)\cdot H_t^{(n)}(\mathbf{r},\tau) + J_s^{(n)}(\mathbf{r},\tau)\cdot E_t^{(n)}(\mathbf{r},\tau)\right]. \tag{57}$$

In summary, evaluating Eq. (52) for $t = 0$ and using Eqs. (56), (57) we obtain the energy-type ordinary optical theorem:

$$V^{(n,n)}(0) = U^{(n)}(0) = \text{Energy}_s^{(n)} + \text{Energy}_d^{(n)}. \tag{58}$$

5. THE REACTIVE OPTICAL THEOREM IN THE TIME DOMAIN

The correlation-type reactive optical theorem, which generalizes the frequency-domain results in [5], sec. V, is obtained from (32) with the substitutions E_1, $H_1 \to E_s^{(n)}$, $H_s^{(n)}$, J_1, $M_1 \to J_s^{(n)}$, $M_s^{(n)}$, $V_1 \to V_s$, and $\partial V \to \partial V_s$. We obtain the result

$$W^{(n)-}(t) = S_e^{(n,n)-}(t) + S_r^{(n,n)-}(t) + \delta S_r^{(n,n)}(t) \tag{59}$$

where

$$W^{(n)-}(t) = \frac{1}{2}\left[W^{(n)}(t) - W^{(n)}(-t)\right] \tag{60}$$

where

$$W^{(n)} = \int_{V_s} dV \left(-H_i^{(n)} \diamond M_s^{(n)} + E_i^{(n)} \diamond J_s^{(n)}\right) = \int_{V_s} dV \left(-\bar{H}_i^{(n)} \odot M_s^{(n)} + \bar{E}_i^{(n)} \odot J_s^{(n)}\right), \tag{61}$$

and

$$\delta S_r^{(n,n)} = -\frac{\partial}{\partial t} \int_{V_s} dV \left[\bar{H}_s^{(n)} \odot (\underline{\mu} \odot H_s^{(n)}) + \bar{E}_s^{(n)} \odot (\underline{\epsilon} \odot E_s^{(n)})\right], \tag{62}$$

$$S_e^{(n,n)-}(t) = \frac{1}{2}\left[S_e^{(n,n)}(t) - S_e^{(n,n)}(-t)\right], \tag{63}$$

and

$$S_r^{(n,n)-}(t) = \frac{1}{2}\left[S_r^{(n,n)}(t) - S_r^{(n,n)}(-t)\right] \tag{64}$$

where

$$S_r^{(n,n)}(t) = \int_{V_s} dV \left(-H_t^{(n)} \diamond M_s^{(n)} + E_t^{(n)} \diamond J_s^{(n)}\right). \tag{65}$$

The result in Eq. (59) is the reactive optical theorem for special cases in which the quantity $W^{(n)}$ is measurable with probes located outside the scattering region V_s. In particular, the quantity $W^{(n)}$ cannot in general be measured as a scattered field measurement of the form (41), (42). However, if either $J_s^{(n)}$ or $M_s^{(n)}$ is zero, e.g., as in a purely dielectric or purely magnetic material, respectively, then the quantity $W^{(n)}$ can be measured. Consider first the particular case of a scatterer made of a nonmagnetic material so that $M_s^{(n)} = 0$, which implies that Eqs. (61), (65) reduce to

$$\begin{aligned} W^{(n)} &= \int_{V_s} dV \left(E_i^{(n)} \diamond J_s^{(n)}\right) = \int_{V_s} dV \left(\bar{E}_i^{(n)} \odot J_s^{(n)}\right) \\ S_r^{(n,n)} &= \int_{V_s} dV \left(E_t^{(n)} \diamond J_s^{(n)}\right). \end{aligned} \tag{66}$$

In this case $W^{(n)}$ can be measured as a scattered field measurement $V^{(n,n)}$ where the probe modes characterized by $h_e^{(n)}$, $h_m^{(n)}$ are such that when $h_e^{(n)}$, $h_m^{(n)}$ act as sources (transmitters) they generate, in the complementary medium, fields $E_h^{C(n)}$, $H_h^{C(n)}$ obeying Eq. (43) in the scattering region V_s. Another relevant special case is that of a nondielectric material for which $J_s^{(n)} = 0$. In this case $W^{(n)}$ can be measured as a scattered field measurement $V^{(n,n)}$ where $h_e^{(n)}$, $h_m^{(n)}$ are such that they generate, in their transmit counterpart function, the fields $E_h^{C(n)} = -\bar{E}_i^{(n)}$, $H_h^{C(n)} = \bar{H}_i^{(n)}$ in V_s when radiating in the complementary medium. These fields are realizable using sources outside V_s, thus for both purely dielectric and purely magnetic materials the result in Eq. (59) is a reactive optical theorem that can be implemented in practice.

6. THE GENERALIZED OPTICAL THEOREMS IN THE TIME DOMAIN

Let $V^{(m,n)}$ be the scattered field measurement defined in Eq. (41) where the probes characterized by the local impulse response functions $h_e^{(m)}$, $h_m^{(m)}$ are such that they generate, in transmit mode, the fields given by Eq. (51), with n substituted by m, in the complementary medium defined in Eq. (10). Then from Eq. (42)

$$V^{(m,n)} = \int_{V_r} dV \left(H_s^{(n)} \odot h_m^{(m)} + E_s^{(n)} \odot h_e^{(m)}\right) = \int_{V_s} dV \left(\bar{H}_i^{(m)} \odot M_s^{(n)} + \bar{E}_i^{(m)} \odot J_s^{(n)}\right). \tag{67}$$

The correlation-type generalized optical theorem follows from (28) after the substitutions $E_1, H_1 \to E_s^{(m)}, H_s^{(m)}$, $J_1, M_1 \to J_s^{(m)}, M_s^{(m)}$, $V_1 \to V_s$, $\partial V \to \partial V_s$, and $E_2, H_2 \to E_s^{(n)}, H_s^{(n)}$, $J_2, M_2 \to J_s^{(n)}, M_s^{(n)}$, and $V_2 \to V_s$. In view of Eqs. (51), (67), it can be conveniently stated directly in terms of the required generalized optical theorem measurement $V^{(m,n)}$ as

$$V^{(m,n)+}(t) = S_e^{(m,n)+}(t) + S_d^{(m,n)+}(t) \tag{68}$$

where

$$V^{(m,n)+}(t) = \frac{1}{2}\left[V^{(m,n)}(t) + V^{(n,m)}(-t)\right], \tag{69}$$

$$S_e^{(m,n)+}(t) = \frac{1}{2}\left[S_e^{(m,n)}(t) + S_e^{(n,m)}(-t)\right] \tag{70}$$

where

$$S_e^{(m,n)} = \int_{\partial V_s} dS\hat{\mathbf{n}} \cdot \left(E_s^{(m)} \star H_s^{(n)}\right), \tag{71}$$

and

$$S_d^{(m,n)+}(t) = \frac{1}{2}\left[S_d^{(m,n)}(t) + S_d^{(n,m)}(-t)\right] \tag{72}$$

where

$$S_d^{(m,n)} = \int_{V_s} dV \left(H_t^{(m)} \diamond M_s^{(n)} + E_t^{(m)} \diamond J_s^{(n)}\right). \tag{73}$$

Evaluating (68) for $t = 0$ gives the energy-type generalized optical theorem. The latter is an important special case since its ordinary counterpart corresponding to $m = n$ characterizes the energy extinction of the scattering by probing field mode "n', as explained earlier.

We obtain a complementary generalized optical theorem based on Eq. (29) via the same procedure used to obtain Eq. (68). For reasons already explained in connection with the discussion of Eq. (59), we assume next nonmagnetic scatterers. Similar results hold for nondielectric scatterers. We get

$$V^{(m,n)-}(t) = S_e^{(m,n)-}(t) + S_r^{(m,n)-}(t) + \delta S_r^{(m,n)} \tag{74}$$

where

$$V^{(m,n)-}(t) = \frac{1}{2}\left[V^{(m,n)}(t) - V^{(m,n)}(-t)\right], \tag{75}$$

$$S_e^{(m,n)-}(t) = \frac{1}{2}\left[S_e^{(m,n)}(t) - S_e^{(m,n)}(-t)\right], \tag{76}$$

$$S_r^{(m,n)-}(t) = \frac{1}{2}\left[S_r^{(m,n)}(t) - S_r^{(m,n)}(-t)\right] \tag{77}$$

where

$$S_r^{(m,n)} = \int_{V_s} dV \left(E_t^{(m)} \diamond J_s^{(n)}\right), \tag{78}$$

and

$$\delta S_r^{(m,n)} = -\frac{\partial}{\partial t}\int_{V_s} dV \left[\bar{H}_s^{(m)} \odot (\underline{\mu} \odot H_s^{(n)}) + \bar{E}_s^{(m)} \odot (\underline{\epsilon} \odot E_s^{(n)})\right]. \tag{79}$$

7. THE SPECIAL CASE OF FREE SPACE AND PLANE WAVES

In this section we show that the optical theorem results derived in this paper are consistent with those of prior work, particularly the pioneering work of Karlsson [20, 21] which focuses on homogeneous media such as free space and plane wave excitation. In particular, we show that the energy-type ordinary optical theorem in Eq. (58) reduces for free space and plane wave excitation to the results Eq. (2.5) in [20] and Eq. (15) in [21].

In this section we focus on the particular case in which the scatterer is probed with a plane wave traveling in the direction of the unit vector $\mathbf{s}_0$. The respective incident electric field ($E_i^{(n)}$) is given by $E_0(t - \mathbf{s}_0 \cdot \mathbf{r}/c)$ where $\mathbf{s}_0 \cdot E_0 = 0$. Next, it is convenient to substitute the generic label "n" of the preceding general theory by the particular label $\mathbf{s}_0$. This conveniently reminds us that the results hold for excitation with a plane wave traveling in the direction $\mathbf{s}_0$. Thus, we shall denote the respective scattered electric field as $E_s(\mathbf{r}, t; \mathbf{s}_0)$ (instead of $E_s^{(n)}$), and similarly the sources induced in the scatterer will be denoted as $J_s(\mathbf{r}, t; \mathbf{s}_0), M_s(\mathbf{r}, t; \mathbf{s}_0)$ (instead of $J_s^{(n)}$, $M_s^{(n)}$).

It is not hard to show that the (scattered) electric field (E_s) that is generated by sources (J_s, M_s) induced in a scatterer that is embedded in free space behaves in the far zone as

$$E_s(r\hat{\mathbf{r}}, t; \mathbf{s}_0) \sim \frac{F_s(\hat{\mathbf{r}}, t - r/c; \mathbf{s}_0)}{r} \tag{80}$$

where F_s is the far-field scattering amplitude, which is given in terms of the sources by ([29], Eqs. (46)–(49))

$$F_s(\hat{\mathbf{r}}, t; \mathbf{s}_0) = -\frac{\mu_0}{4\pi}\frac{\partial}{\partial t}(1 - \hat{\mathbf{r}}\hat{\mathbf{r}}\cdot)\left[\hat{J}_s(\hat{\mathbf{r}}, t; \mathbf{s}_0) - \hat{\mathbf{r}} \times \hat{M}_s(\hat{\mathbf{r}}, t; \mathbf{s}_0)/\eta)\right] \tag{81}$$

where $\hat{J}_s$ and $\hat{M}_s$ are the slant-stack transforms of J_s and M_s, respectively, which are given by

$$\begin{aligned}\hat{J}_s(\hat{\mathbf{r}}, t; \mathbf{s}_0) &= \int_{V_s} dV J_s(\mathbf{r}, t + \hat{\mathbf{r}}\cdot\mathbf{r}/c; \mathbf{s}_0)\\ \hat{M}_s(\hat{\mathbf{r}}, t; \mathbf{s}_0) &= \int_{V_s} dV M_s(\mathbf{r}, t + \hat{\mathbf{r}}\cdot\mathbf{r}/c; \mathbf{s}_0).\end{aligned} \tag{82}$$

Now, it follows from Eq. (81) with the substitution $\hat{\mathbf{r}} = \mathbf{s}_0$ and the fact that $\mathbf{s}_0 \cdot E_0 = 0$ that

$$E_0(t)\cdot\left[\hat{J}_s(\mathbf{s}_0, t; \mathbf{s}_0) - \mathbf{s}_0 \times \hat{M}_s(\mathbf{s}_0, t; \mathbf{s}_0)/\eta\right] = -\frac{4\pi}{\mu_0}E_0(t)\cdot\int_{-\infty}^{t} dt' F(\mathbf{s}_0, t'). \tag{83}$$

It follows from Eqs. (54), (58) that the extinct energy under this plane wave excitation case is given by

$$\begin{aligned} U &= \int_{V_s} dV \int_{-\infty}^{\infty} dt E_0(t - \mathbf{s}_0\cdot\mathbf{r}/c)\cdot J_s(\mathbf{r}, t; \mathbf{s}_0) + \mathbf{s}_0 \times H_0(t - \mathbf{s}_0\cdot\mathbf{r}/c)\cdot M_s(\mathbf{r}, t; \mathbf{s}_0)/\eta\\ &= \int_{V_s} dV \int_{-\infty}^{\infty} dt' E_0(t')\cdot J_s(\mathbf{r}, t' + \mathbf{s}_0\cdot\mathbf{r}/c; \mathbf{s}_0) + \mathbf{s}_0 \times E_0(t')\cdot M_s(\mathbf{r}, t' + \mathbf{s}_0\cdot\mathbf{r}/c; \mathbf{s}_0)/\eta\\ &= \int_{-\infty}^{\infty} dt' E_0(t')\cdot\left[\hat{J}_s(\hat{\mathbf{s}}_0, t'; \mathbf{s}_0) - \mathbf{s}_0 \times \hat{M}_s(\mathbf{s}_0, t'; \mathbf{s}_0)/\eta\right]\\ &= -\frac{4\pi}{\mu_0}\int_{-\infty}^{\infty} dt' E_0(t')\cdot\int_{-\infty}^{t'} dt'' F_s(\mathbf{s}_0, t''; \mathbf{s}_0), \end{aligned} \tag{84}$$

where we used the fact that the incident magnetic field is

$$H_i(\mathbf{r}, t; \mathbf{s}_0) = \mathbf{s}_0 \times E_0(t - \mathbf{s}_0\cdot\mathbf{r}/c)/\eta,$$

followed by the change of variable $t \to t' + \mathbf{s}_0\cdot\mathbf{r}/c$, followed by a well known vector identity ($A\cdot(B\times C) = C\cdot(A\times B)$), followed by Eq. (82), and finally by Eq. (83). The last equation in Eq. (84) is identical to the statement of the optical theorem for free space and plane waves as derived by Larsson in [20, 21], as we wanted to show.

8. NUMERICAL ILLUSTRATION

In this section we consider in two-dimensional (2D) space the scattering of an electromagnetic pulse by a uniform circular cylinder of radius R. The scattering cylinder is surrounded by a background medium consisting of two perfect electric conductor (PEC) plates that act as a corner reflector. In this example, the background is the free space medium including the corner reflector. We focus on transverse magnetic (TM_z) modes having nonzero electric field component E_z and magnetic field components H_x and H_y. The relevant scattering geometry is illustrated in Fig. 1. The corresponding propagation and scattering is simulated computationally with the FDTD method. In this example, we consider a $9\,\mathrm{m}\times 9\,\mathrm{m}$ scattering region or region of interest (ROI). The boundaries located under and to the right of the ROI are assumed to be open. Computationally, this is modeled via perfectly matched layer (PML) absorbing boundary conditions, as shown in the figure. The probing field is generated by a point source (J_z) whose position is shown in the figure. Its time-dependence is that of the modulated Gaussian

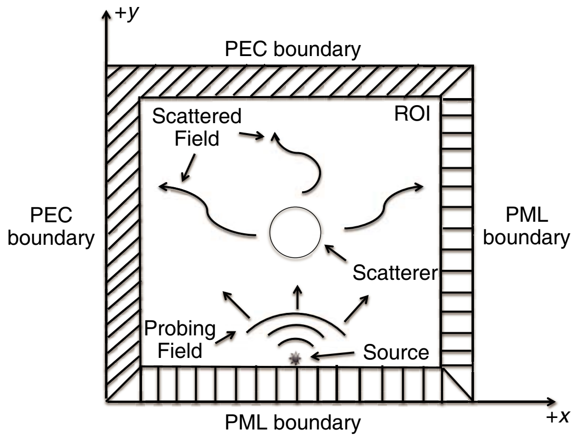

Figure 1. Scattering geometry, showing the probing source, the scatterer, and the corner reflector which acts as the background. PEC boundary conditions are applied to the top and left sides (this is the corner reflector), while PML boundary conditions are implemented at the bottom and right sides of the FDTD computational grid.

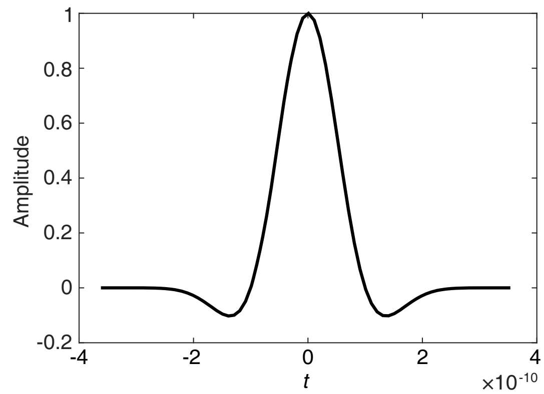

Figure 2. Modulated Gaussian pulse used in the simulation. Time t is in seconds.

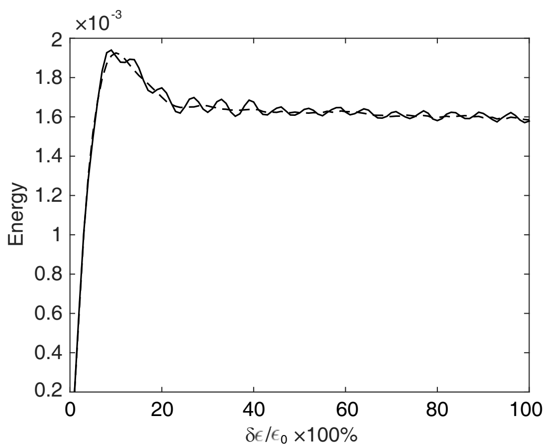

Figure 3. Extinct energy U and scattered energy S_e versus permittivity perturbation $\delta\epsilon$. The dashed line represents U while the solid line corresponds to S_e.

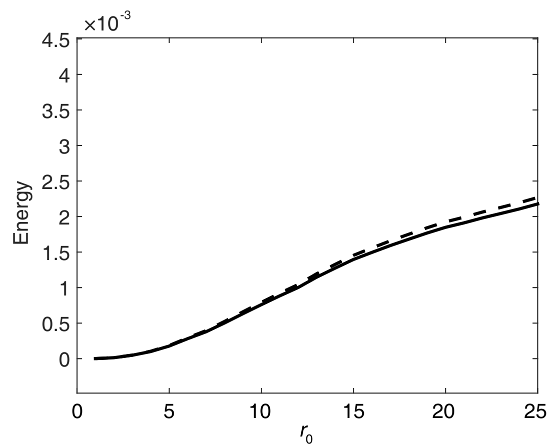

Figure 4. Extinct energy U and scattered energy S_e for fixed $\delta\epsilon = 0.1\epsilon_0$ versus the normalized radius $r_0 = R/\Delta$. The dashed line represents U while the solid line corresponds to S_e.

pulse shown in Fig. 2, which has 2.5 GHz center frequency and fractional bandwidth of 2. In the FDTD simulations, we used a computational grid having space step $\Delta x = \Delta y = \Delta = 0.02\,\text{m}$, and time step $\Delta t = \Delta/(\sqrt{2}c)$ where c is the free space speed of light. In the simulations, we considered scatterer radii (R) in the range [0.02 m, 0.5 m].

To validate the ordinary optical theorem Eq. (58), we computed numerically the key terms appearing in that theorem, namely, the total extinct energy $V = U$, the scattered energy $S_e = \text{Energy}_s$, and the dissipated energy $S_d = \text{Energy}_d$, versus different scattering parameters. Figure 3 shows, for fixed scatterer radius $R = 0.4\,\text{m}$, the variation of U and S_e versus permittivity perturbation $\delta\epsilon$ ranging from $0.01\epsilon_0$ to ϵ_0. In this example the scatterer is lossless so the total extinct energy must be equal to the scattered energy. The plots of U and S_e are, indeed, very similar, as expected. In another set of

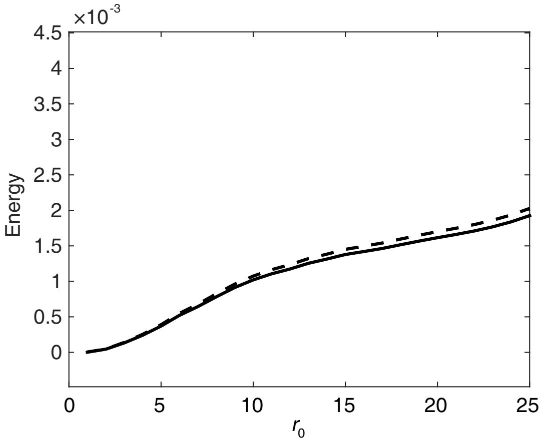

Figure 5. Extinct energy U and scattered energy S_e for fixed $\delta\epsilon = 0.2\epsilon_0$ versus the normalized radius $r_0 = R/\Delta$. The dashed line represents U while the solid line corresponds to S_e.

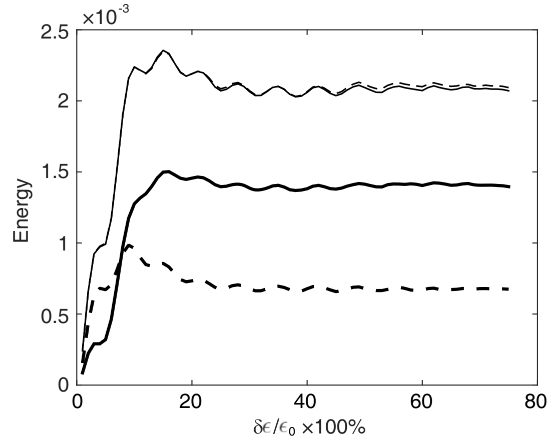

Figure 6. Computed extinct energy versus $\delta\epsilon$ for two scatterers having $R = 0.4\,\mathrm{m}$, and centered at $(x, y) = (-1\,\mathrm{m}, 0)$ and $(1\,\mathrm{m}, 0)$. Bold dashed line: right scatterer. Bold solid line: left scatterer. Thin solid line: sum of the individual extinct energies. Dashed line: extinct energy computed for the two scatterers together.

plots, Figs. 4 and 5 show U and S_e as functions of the normalized scatterer radius $r_0 = R/\Delta$, which was chosen to vary in the range [1, 25]. In these simulations we considered a lossless nonmagnetic scatterer having $\delta\epsilon = 0.1\epsilon_0$ (in Fig. 4) and $\delta\epsilon = 0.2\epsilon_0$ (in Fig. 5). Again we find that, as expected, the plots of the extinct and scattered energies based on the results of Section 4 are very similar. The minor differences between the corresponding plots are attributed to the numerical truncation errors of the FDTD method.

We conclude with another example, which is inspired by an intriguing implication of the time-domain optical theorem discussed in [20, 21]. In particular, the energy extincted by two scatterers is not, in general, equal to the sum of the individual extinct energies of the scatterers. However, the extinction is governed by the interaction of the incident field with the induced source in the scatterer only in the time window in which the probing field passes by the scatterer's support. Therefore, if two scatterers are sufficiently far so that the induced source (in each scatterer), in said time window, is due only to the incident field excitation, and not due to the field scattered by the neighboring scatterer, then the extinction resulting from each scatterer remains identical to that which would be obtained if that scatterer is alone. Thus for such well-separated scatterers the total extinct energy is equal to the sum of the individual extinct energies of the scatterers. This was shown in [20, 21] for the case of an unbounded homogeneous background probed by plane waves. The same principle applies to arbitrary backgrounds and probing fields, as can be easily shown from the results in Section 6 (see the discussion in Eqs. (51), (54). In the next set of figures we illustrate these ideas for the corner reflector background medium. Figure 6 shows the results for two well-separated circular scatterers of radius $R = 0.4\,\mathrm{m}$. In this case the scatterers are centered at positions $(x, y) = (-1\,\mathrm{m}, 0)$ (left scatterer) and $(1\,\mathrm{m}, 0)$ (right scatterer). In this configuration the initial or early contribution to the incident pulse arrives at the same time at both scatterers. The time window associated to the passing of this early probing pulse by each scatterer is approximately equal to $2R/c$. The distance $2R = 0.8\,\mathrm{m}$ is shorter than the separation distance of the two scatterers (equal to $1.2\,\mathrm{m}$ in this example). Thus the multiple scattering interaction between the scatterers happens only after the early part of the probing pulse has passed their supports. In addition to this early probing pulse contribution, there is a late contribution to the probing pulse due to the reverberations at the reflector. Figure 6 shows that for small values of the scatterer permittivity (weak scattering regime) the extinct energy is equal to the sum of the individual extinct energies, as expected since in this limit the scatterers do not interact. The same figure also shows that the effect of the late probing pulse contribution becomes more noticeable as the scatterer permittivity increases, as expected. In particular, for sufficiently large scatterer permittivity ($\delta\epsilon > 0.4\epsilon_0$), the extinct energy

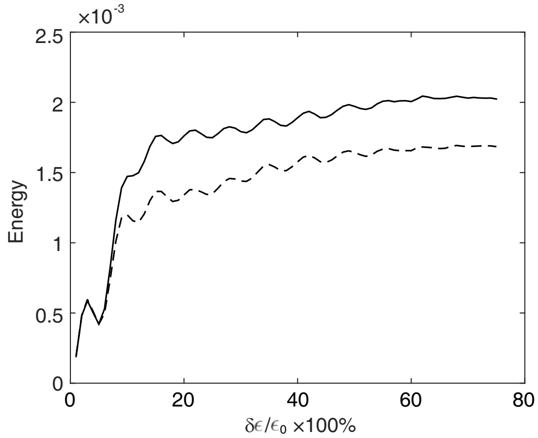

Figure 7. Computed extinct energy versus $\delta\epsilon$ for two scatterers next to each other having $R = 0.4\,\mathrm{m}$, and centered at $(x, y) = (-1\,\mathrm{m}, 0)$ and $(-0.2\,\mathrm{m}, 0)$. The solid line is the sum of the individual extinct energies. The dashed line corresponds to the extinct energy computed for the two scatterers together.

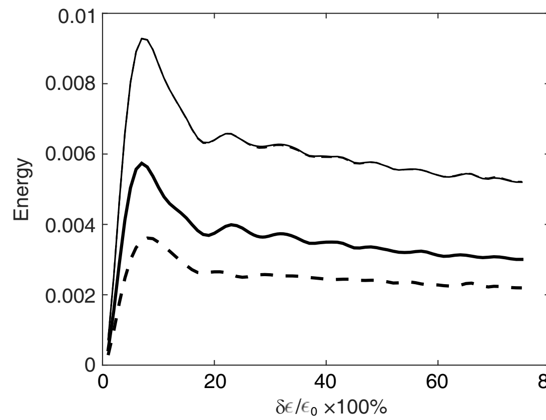

Figure 8. Computed extinct energy versus $\delta\epsilon$ for two scatterers having $R = 0.4\,\mathrm{m}$, and centered at $(x, y) = (-1\,\mathrm{m}, 4\,\mathrm{m})$ and $(1\,\mathrm{m}, 4\,\mathrm{m})$. The bold dashed line applies to the right scatterer only. The bold solid line applies to the left scatterer only. The thin solid line is the sum of the individual extinct energies. The dashed line corresponds to the extinct energy computed for the two scatterers together.

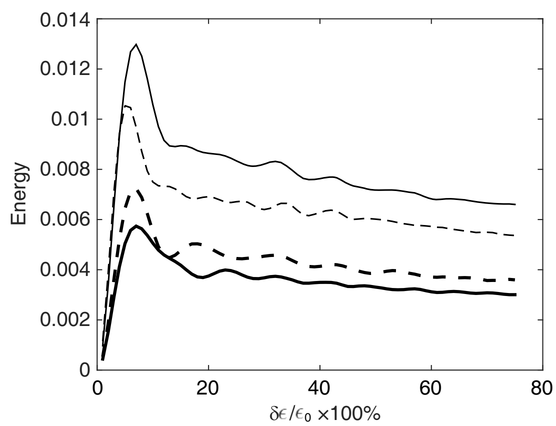

Figure 9. Computed extinct energy versus $\delta\epsilon$ for two scatterers having $R = 0.4\,\mathrm{m}$, and centered at $(x, y) = (-1\,\mathrm{m}, 4\,\mathrm{m})$ and $(-0.2\,\mathrm{m}, 4\,\mathrm{m})$. Bold dashed line: right scatterer. Bold solid line: left scatterer. Thin solid line: sum of the individual extinct energies. Dashed line: extinct energy computed for the two scatterers together.

of the two-scatterer compound is no longer equal to the sum of the extinct energies of the individual scatterers. Figure 7 shows the results for two scatterers of radius $R = 0.4\,\mathrm{m}$ that are placed next to each other. In this case the left scatterer is centered at $(-1\,\mathrm{m}, 0)$ while the right scatterer is centered at $(-0.2\,\mathrm{m}, 0)$. In this case the scatterers are in close proximity, so that the extinct energy is approximately given by the sum of the individual extinct energies only for small $\delta\epsilon$ (corresponding to weak multiple scattering). Figure 8 shows the corresponding results for well-separated scatterers that are placed near the top PEC boundary, at positions $(-1\,\mathrm{m}, 4\,\mathrm{m})$ and $(1\,\mathrm{m}, 4\,\mathrm{m})$. In this case the late reverberating contribution of the probing field passes by the scatterer's support shortly after the early pulse, and the multiple scattering interaction is insignificant within the passing window of the entire probing pulse.

Consequently, the extinct energy is given by the sum of the individual extinct energies. Figure 9 illustrates the corresponding results when the two scatterers are placed next to each other, at positions $(-1\,\text{m}, 4\,\text{m})$ and $(-0.2\,\text{m}, 4\,\text{m})$. As expected, in this case the total extinct energy is in general different from the sum of the individual extinct energies. These two quantities are equal only for small values of the scatterer permittivity $\delta\epsilon$, since then multiple scattering becomes negligible.

9. CONCLUSION

We have developed a very general formulation of the optical theorem in the time domain that applies to arbitrary probing fields and media. The derived theoretical framework for the optical theorem in the time domain is applicable to the most general scatterer, which can be time-varying and nonlinear, and has important envisioned applications such as the validation of computational electromagnetics codes, the fast computation of energy and field correlations, and the design of broadband detectors based on the time domain optical theorem. The derived formulation covers not only the ordinary optical theorem (which has been the focus of past work on the time-domain optical theorem), but also the most general form of this result, known as the generalized optical theorem. Furthermore, two classes of time-domain optical theorems were developed: correlation-type optical theorems, which are the most general relations, as well as more specialized versions called autocorrelation- or energy-type optical theorems. In the discussion of the practical implementation of the optical theorem to measure the extinct energy using external electromagnetic probes or sensors, it was emphasized that the sensors in question are inherently nonunique. Thus even though the statements of the optical theorems derived in this work are universal, their implementations can take in practice on an infinitude of alternative, equally viable forms (see [28]). To demonstrate the link between our general formulation and prior work in this area, we showed that the ordinary form of the time-domain optical theorem presented in this work renders the particular time-domain optical theorem for free space and plane wave excitation derived in previous papers. The derived ordinary optical theorem results were illustrated with the help of numerical examples in which the background medium is a corner reflector. The numerical results confirmed the validity of the formulation and shed insight on important implications of the time-domain optical theorem pointed out in this work and in previous papers in this area.

An interesting avenue for future research that is related to the work reported in this paper is the application of the time-domain optical theorem to change detection with broadband fields. Prior related work, within the simpler frequency-domain formulation, is reported in [23]. We are currently exploring this area and plan to report on the associated research developments in the future.

APPENDIX A.

In the far zone

$$\begin{aligned} E_1(r\hat{\mathbf{r}}, t) &\sim \frac{F_E(\hat{\mathbf{r}}, t - r/c)}{r} \\ H_1(r\hat{\mathbf{r}}, t) &\sim \frac{F_H(\hat{\mathbf{r}}, t - r/c)}{r} \end{aligned} \tag{A1}$$

where $r \equiv |\mathbf{r}|$, $\hat{\mathbf{r}} \equiv \mathbf{r}/r$, and $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the free space speed of light where ϵ_0 and μ_0 are the free space permittivity and permeability, respectively. In addition, F_E and F_H are the far-field radiation patterns corresponding to the electric and magnetic fields, respectively, both of which are perpendicular to $\hat{\mathbf{r}}$, and which are mutually related via

$$F_H(\hat{\mathbf{r}}, t) = \hat{\mathbf{r}} \times F_E(\hat{\mathbf{r}}, t)/\eta \tag{A2}$$

where $\eta = \sqrt{\mu_0/\epsilon_0}$ is the free space impedance.

Using Eqs. (A1), (A2) we find that if the bounding surface ∂V is a large origin-centered sphere in the far zone then

$$\int_{\partial V} dS \hat{\mathbf{n}} \cdot (E_1 \star H_1)(\mathbf{r}, t) = \frac{1}{\eta} \int_{4\pi} d\hat{\mathbf{r}} (F_E \diamond F_E)(\hat{\mathbf{r}}, t) \tag{A3}$$

which is the integral over the unit sphere of the autocorrelation of the electric far-field radiation pattern. In view of a well-known property of the autocorrelation operation,

$$\max \int_{\partial V} dS\hat{\mathbf{n}}\cdot(E_1 \star H_1)(\mathbf{r},t) = \int_{\partial V} dS\hat{\mathbf{n}}\cdot(E_1 \star H_1)(\mathbf{r},0) = \frac{1}{\eta}\int_{-\infty}^{\infty} dt \int_{4\pi} d\hat{\mathbf{r}}[F_E(\hat{\mathbf{r}},t)]^2 = \text{Energy}_{rad} \quad \text{(A4)}$$

where Energy_{rad} denotes the radiated energy. Finally, it follows from Eqs. (30), (A4) and the correlation operation properties mentioned after Eq. (6) that

$$\begin{aligned} \text{Energy}_{rad} &= -\frac{1}{2}\max\left\{\int_{V_1} dV(M_1 \diamond H_1 + J_1 \diamond E_1 + H_1 \diamond M_1 + E_1 \diamond J_1)\right\} \\ &= -\int_{-\infty}^{\infty} dt \int_{V_1} dV\left[(M_1 \cdot H_1)(\mathbf{r},t) + (J_1 \cdot E_1)(\mathbf{r},t)\right]. \end{aligned} \quad \text{(A5)}$$

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