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Definition for Polarization P and Magnetization M Fully Consistent with Maxwell's Equations

Carlo A. Gonano*, Riccardo E. Zich, and Marco Mussetta

Abstract—Dealing with the project of metamaterials scientists often have to design circuit elements at a sub-wavelength (or "microscopic") scale. At that scale, they use the set of Maxwell's equations in free-space, and neither permittivity ε nor permeability μ are formally defined. However, the objective is to use the unit cells in order to build a bulk material with some desired "macroscopic" properties. At that scale the set of Maxwell's equations in matter is adopted. To pass from one approach to the other is not obvious. In this paper we analyse the classic definitions of polarization P and magnetization M, highlighting their limits. Then we propose a definition for P and M fully consistent with Maxwell's equations at any scale.

1. INTRODUCTION

Maxwell's equations are the well-known fundamental laws describing the ElectroMagnetic field [1]. Depending on the system, they can be written in "free-space" or in "matter". Usually, if you are dealing with a "microscopic" system, you use free-space Maxwell's equations. Otherwise, if the system is composed by "macroscopic" bulk materials, you have to use the Maxwell's equations in matter.

In the field of metamaterials [2–4], engineers often have to project small, "microscopic", sub-wavelength unitary cells in order to create a "macroscopic" bulk material with some desired properties. Designing the sub-wavelength circuit or device, you have to deal with free-space Maxwell's equations, while if you want to describe the macroscopic behaviour you are going to use the in-matter set. So the problem of how to council the two different models (approaches), microscopic and macroscopic ones, arises [5]. In particular, polarization P and magnetization M fields should be rigorously defined in order to avoid paradoxes and contradictions.

In the first part of this work we make a comparison between the two sets of Maxwell's Equations, in free-space and in matter respectively, and we derive a third set in order to enlighten the role of P and M.

In the second part we explore the limits of the distinction between "free" and "bound" charges and currents.

In the third part of this work we analyse the classic definitions of P and M, showing they cannot be easily extended to the microscopic case.

In the fourth and fifth parts we fix some conditions and then formally define polarization P and magnetization M for a generic system, no matter if "microscopic" or macroscopic", in a scale-invariant way.

In the sixth we discuss the main results and in the seventh part we resume some conclusions.

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* Corresponding author: Carlo Andrea Gonano (carloandrea.gonano@polimi.it).
The authors are with the Energy Department, Politecnico di Milano, via La Masa 34, 20156 Milan, MI, Italy.

2. COMPARISON OF MAXWELL'S EQUATIONS SETS

2.1. Maxwell's Equations in Free Space in the Time Domain

Maxwell's Equations in free-space or in "vacuum" can be written as:

$$\begin{cases} \overrightarrow{\nabla}^{T}\cdot\left(\varepsilon_0\vec{E}\right)=\rho_e \\ \overrightarrow{\nabla}\times\left(\dfrac{1}{\mu_0}\vec{B}\right)=\vec{J}_e+\dfrac{\partial\left(\varepsilon_0\vec{E}\right)}{\partial t} \end{cases} \qquad \begin{cases} \overrightarrow{\nabla}^{T}\cdot\vec{B}=0 \\ \overrightarrow{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t} \end{cases} \tag{1}$$

where: $\vec{E}$ is the electric field; $\vec{B}$ is the magnetic (induction) field; ρ_e is the electric charge density; $\vec{J}_e$ is the electric current per unit of surface; ε_0 and μ_0 are the free space electric permittivity and magnetic permeability respectively. Here we adopt the superscript T to indicate the transposed (horizontal) vectors and $\overrightarrow{\nabla}$, ensuring consistency with further matrix equations.

2.2. Maxwell's Equations in Matter in the Time Domain

Maxwell's Equations in matter have a similar structure to free-space ones, but they involve the displacement field $\vec{D}$, the magnetic field $\vec{H}$, and *free* charge ρ_f and current $\vec{J}_f$ densities. All of these have to be expressed in terms of $\vec{E}$ and $\vec{B}$ via the constitutive relations in order to be tackled.

$$\begin{cases} \overrightarrow{\nabla}^{T}\cdot\vec{D}=\rho_f \\ \overrightarrow{\nabla}\times\vec{H}=\vec{J}_f+\dfrac{\partial\vec{D}}{\partial t} \end{cases} \qquad \begin{cases} \overrightarrow{\nabla}^{T}\cdot\vec{B}=0 \\ \overrightarrow{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t} \end{cases} \tag{2}$$

The second pair of equations is the same for the two sets, and it can be rephrased in terms of scalar φ_A and vector $\vec{A}$ potentials.

$$\begin{cases} \vec{B}=\overrightarrow{\nabla}\times\vec{A} \\ \vec{E}=-\dfrac{\partial\vec{A}}{\partial t}-\overrightarrow{\nabla}\varphi_A \end{cases} \tag{3}$$

2.3. Maxwell's Equations for P and M

Now we focus the attention on the sets involving the *sources* of the EM fields, and we will try to derive an analogous set in order to enlighten the polarization P and magnetization M. The sources of the EM fields are electric charges and currents, and usually they are divided in *free* and *bound* ones:

$$\rho_e = \rho_f+\rho_b \tag{4}$$

$$\vec{J}_e = \vec{J}_f+\vec{J}_b \tag{5}$$

The displacement $\vec{D}$ and magnetic $\vec{H}$ fields are related through their own definitions to $\vec{P}$ and $\vec{M}$ respectively, in fact:

$$\vec{D}\triangleq\varepsilon_0\vec{E}+\vec{P}\implies\varepsilon_0\vec{E}=\vec{D}-\vec{P} \tag{6}$$

$$\vec{H}\triangleq\frac{1}{\mu_0}\vec{B}-\vec{M}\implies\frac{1}{\mu_0}\vec{B}=\vec{H}+\vec{M} \tag{7}$$

where the symbol $\triangleq$ stands for "is defined to be equal to". Subtracting the first sets of (1) and (2) one from the other, we obtain the Maxwell's Equations for $\vec{P}$ and $\vec{M}$:

$$\begin{cases} \overrightarrow{\nabla}^{T}\cdot(-\vec{P})=\rho_b \\ \overrightarrow{\nabla}\times\vec{M}=\vec{J}_b+\dfrac{\partial(-\vec{P})}{\partial t} \end{cases} \tag{8}$$

The structure is the same as the ones for $\vec{E}$, $\vec{B}$ and $\vec{D}$, $\vec{H}$. However we underline that we have not yet given a definition neither for polarization $\vec{P}$ nor for magnetization $\vec{M}$: anyway they are required to respect (8), otherwise a contradiction between microscopic and macroscopic Maxwell's Equations would arise.

3. FREE AND BOUND CHARGES

Till now we have mentioned *free* and *bound* charges, without saying how to distinguish the former from the latter ones. Here we are going to show that subdivision is not so rigid, and unfortunately those two kinds of charge are usually defined in many different, non-equivalent ways, depending on the context. We analyse some of those "definitions" of *free* and *bound* charges, highlighting their limits.

3.1. Free Charges as Charges Free to Move

A possible definition of *free* charges is:

Free charges are those free to move, instead of bound ones which have limited displacements.

This could be a reasonable definition [6–9], since in metals and conductors electrons can move quite easily in the crystal lattice. In polarized dielectrics, instead, the charges are restrained by strong internal forces and so they are bound to their position. However, that definition does not work in some contexts.

Let's consider a plasma, that is a ionized gas of *charged unbound particles* [10]. Formally, all the charges are *free*, since the molecular forces are negligible and the conductivity is very high. At a microscopic level — that in this case correspond to sizes smaller than the Debye's length λ_D — the plasma is thus usually modelled as system of *free* charged particles.

$$\Delta x \ll \lambda_D \Longrightarrow \text{system of } \textit{free} \text{ charged particles} \tag{9}$$

However, charges produce and are subjected to the EM forces. At a macroscopic level, that is for systems larger than λ_D, the plasma is modelled as a continuous medium with average properties. For example, in the Drude model [11] it is possible to calculate the effective permittivity ε of the plasma as:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \frac{\omega_P^2}{\omega^2} \tag{10}$$

where ω_P is the characteristic plasma "frequency". Since $\vec{D}(\omega) = \varepsilon(\omega)\vec{E}(\omega)$, the polarization of plasma can be different from zero, in fact:

$$\vec{P}(\omega) = (\varepsilon(\omega) - \varepsilon_0)\,\vec{E}(\omega) \tag{11}$$

Since $\vec{P} \neq \vec{0}$, except as $\omega \to \infty$, both ρ_b and $\vec{J}_b$ can be different from zero and so the plasma can contain bound charges.

$$\Delta x \gg \lambda_D \Longrightarrow \text{continuous system with } \textit{bound} \text{ charges} \tag{12}$$

Also in this case the distinction between free and bound charges seems to be related to the *scale* of the system.

Let's now make an other example. Consider a group of *metallic* spheres: if they are invested by an electric field, there will be a displacement of *free* charges on the single surfaces (see Fig. 1). If we look at the same system at a larger scale, interpreting the group of spheres as a bulk metamaterial, then the displacement of charges looks quite limited. Actually, they are *bound* on the surface of the little spheres and cannot move far beyond. Actually, at that macroscopic scale the system looks as a polarized *dielectric*.

3.2. Free Charges as Charges in Conductors

An alternative definition of *free* charge could be based on the distinction between conductors and dielectrics:

Free charges and currents are those in conductors (metals), instead of bound ones which are in dielectrics.

In some ways this definition is even worse than the previous one, since it requires to distinguish between metals and dielectrics. It would be hard to establish if the charge in semiconductors like silicon and germanium is free or bound. Anyway, also this definition does not work in some contexts.

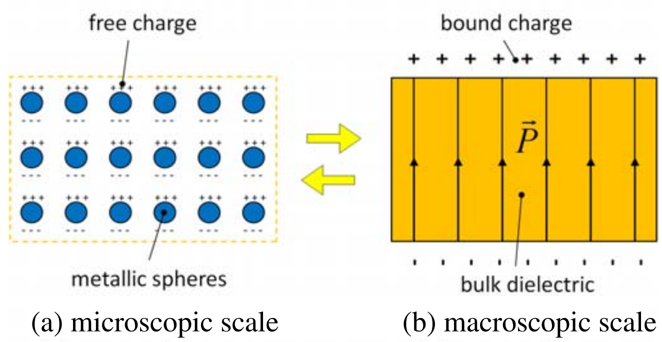

Figure 1. (a) At a microscopic scale, the charge on the single metallic spheres is regarded as *free*; (b) At macroscopic scale, the same system looks as a polarized dielectric since the charges are *bound* and their displacements are small.

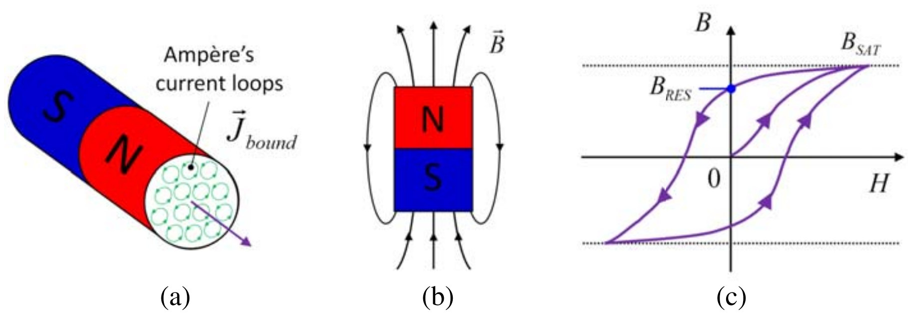

Figure 2. (a) At a microscopic scale, a magnet can be modelled through Amperian loops of *bound* current; (b) Magnetic field produced by a permanent magnet; (c) Hysteresis curve. At $H = 0$ there are residual B field and a net magnetization M.

Let's consider a static permanent magnet: it generates a magnetic field $\vec{B}$ even if there are no visible macroscopic currents. However, a magnet is usually modelled as an ensemble of microscopic current loops (Amperian currents, see Fig. 2). Many permanent magnets are made of conducting materials, like iron, cobalt, nickel etcetera, so the currents inside the magnet should be *free*. Hence, bound currents should be negligible: $|\vec{J}_b| \ll |\vec{J}_f|$. Surprisingly, that's false, and the opposite is true.

In the static case ($\frac{\partial}{\partial t} = 0$), $\vec{H}$ and $\vec{M}$ can be calculated as:

$$\vec{\nabla} \times \vec{H} = \vec{J}_f; \qquad \vec{\nabla} \times \vec{M} = \vec{J}_b \tag{13}$$

Since we are considering a permanent magnet with no external currents, the $\vec{H}$ field inside is zero (see Fig. 3) and the magnet will produce a residual $\vec{B}$ field which can be determined from the hysteresis curve at $H = 0$. In other words, inside the permanent magnet the H field is much smaller than the magnetization M, and so the free currents are much less intense than bound ones:

$$\text{inside permanent magnet} \quad |\vec{H}| \ll |\vec{M}| \Longrightarrow |\vec{J}_f| \ll |\vec{J}_b| \tag{14}$$

Thus, even if the permanent magnet is made of a conducting material, the currents generating its magnetic field $\vec{B}$ are not free, but *bound* ones. Differently, the magnetization field $\vec{M}$ would be identically null.

The case of a empty inductor filled with current (Fig. 4) can be regarded as complementary to the permanent magnet. The current flowing in the windings is usually considered as *free* (so $\vec{J}_e = \vec{J}_f$) and since the inductor is *empty* inside, its magnetization $\vec{M}$ is actually zero: $\vec{M} = \vec{0}$. Field H instead is comparable to B/μ_0:

$$\text{inside inductor} \quad |\vec{H}| \gg |\vec{M}| \Longrightarrow |\vec{J}_f| \gg |\vec{J}_b| \tag{15}$$

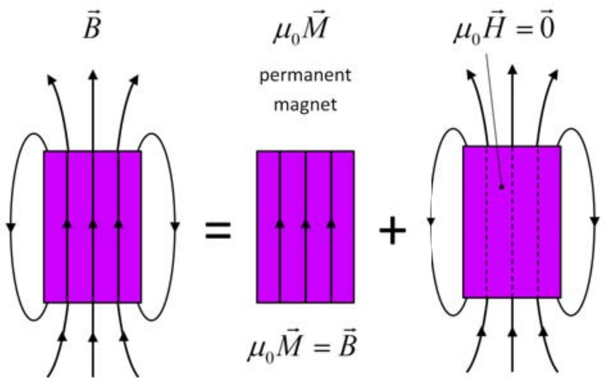

Figure 3. Fields for a permanent magnet. The H field is zero inside, while the magnetization M is high and equal to B/μ_0.

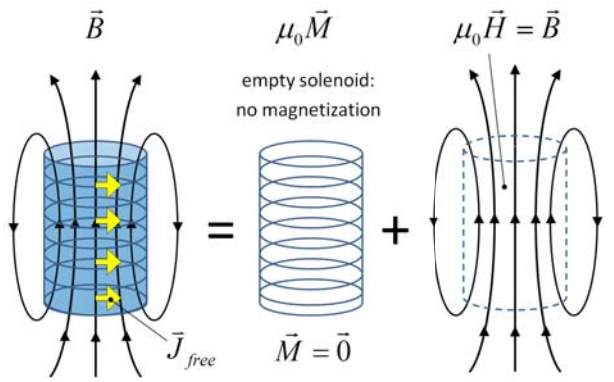

Figure 4. Fields for an empty inductor. The magnetization M is zero inside the solenoid, since it is empty, while the H field is equal to B/μ_0. This case is complementary to that of the permanent magnet.

Summarizing:

$$\text{Inside a permanent magnet} \quad \vec{B}=\mu_0\vec{M}; \quad \vec{H}=\vec{0} \Longrightarrow \vec{J}_e=\vec{J}_b \tag{16}$$

$$\text{Inside an empty inductor} \quad \vec{B}=\mu_0\vec{H}; \quad \vec{M}=\vec{0} \Longrightarrow \vec{J}_e=\vec{J}_f \tag{17}$$

3.3. Concluding Remarks on Free and Bound Charges

As we have seen in the previous paragraphs, to distinguish free charges from bound ones is not so obvious and that subdivision appears to rely on the system to be analysed, and in particular on its scale. Generally speaking, at microscopic scales charges look to be *free* and discrete, while at macroscopic scales they appear to be *bound* and continuous. Moreover, in numerical simulation free charges and currents are considered instead as the *known* or *assigned* sources for the ElectroMagnetic problem. On the contrary, bound charges and currents are considered as *unknown* variables to be calculated, once the constitutive relations for the media are assigned.

Shortly, the concepts of *free* and *bound* charges sound quite fuzzy and arbitrary, and trying to impose a rigid definition of them seems useless. Actually scientists working on different topics will adopt different definitions, suitable to their models and objectives. Here we do not mean to re-define free and bound charges, but if we want to develop and work with coherent models we must know the objects we are dealing with. Otherwise, we risk to achieve contradictory conclusions and misleading results, without being aware of that.

Hereafter we assume the *free* and *bound* charges have been defined in some way by the "user": we are just going to require that Maxwell's Equations have to be satisfied.

3.4. Conservation of Charge

It could be easily verified that electric charges are conserved for each set in (3), (8). Let's consider the "microscopic" case: we take the derivative in time for the divergence of $\vec{E}$, apply the divergence to the curl of $\vec{B}$ (identically zero) and then sum the equations together:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\vec{\nabla}^T\cdot\left(\varepsilon_0\vec{E}\right)\right)=\dfrac{\partial}{\partial t}\rho_e \\ 0=\vec{\nabla}^T\cdot\left(\vec{\nabla}\times\left(\dfrac{1}{\mu_0}\vec{B}\right)\right)=\vec{\nabla}^T\cdot\left(\vec{J}_e+\dfrac{\partial\left(\varepsilon_0\vec{E}\right)}{\partial t}\right)\Longrightarrow \end{cases} \tag{18}$$

$$\frac{\partial\rho_e}{\partial t}+\vec{\nabla}^T\cdot\vec{J}_e=0 \quad \text{conservation of charge} \tag{19}$$

With an analogous procedure, conservation laws for *free* and *bound* charge can be deduced:

$$\frac{\partial \rho_f}{\partial t} + \vec{\nabla}^T \cdot \vec{J}_f = 0 \quad \text{conservation of free charge} \tag{20}$$

$$\frac{\partial \rho_b}{\partial t} + \vec{\nabla}^T \cdot \vec{J}_b = 0 \quad \text{conservation of bound charge} \tag{21}$$

3.5. Fields Produced by Free and Bound Charges

Thanks to the linearity of Maxwell's Equations, we can consider separately free charges from bound ones and decompose the fields $\vec{E}$ and $\vec{B}$ on the basis of their sources. We call:

- $\vec{E}_f, \vec{B}_f, \vec{A}_f, \varphi_{A,f}$ the fields produced by *free* charges and currents.
- $\vec{E}_b, \vec{B}_b, \vec{A}_b, \varphi_{A,b}$ the fields produced by *bound* charges and currents.

The Maxwell's Equations for free charges will look so:

$$\begin{cases} \vec{\nabla}^T \cdot \left(\varepsilon_0 \vec{E}_f\right) = \rho_f \\ \vec{\nabla} \times \left(\dfrac{1}{\mu_0} \vec{B}_f\right) = \vec{J}_f + \dfrac{\partial\left(\varepsilon_0 \vec{E}_f\right)}{\partial t} \end{cases} \qquad \begin{cases} \vec{B}_f = \vec{\nabla} \times \vec{A}_f \\ \vec{E}_f = -\dfrac{\partial \vec{A}_f}{\partial t} - \vec{\nabla} \varphi_{A,f} \end{cases} \tag{22}$$

In the same way, for bound charges will hold:

$$\begin{cases} \vec{\nabla}^T \cdot \left(\varepsilon_0 \vec{E}_b\right) = \rho_b \\ \vec{\nabla} \times \left(\dfrac{1}{\mu_0} \vec{B}_b\right) = \vec{J}_b + \dfrac{\partial\left(\varepsilon_0 \vec{E}_b\right)}{\partial t} \end{cases} \qquad \begin{cases} \vec{B}_b = \vec{\nabla} \times \vec{A}_b \\ \vec{E}_b = -\dfrac{\partial \vec{A}_b}{\partial t} - \vec{\nabla} \varphi_{A,b} \end{cases} \tag{23}$$

The sets (22) and (23) are quite similar to (2) and (8) respectively. In fact, the sources of the fields are unchanged. Summing together the equations in (22) with those in (23) we get the set of Maxwell's Equations in vacuum (1). Obviously the global fields produced by all charges and currents ρ_e and $\vec{J}_e$ will be the sum of those originated by free and bound ones.

$$\vec{E} = \vec{E}_f + \vec{E}_b \qquad \varphi_A = \varphi_{A,f} + \varphi_{A,b} \tag{24}$$

$$\vec{B} = \vec{B}_f + \vec{B}_b \qquad \vec{A} = \vec{A}_f + \vec{A}_b \tag{25}$$

It should be noticed that all the fields in (22) and (23) can propagate also in vacuum, while $\vec{P}$ and $\vec{M}$ are identically zero in empty space. In fact, in vacuum the permittivity and permeability are respectively ε_0 and μ_0, so:

$$\vec{D} = \varepsilon_0 \vec{E} \Longrightarrow \vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \vec{0} \tag{26}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} \Longrightarrow \vec{M} = -\vec{H} + \frac{1}{\mu_0} \vec{B} = \vec{0} \tag{27}$$

In general, $\vec{P}$ and $\vec{M}$ are required to be zero outside any body.

4. ANALYSING THE CLASSIC DEFINITIONS FOR *P* AND *M*

Let's now try to define polarization $\vec{P}$ and magnetization $\vec{M}$. Suppose we are considering a domain Ω_x which could contain some charges, no matter if they *free* or *bound*. We want to determine $\vec{P}$ and $\vec{M}$ on that domain.

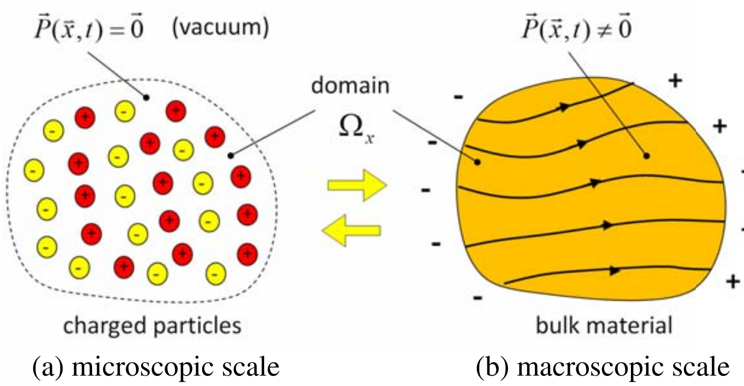

Figure 5. (a) At microscopic scale, the polarization field P among the charged particles should be zero, since that space is vacuum ($\varepsilon = \varepsilon_0$); (b) At macroscopic scale, the same system looks as a bulk material with a non-zero polarization field P.

4.1. Limits of Classic Definition for Polarization P

Classically [6–9, 12] the polarization $\vec{P}$ on the domain Ω_x is defined as the net electric dipole per unit of volume V. If all the electric dipoles $\vec{p}_i$ on Ω_x are known, then $\vec{P}$ is:

$$\vec{P} = \frac{1}{V} \sum_{\Omega_x} (\vec{p}_i) \tag{28}$$

Unfortunately, with this definition $\vec{P}$ is not a *field* $\vec{P}(\vec{x}, t)$, but an *average quantity*: in fact it is not associated to a point $\vec{x}$, but to a whole domain Ω_x.

Let's suppose we consider a system at microscopic scale, made by many charged particles separated from vacuum (Fig. 5). What is the polarization field $\vec{P}(\vec{x}, t)$ on this domain Ω_x? Since in vacuum holds $\vec{D} = \varepsilon_0 \vec{E}$, rigorously $\vec{P}(\vec{x}, t)$ should be zero wherever there are no charges.

$$\vec{P}(\vec{x}, t) = \vec{0} \qquad \forall \vec{x} \in \text{ vacuum} \tag{29}$$

However, if we consider the same system at a larger scale, we could see a *bulk* material with net dipole and polarization. So defining $\vec{P}$ appears to be again a *scale* question.

4.2. Calculating a Net Electric Dipole

Another problem to face in defining $\vec{P}$, also as an average quantity, is how to calculate the net dipole on a domain.

For a group of point charges Q_i, the net dipole $\vec{p}$ is equal to:

$$\vec{p} = \sum_i \vec{p}_i = \sum_i Q_i \cdot (\vec{x}_i - \vec{x}_0) \tag{30}$$

where $\vec{x}_i$ is the ith charge position, while $\vec{x}_0$ is the reference point. More generally, for a domain Ω_x the net electric dipole $\vec{p}$ can be calculated as:

$$\vec{p} = \int_{\Omega_x} \rho \cdot (\vec{x} - \vec{x}_0) \, d\Omega_x \tag{31}$$

The problem resides in the arbitrary point $\vec{x}_0$: if the net charge Q on the domain is different from zero, the value of $\vec{p}$ can change depending on the choice of $\vec{x}_0$, in fact:

$$Q = \int_{\Omega_x} \rho \, d\Omega_x \Longrightarrow \vec{p} = \int_{\Omega_x} (\rho \cdot \vec{x}) \, d\Omega_x - Q \, \vec{x}_0 \tag{32}$$

So the net dipole $\vec{p}$ is independent from $\vec{x}_0$ only if $Q = 0$.

Supposing the point $\vec{x}_0$ is fixed, holding (28), (31) the *average* polarization can be written as:

$$\vec{P} = \frac{1}{V}\int_{\Omega_x} \rho \cdot (\vec{x} - \vec{x}_0)\, d\Omega_x \tag{33}$$

Now we are going to analyse the limits of the definition for magnetization $\vec{M}$, following an analogous procedure.

4.3. Limits of Classic Definition for Magnetization M

Classically [6–9, 12], the magnetization $\vec{M}$ on the domain Ω_x is defined as the net magnetic dipole per unit of volume V. If all the magnetic dipoles $\vec{m}_i$ on Ω_x are known, then $\vec{M}$ is:

$$\vec{M} = \frac{1}{V}\sum_{\Omega_x} (\, \vec{m}_i \,) \tag{34}$$

Similarly to (28), that quantity is not a *field* $\vec{M}(\vec{x}, t)$ but an *average value*: in fact it is not associated to a point $\vec{x}$, but to a whole domain Ω_x.

Let's suppose we consider a system at microscopic scale, made by many circular current loops (amperian currents). What is the magnetization field $\vec{M}(\vec{x}, t)$ on this domain Ω_x?

Since in vacuum holds $\vec{H} = \frac{1}{\mu_0}\vec{B}$, rigorously $\vec{M}(\vec{x}, t)$ should be zero wherever there are no currents.

$$\vec{M}(\vec{x}, t) = \vec{0} \qquad \forall \vec{x} \in \text{ vacuum} \tag{35}$$

However, if we consider the same system at a larger scale, we could see a *bulk* material with net dipole and magnetization. For example, it could be an ordinary magnet. So defining $\vec{M}$ appears to be also a *scale* question.

4.4. Calculating a Net Magnetic Dipole

Another problem to face in defining $\vec{M}$, also as an average quantity, is how to calculate the net magnetic dipole on a domain.

For a group of point charges Q_i, the net magnetic dipole $\vec{m}$ is equal to:

$$\vec{m} = \frac{1}{2}\sum_i (\vec{x}_i - \vec{x}_0) \times (Q_i \vec{v}_i) \tag{36}$$

where $\vec{x}_i$ is the ith charge position, $\vec{v}_i$ is its velocity and $\vec{x}_0$ is the reference point.

More generally, for a domain Ω_x net magnetic dipole $\vec{m}$ can be calculated as:

$$\vec{m} = \frac{1}{2}\int_{\Omega_x} (\vec{x} - \vec{x}_0) \times (\rho\vec{v})\, d\Omega_x = \frac{1}{2}\int_{\Omega_x} (\vec{x} - \vec{x}_0) \times \vec{J}\, d\Omega_x \tag{37}$$

Even in this case, the value of $\vec{m}$ can change depending on the choice of reference point $\vec{x}_0$, so it is not uniquely defined. Supposing $\vec{x}_0$ to be fixed, holding (34), (37) the *average* magnetization can be written as:

$$\vec{M} = \frac{1}{V}\,\frac{1}{2}\int_{\Omega_x} (\vec{x} - \vec{x}_0) \times \vec{J}\, d\Omega_x \tag{38}$$

5. CONDITIONS FOR THE DEFINITION OF *P* AND *M*

Now we are going to give two definitions, one for polarization $\vec{M}$, one for magnetization $\vec{P}$, requiring that these respect some conditions:

- both $\vec{P}$ and $\vec{M}$ must satisfy Maxwell's Equations inside any material Ω_x, that is:

$$\begin{cases} \vec{\nabla}^{T}\!\cdot(-\vec{P}) = \rho_b \\ \vec{\nabla} \times \vec{M} = \vec{J}_b \, + \dfrac{\partial(-\vec{P})}{\partial t} \end{cases} \qquad \forall \vec{x} \in \Omega_x \tag{39}$$

- both $\vec{P}$ and $\vec{M}$ must be zero in free-space, that is outside the materials:

$$\vec{P} = \vec{0}; \quad \vec{M} = \vec{0} \quad \forall \vec{x} \notin \Omega_x \tag{40}$$

- The average values of $\vec{P}$ and $\vec{M}$ should be equal, respectively, to the net electric and magnetic dipole per unit of volume:

$$\begin{cases} \int_{\Omega_x} \vec{P}\, d\Omega_x = \vec{p} = \int_{\Omega_x} \rho_b \cdot (\vec{x} - \vec{x}_0)\, d\Omega_x \\ \int_{\Omega_x} \vec{M}\, d\Omega_x = \vec{m} = \frac{1}{2} \int_{\Omega_x} (\vec{x} - \vec{x}_0) \times \vec{J}_b\, d\Omega_x \end{cases} \tag{41}$$

- Moreover, the definitions for $\vec{P}$ and $\vec{M}$ must be valid for any kind of materials, even for a non-linear or hysteretic one.

We cannot guarantee *a priori* that all these requirements can be accomplished, but we are going to verify if that is possible or not.

5.1. Bound Charges and Currents are Null Outside Materials

Here we prove that, holding conditions (39) and (40), bound charges and currents are identically zero *outside* any material Ω_x. In fact:

$$\begin{cases} \vec{P} = \vec{0} \\ \vec{M} = \vec{0} \end{cases} \quad \forall \vec{x} \notin \Omega_x \Longrightarrow \begin{cases} \vec{\nabla}^T \cdot (-\vec{0}) = \rho_b \\ \vec{\nabla} \times \vec{0} = \vec{J}_b + \dfrac{\partial(-\vec{0})}{\partial t} \end{cases} \Longrightarrow \begin{cases} \rho_b = 0 \\ \vec{J}_b = \vec{0} \end{cases} \quad \forall \vec{x} \notin \Omega_x \tag{42}$$

So the bound charges and currents ρ_b and $\vec{J}_b$ can exist just inside the material Ω_x or at least on its boundary $\partial\Omega_x$.

5.2. Consistency for P Definition

Here we check if the classic definition of $\vec{P}$ as average quantity is coherent with Maxwell's Equation (39). We take the equation linking the divergence for $\vec{P}$ to ρ_b and try to derive the electric dipole's definition.

$$\vec{\nabla}^T \cdot (-\vec{P}) = \rho_b \tag{43}$$

$$\left(\vec{\nabla}^T \cdot \vec{P}\right) \cdot (\vec{x} - \vec{x}_0) = -\rho_b \cdot (\vec{x} - \vec{x}_0) \tag{44}$$

$$\Delta\vec{x} \cdot \left(\vec{\nabla}^T \cdot \vec{P}\right) = -\rho_b \cdot \Delta\vec{x} \tag{45}$$

We place $\Delta\vec{x} = \vec{x} - \vec{x}_0$ in order to simplify the notation, and exploit a differential identity:

$$\Delta\vec{x} \cdot \left(\vec{\nabla}^T \cdot \vec{P}\right) = [\overline{\overline{\Delta x P^T}}] \cdot \vec{\nabla} - \vec{P} \tag{46}$$

More explicitly:

$$\Delta x_i \cdot \sum_{j=1}^{3} \left(\frac{\partial P_j}{\partial x_j}\right) = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (\Delta x_i P_j) - P_i \tag{47}$$

Replacing (46) in (45), it follows:

$$\vec{P} - [\overline{\overline{\Delta x P^T}}] \cdot \vec{\nabla} = \rho_b \cdot \Delta\vec{x} \tag{48}$$

Integrating on the domain Ω_x it yields:

$$\int_{\Omega_x} \vec{P}\, d\Omega_x - \int_{\Omega_x} \left([\overline{\overline{\Delta x P^T}}] \cdot \vec{\nabla}\right) d\Omega_x = \int_{\Omega_x} (\rho_b \cdot \Delta\vec{x})\, d\Omega_x \tag{49}$$

Using an extension of the Gauss' Theorem for Divergence [13], the integral on Ω_x can be rephrased as an integral on its boundary $\partial\Omega_x$, so:

$$\int_{\Omega_x} \vec{P}\, d\Omega_x - \oint_{\partial\Omega_x} \left([\overline{\overline{\Delta x P^T}}] \cdot \vec{n} \right) dS_x = \int_{\Omega_x} (\rho_b \cdot \Delta\vec{x})\, d\Omega_x \tag{50}$$

Remembering that the electric dipole $\vec{p}$ on Ω_x is defined as:

$$\vec{p} = \int_{\Omega_x} (\rho_b \cdot \Delta\vec{x})\, d\Omega_x \tag{51}$$

we get:

$$\int_{\Omega_x} \vec{P}\, d\Omega_x = \vec{p} + \oint_{\partial\Omega_x} \left(\Delta\vec{x} \cdot \left(\vec{P}^T \cdot \vec{n} \right) \right) dS_x \tag{52}$$

We can notice that this result is quite different from the classic definition of $\vec{P}$, which would imply:

$$\int_{\Omega_x} \vec{P}\, d\Omega_x = \vec{p} \tag{53}$$

However, the residual term can be considered as a net dipole produced by the charges on the domain's surface $\partial\Omega_x$.

$$\sigma_b = \vec{n}^T \cdot \vec{P} \implies \oint_{\partial\Omega_x} \left(\Delta\vec{x} \cdot \left(\vec{P}^T \cdot \vec{n} \right) \right) dS_x = \oint_{\partial\Omega_x} (\sigma_b \cdot \Delta\vec{x})\, dS_x \tag{54}$$

In fact, even if inside a body the density of bound charge is zero ($\rho_b = 0$), the net dipole can be different from zero thanks to the surface charge density σ_b.

Moreover, it should be remembered that (39) are required to be valid just *inside* the domain Ω_x and that the value of global dipole depends also on the choice of the reference point $\vec{x}_0$.

Shortly, we can say that the "classic" definition of polarization $\vec{P}$ is consistent with Maxwell's Equations only if we consider also the dipole related to the boundary term (54).

5.3. Consistency for M Definition

Here we check if the classic definition of $\vec{M}$ as average quantity is coherent with Maxwell's Equation (39). We take the equation linking the curl for $\vec{M}$ to $\vec{J}_b$ and try to derive the magnetic dipole's definition.

$$\vec{\nabla} \times \vec{M} = \vec{J}_b + \frac{\partial(-\vec{P})}{\partial t} \tag{55}$$

$$\vec{J}_B = \vec{J}_b + \frac{\partial(-\vec{P})}{\partial t} \implies \vec{\nabla} \times \vec{M} = \vec{J}_B \tag{56}$$

We introduce the equivalent current density $\vec{J}_B$ just to simplify the notation. Now we calculate the moment with respect to a reference point $\vec{x}_0$:

$$(\vec{x} - \vec{x}_0) \times \left(\vec{\nabla} \times \vec{M} \right) = (\vec{x} - \vec{x}_0) \times \vec{J}_B \tag{57}$$

$$\Delta\vec{x} \times \left(\vec{\nabla} \times \vec{M} \right) = \Delta\vec{x} \times \vec{J}_B \tag{58}$$

Again, $\Delta\vec{x} = \vec{x} - \vec{x}_0$ in order to simplify the notation. We exploit a differential identity for curl:

$$\Delta\vec{x} \times \left(\vec{\nabla} \times \vec{M} \right) = 2\vec{M} + \left(\overline{\overline{I}} \left(\vec{M}^T \cdot \Delta\vec{x} \right) - [\overline{\overline{M \Delta x^T}}] \right) \cdot \vec{\nabla} \tag{59}$$

More explicitly:

$$\left(\Delta\vec{x} \times \left(\vec{\nabla} \times \vec{M} \right) \right)_i = 2M_i + \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{3} M_j \Delta x_j \right) - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (M_i \Delta x_j) \tag{60}$$

Replacing (59) in (58), it follows:

$$2\vec{M} + \left(\overline{\overline{I}}\left(\vec{M}^T \cdot \Delta\vec{x}\right) - \overline{\overline{[M\Delta x^T]}}\right) \cdot \overrightarrow{\nabla} = \Delta\vec{x} \times \vec{J}_B \tag{61}$$

Integrating on the domain Ω_x it yields:

$$2\int_{\Omega_x} \vec{M}\, d\Omega_x = \int_{\Omega_x} \Delta\vec{x} \times \vec{J}_B\, d\Omega_x - \int_{\Omega_x} \left(\overline{\overline{I}}\left(\vec{M}^T \cdot \Delta\vec{x}\right) - \overline{\overline{[M\Delta x^T]}}\right) \cdot \overrightarrow{\nabla}\, d\Omega_x \tag{62}$$

In order to transform the integral on Ω_x in an integral on its boundary $\partial\Omega_x$, we use the theorem:

$$\int_{\Omega_x} \left(\overline{\overline{A}} \cdot \overrightarrow{\nabla}\right) d\Omega_x = \oint_{\partial\Omega_x} \left(\overline{\overline{A}} \cdot \vec{n}\right) dS_x \tag{63}$$

where $\overline{\overline{A}}$ is a generic matrix field, while $\vec{n}$ is the normal pointing outward the domain Ω_x. So if follows:

$$\int_{\Omega_x} \left(\overline{\overline{I}}\left(\vec{M}^T \cdot \Delta\vec{x}\right) - \overline{\overline{[M\Delta x^T]}}\right) \cdot \overrightarrow{\nabla}\, d\Omega_x = \oint_{\partial\Omega_x} \left(\overline{\overline{I}}\left(\vec{M}^T \cdot \Delta\vec{x}\right) - \overline{\overline{[M\Delta x^T]}}\right) \cdot \vec{n}\, dS_x \tag{64}$$

Now Equation (62) can be rephrased as:

$$2\int_{\Omega_x} \vec{M}\, d\Omega_x = \int_{\Omega_x} \Delta\vec{x} \times \vec{J}_B\, d\Omega_x - \oint_{\partial\Omega_x} \left(\overline{\overline{I}}\left(\vec{M}^T \cdot \Delta\vec{x}\right) - \overline{\overline{[M\Delta x^T]}}\right) \cdot \vec{n}\, dS_x \tag{65}$$

$$2\int_{\Omega_x} \vec{M}\, d\Omega_x = \int_{\Omega_x} \Delta\vec{x} \times \vec{J}_B\, d\Omega_x - \oint_{\partial\Omega_x} \left(\left(\vec{M}^T \cdot \Delta\vec{x}\right)\vec{n} - \vec{M}\left(\Delta\vec{x} \cdot \vec{n}\right)\right) dS_x \tag{66}$$

The term inside the boundary integral can be compacted using an algebraic identity:

$$\left(\vec{M}^T \cdot \Delta\vec{x}\right)\vec{n} - \vec{M}\left(\Delta\vec{x}^T \cdot \vec{n}\right) = \Delta\vec{x} \times \left(\vec{n} \times \vec{M}\right) = -\Delta\vec{x} \times \left(\vec{M} \times \vec{n}\right) \tag{67}$$

Replacing the last cross product in the integral, we obtain:

$$2\int_{\Omega_x} \vec{M}\, d\Omega_x = \int_{\Omega_x} \Delta\vec{x} \times \vec{J}_B\, d\Omega_x + \oint_{\partial\Omega_x} \left(\Delta\vec{x} \times \left(\vec{M} \times \vec{n}\right)\right) dS_x \tag{68}$$

Remembering that the magnetic dipole $\vec{m}$ on Ω_x is defined as:

$$\vec{m} = \frac{1}{2}\int_{\Omega_x} \Delta\vec{x} \times \vec{J}_b\, d\Omega_x \tag{69}$$

we can notice that (68) is quite different from the classic definition of $\vec{M}$, which would imply:

$$2\int_{\Omega_x} \vec{M}\, d\Omega_x = 2\,\vec{m} = \int_{\Omega_x} \Delta\vec{x} \times \vec{J}_b\, d\Omega_x \tag{70}$$

In fact:

- there is an additional term related to the value of $\vec{M}$ on the boundary $\partial\Omega_x$,
- the current density $\vec{J}_B$ is *different* from the real bound one $\vec{J}_b$. Since $\vec{J}_B = \vec{J}_b + \frac{\partial(-\vec{P})}{\partial t}$, those two quantities are equal just if the polarization $\vec{P}$ is constant in time.

Anyway, the residual term can be interpreted as a net magnetic dipole produced by the currents on the domain's surface $\partial\Omega_x$.

$$\vec{J}_{S,B} = \vec{M} \times \vec{n} \Longrightarrow \oint_{\partial\Omega_x} \left(\Delta\vec{x} \times \left(\vec{M} \times \vec{n}\right)\right) dS_x = \oint_{\partial\Omega_x} \left(\Delta\vec{x} \times \vec{J}_{S,B}\right) dS_x \tag{71}$$

In fact, even if inside a body the density of generalized bound currents is zero ($\vec{J}_B = \vec{0}$), the net magnetic dipole can be different from zero thanks to the surface current density $\vec{J}_{S,B}$.

Moreover, it should be remembered that (39) are required to be valid just *inside* the domain Ω_x and that the value of the global dipole depends also on the choice of the reference point $\vec{x}_0$.

As a consequence, we can say that the "classic" definition of magnetization $\vec{M}$ is not fully consistent with Maxwell's Equations. In fact, it could be considered as a particular case, where the polarization $\vec{P}$ is supposed to be stationary and the surface integral is neglected.

6. DEFINITION OF *P* AND *M* FIELDS

Now let's observe and compare the set of Maxwell's Equations involving ρ_b and $\vec{J}_b$. We rewrite here (23) and (39) for clarity:

$$\begin{cases} \vec{\nabla}^T \cdot (-\vec{P}) = \rho_b \\ \vec{\nabla} \times \vec{M} = \vec{J}_b + \dfrac{\partial(-\vec{P})}{\partial t} \end{cases} \quad \forall \vec{x} \in \Omega_x \quad \begin{cases} \vec{\nabla}^T \cdot \left(\varepsilon_0 \vec{E}_b\right) = \rho_b \\ \vec{\nabla} \times \left(\dfrac{1}{\mu_0} \vec{B}_b\right) = \vec{J}_b + \dfrac{\partial\left(\varepsilon_0 \vec{E}_b\right)}{\partial t} \end{cases} \quad \text{(everywhere)} \tag{72}$$

We can notice that the two systems are almost identical, but the first one is required to be valid just *inside* the domain Ω_x occupied by the bodies: in fact, $\vec{P}$ and $\vec{M}$ are required to be zero outside. The second system instead is valid *everywhere*, since the electric $\vec{E}_b$ and magnetic $\vec{B}_b$ fields can propagate and extend also in vacuum, so outside any material.

It could be useful to define a "belonging function" $\in_\Omega$ such that:

$$\in_\Omega (\vec{x}, t) = \begin{cases} 1 & \text{for } \vec{x} \in \Omega_x(t) \\ 0 & \text{for } \vec{x} \notin \Omega_x(t) \end{cases} \tag{73}$$

This function is equal to 1 *inside* the domain $\Omega_x(t)$, while it is zero *outside*. On the boundary $\partial\Omega_x(t)$ the belonging function is discontinuous, but it could be placed equal to 1/2 if desired. Let's note that $\Omega_x(t)$ can change in time, so we are taking into account also the possibility of a *moving* (or *lagrangian*) domain. Finally, the polarization $\vec{P}$ and magnetization $\vec{M}$ can be related to $\vec{E}_b$ and $\vec{B}_b$ through the belonging function:

$$\begin{cases} -\vec{P} = \in_\Omega \varepsilon_0 \vec{E}_b & -\Delta\vec{P} \\ \vec{M} = \in_\Omega \dfrac{1}{\mu_0} \vec{B}_b & +\Delta\vec{M} \end{cases} \tag{74}$$

The terms $\Delta\vec{P}$ and $\Delta\vec{M}$ were included for the sake of completeness, since they are associated to a kind of "gauge transformation". In fact, they are required to respect two conditions, deriving from (40) and (72):

$$\begin{cases} \vec{\nabla}^T \cdot \Delta\vec{P} = 0 \\ \vec{\nabla} \times \Delta\vec{M} = \vec{0} \end{cases} \quad \forall \vec{x} \in \Omega_x \qquad \begin{cases} \Delta\vec{P} = \vec{0} \\ \Delta\vec{M} = \vec{0} \end{cases} \quad \forall \vec{x} \notin \Omega_x \tag{75}$$

This ensures that Maxwell's Equations are not affected. Hence, here we set both $\Delta\vec{P}$ and $\Delta\vec{M}$ equal to zero.

6.1. Formal Definition of *P* and *M*

We propose these formal definitions for polarization $\vec{P}$ and magnetization $\vec{M}$:

6.1.1. Definition of Polarization Field $\vec{P}$

Given a system of material bodies on a domain Ω_x, the inner polarization field $\vec{P}$ equals the field $-\varepsilon_0 \vec{E}_b$ generated by the bound charges and currents inside the domain itself. Outside the domain Ω_x, the associated polarization field $\vec{P}$ is null.

6.1.2. Definition of Magnetization Field $\vec{M}$

Given a system of material bodies on a domain Ω_x, the inner magnetization field $\vec{M}$ equals the field $\vec{B}_b/\mu_0$ generated by the bound charges and currents inside. Outside the domain Ω_x, the associated magnetization field $\vec{M}$ is null.

Mathematically:

$$\begin{cases} \vec{P} \triangleq - \in_\Omega \varepsilon_0 \vec{E}_b \\ \vec{M} \triangleq \in_\Omega \dfrac{1}{\mu_0} \vec{B}_b \end{cases} \tag{76}$$

The definitions (76) can be rephrased also in function of the EM potentials $\varphi_{A,b}$, $\vec{A}_b$ associated to bound charges and currents:

$$\begin{cases} \vec{P} \triangleq \in_\Omega \ \varepsilon_0 \left(\dfrac{\partial \vec{A}_b}{\partial t} + \vec{\nabla} \varphi_{A,b} \right) \\ \vec{M} \triangleq \in_\Omega \ \dfrac{1}{\mu_0} \left(\vec{\nabla} \times \vec{A}_b \right) \end{cases} \tag{77}$$

These definitions guarantee that $\vec{P}$ and $\vec{M}$ are null outside the domain Ω_x and that Maxwell's Equations are satisfied inside it.

7. DISCUSSION OF THE RESULTS

In this section we discuss some consequences descending from the definitions (76) we have proposed.

7.1. Definition of D and H Fields

Once we have defined $\vec{P}$ and $\vec{M}$, we can express straightforward the electric displacement $\vec{D}$ and the magnetic field $\vec{H}$. Substituting (76) in their own definition we find:

$$\begin{cases} \vec{D} \triangleq \ \varepsilon_0 \vec{E} + \vec{P} \\ \vec{H} \triangleq \dfrac{1}{\mu_0} \vec{B} - \vec{M} \end{cases} \Longrightarrow \begin{cases} \vec{D} = \varepsilon_0 \left(\vec{E} - \in_\Omega \vec{E}_b \right) \\ \vec{H} = \dfrac{1}{\mu_0} \left(\vec{B} - \in_\Omega \vec{B}_b \right) \end{cases} \tag{78}$$

We can express $\vec{D}$ and $\vec{H}$ for points outside or inside the chosen domain Ω_x:

(i) If $\vec{x} \notin \Omega_x$, so for points *outside* the body, then $\in_\Omega = 0$ and thus:

$$\begin{cases} \vec{D} = \varepsilon_0 \vec{E} \\ \vec{H} = \dfrac{1}{\mu_0} \vec{B} \end{cases} \tag{79}$$

Thus $\vec{D}$ and $\vec{H}$ result to be respectively proportional to the global electric $\vec{E}$ and magnetic $\vec{B}$ in vacuum.

(ii) If $\vec{x} \in \Omega_x$, so for points *inside* the body, then $\in_\Omega = 1$ and thus:

$$\begin{cases} \vec{D} = \varepsilon_0 \left(\vec{E} - \vec{E}_b \right) = \varepsilon_0 \vec{E}_f \\ \vec{H} = \dfrac{1}{\mu_0} \left(\vec{B} - \vec{B}_b \right) = \dfrac{1}{\mu_0} \vec{B}_f \end{cases} \tag{80}$$

Thus $\vec{D}$ and $\vec{H}$ result to be respectively proportional to the electric $\vec{E}_f$ and magnetic $\vec{B}_f$ fields produced by *free* charges and currents.

7.2. Multibody Systems

As we have highlighted, the definition of $\vec{P}$ and $\vec{M}$ for a body or a system of bodies depends on the space occupied by the system itself. Actually, the domain Ω_x is usually chosen on the basis of the system's *scale*. Once you fix the space occupied by the system you are interested in, you are implicitly choosing the problem's scale. Now let's consider a multi-body system, made of different domains Ω_i, each one associated to the i^{th} body.

$$\Omega_x = \bigcup_{i=1}^{N} (\Omega_i) \tag{81}$$

$$\Omega_i \cap \Omega_j = \emptyset \qquad \forall i \neq j \tag{82}$$

The belonging functions $\in_{\Omega i}$ for the sub-domains Ω_i will be so related:

$$\in_{\Omega} = \sum_i \in_{\Omega i} = \in_{\Omega 1} (\vec{x}, t) + \in_{\Omega 2} (\vec{x}, t) + \ldots + \in_{\Omega N} (\vec{x}, t) \tag{83}$$

We suppose to know all the charges and currents ρ_b, $\vec{J}_b$, and we want to determine the global polarization $\vec{P}$ and magnetization $\vec{M}$ fields. In order to do that, we must be aware that:

- charges $\rho_{b,i}$ and currents $\vec{J}_{b,i}$ inside the domain Ω_i can generate $\vec{E}_{b,i}$, $\vec{B}_{b,i}$ fields extending also in other domains Ω_j.
- the global electric dipole $\vec{p}$ on the whole domain can be different from the sum of the single domains' dipoles $\vec{p}_i$. In fact, the electric dipole is not a simple additive quantity.

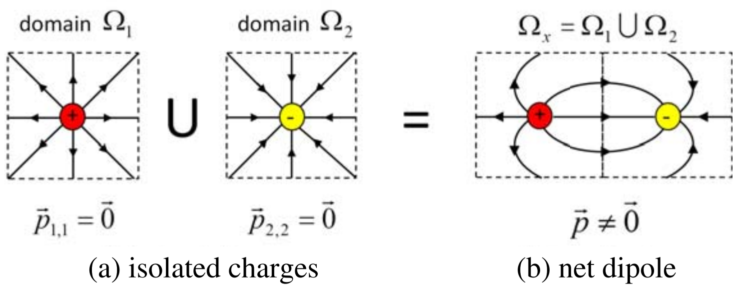

Figure 6. (a) The electric dipoles $\vec{p}_{1,1}$ and $\vec{p}_{2,2}$ — calculated separately for two charges on different domains Ω_1 and Ω_2 — are zero; (b) If the domains are united and considered as a global system, then the net dipole $\vec{p}$ can result to be different from the sum of $\vec{p}_{1,1}$ and $\vec{p}_{2,2}$.

For example, we can consider two single, isolated charges on domains Ω_1 and Ω_2 respectively (Fig. 6). If we calculate separately the dipoles $\vec{p}_{1,1}$ and $\vec{p}_{2,2}$, they can be identically zero:

$$\vec{p}_{1,1} = \vec{0}; \quad \vec{p}_{2,2} = \vec{0} \tag{84}$$

But if we look at the whole system, joining together the domains Ω_1 and Ω_2, we obtain a net dipole different from zero:

$$\vec{p} \neq \vec{0} \Longrightarrow \vec{p} \neq \vec{p}_{1,1} + \vec{p}_{2,2} \tag{85}$$

So the global electric dipole should be calculated considering also the interaction between the charges on different domains.

- the global magnetic dipole $\vec{m}$ on the whole domain can be different from the sum of the single domains' dipoles $\vec{m}_i$. In fact, the magnetic dipole is not a simple additive quantity.
For example, we can consider two isolated currents flowing across the domains Ω_1 and Ω_2 respectively (Fig. 7). If we calculated separately the magnetic dipoles $\vec{m}_{1,1}$ and $\vec{m}_{2,2}$, they can be identically zero:

$$\vec{m}_{1,1} = \vec{0}; \quad \vec{m}_{2,2} = \vec{0} \tag{86}$$

But if we look at the whole system, joining together the domains Ω_1 and Ω_2, we obtain a net magnetic dipole different from zero:

$$\vec{m} \neq \vec{0} \Longrightarrow \vec{m} \neq \vec{m}_{1,1} + \vec{m}_{2,2} \tag{87}$$

So the global magnetic dipole should be calculated considering also the interaction between the currents on different domains.

Shortly, in order to calculate the global $\vec{P}$ and $\vec{M}$ on a multi-body systems we must take into account the *mutual interactions* among the single sub-domains Ω_i.

Here we report a possible procedure aiming to determine the fields produced and induced on the sub-domains Ω_i, and the global fields on the whole domain Ω_x.

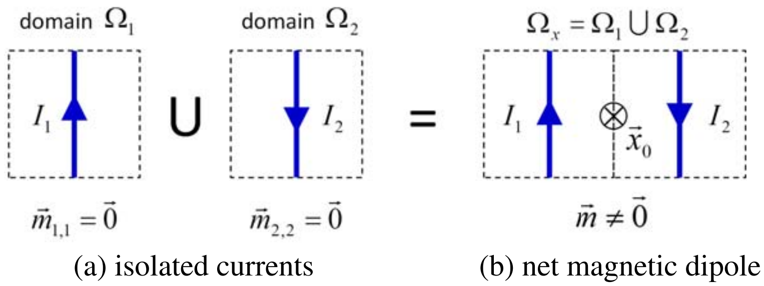

Figure 7. (a) The magnetic dipoles $\vec{m}_{1,1}$ and $\vec{m}_{2,2}$ — calculated separately for two currents on different domains Ω_1 and Ω_2 — are zero; (b) If the domains are united and considered as a global system, then the net dipole $\vec{m}$ can result to be different from the sum of $\vec{m}_{1,1}$ and $\vec{m}_{2,2}$.

(i) Determine the densities of bound charge $\rho_{b,i}$ and current $\vec{J}_{b,i}$ inside single Ω_i. If the global distributions are already known, then it holds:

$$\begin{cases} \rho_{b,i} = \in_{\Omega i} \ \rho_b \\ \vec{J}_{b,i} = \in_{\Omega i} \ \vec{J}_b \end{cases} \tag{88}$$

(ii) Using the Maxwell's Equation (23), calculate the electric and magnetic fields generated by the sources $\rho_{b,i}$, $\vec{J}_{b,i}$.

(iii) Calculate the polarization $\vec{P}_{i,j}$ and magnetization $\vec{M}_{i,j}$ on the sub-domain Ω_i induced by the sources in Ω_j:

$$\begin{cases} \vec{P}_{i,j} = - \in_{\Omega i} \ \varepsilon_0 \vec{E}_{b,j} \\ \vec{M}_{i,j} = \in_{\Omega i} \ \dfrac{1}{\mu_0} \vec{B}_{b,j} \end{cases} \tag{89}$$

The global polarization $\vec{P}_i$ and magnetization $\vec{M}_i$ on the sub-domain Ω_i will be equal to the sum of all the single contributions:

$$\begin{cases} \vec{P}_i = \displaystyle\sum_j \vec{P}_{i,j} = - \in_{\Omega i} \ \varepsilon_0 \vec{E}_b \\ \vec{M}_i = \displaystyle\sum_j \vec{M}_{i,j} = \in_{\Omega i} \ \dfrac{1}{\mu_0} \vec{B}_b \end{cases} \tag{90}$$

(iv) The global polarization $\vec{P}$ and magnetization $\vec{M}$ on the whole domain Ω are given by the sum of the single fields $\vec{P}_i$, $\vec{M}_i$ respectively:

$$\begin{cases} \vec{P} = \displaystyle\sum_i \vec{P}_i \\ \vec{M} = \displaystyle\sum_i \vec{M}_i \end{cases} \tag{91}$$

More explicitly:

$$\begin{cases} \vec{P} = \displaystyle\sum_i \sum_j \vec{P}_{i,j} = - \sum_i \sum_j \in_{\Omega i} \ \varepsilon_0 \vec{E}_{b,j} \\ \vec{M} = \displaystyle\sum_i \sum_j \vec{M}_{i,j} = \sum_i \sum_j \in_{\Omega i} \ \dfrac{1}{\mu_0} \vec{B}_{b,j} \end{cases} \tag{92}$$

The global electric and magnetic field are sum of the fields produced in the single domain. Moreover, the global belonging function is equal to the sum of the other ones, so:

$$\in_\Omega = \sum_i \in_{\Omega i} \qquad \begin{cases} \vec{E}_b = \displaystyle\sum_j \vec{E}_{b,j} \\ \vec{B}_b = \displaystyle\sum_j \vec{B}_{b,j} \end{cases} \tag{93}$$

Finally, as required we obtain:

$$\begin{cases} \vec{P} = -\in_\Omega\ \varepsilon_0 \vec{E}_b \\ \vec{M} = \in_\Omega\ \dfrac{1}{\mu_0} \vec{B}_b \end{cases} \tag{94}$$

So we have verified that the proposed definition works also for multi-body systems.

This property is quite important if you have to implement a numerical simulation in order to determine the EM fields in a complex system. For example, you might be interested to determine the effective, global permittivity ε and permeability μ for a macroscopic, bulk meta-material made of elements with different EM properties [2, 14, 15, 25, 26].

7.3. Relativistic Body

Our definition for $\vec{P}$ and $\vec{M}$ was conceived to be consistent with the Special Relativity Theory, too.

Here we are going to give just some hints, since treating rigorously the problem of a body moving at relativistic speed would require much longer explanations.

A body at rest in a reference frame A is supposed to have a certain permittivity ε_A and permeability μ_A. More generally, it can be characterized by some polarization $\vec{P}_A$ and magnetization $\vec{M}_A$. If we look at the same body but in a different reference frame L, its permittivity ε_L and permeability μ_L could be different [16–19]. Moreover, if in the ref. frame A the body's material was *isotropic*, in the moving ref. frame L it could turn out to be *anisotropic*. That could give rise to a formal problem, since the constitutive relation for the material appear to be different with the change of the reference frame [8, 9, 20, 24]. In other words, we have to face a physical law which seems not to be invariant with the reference frame, and that is usually to be avoided in Relativity [23].

We tried to rephrase part of the problem focusing the attention on the transformation for $\vec{P}$ and $\vec{M}$. Suppose that the polarization $\vec{P}_A$ and the magnetization $\vec{M}_A$ were known in the ref. frame A. We want to determine $\vec{P}_L$ and $\vec{M}_L$ in a ref. frame L, moving at constant speed $\vec{v}$ with respect to A.

7.3.1. Electromagnetic Tensor and Lorentz Transformation

From the definition (76) of $\vec{P}$ and $\vec{M}$ we can notice that these are strongly related to electric and magnetic fields. Now, in Relativity it is possible to build the ElectroMagnetic tensor $F^{\mu\nu}$, containing the electric $\vec{E}$ and magnetic $\overline{\overline{B}}$ fields [8]:

$$F^{\mu\nu} = \overline{\overline{\mathcal{F}}} = \begin{bmatrix} 0 & -\vec{E}^T/c_0 \\ \vec{E}/c_0 & \overline{\overline{B}} \end{bmatrix} = \begin{bmatrix} 0 & -E_1/c_0 & -E_2/c_0 & -E_3/c_0 \\ E_1/c_0 & 0 & -B_3 & B_2 \\ E_2/c_0 & B_3 & 0 & -B_1 \\ E_3/c_0 & -B_2 & B_1 & 0 \end{bmatrix} \tag{95}$$

where c_0 is the speed of light in vacuum and $\overline{\overline{B}}$ the magnetic tensor.

The EM tensor $\overline{\overline{\mathcal{F}}}_L$ in ref. frame L can be obtained applying the Lorentz Transformation to the original one in A. The matrix (tensor) $\overline{\overline{\Lambda}}_{LA}$ associated to the Lorentz Transformation from A to L is:

$$\Lambda^\mu{}_\nu = \overline{\overline{\Lambda}}_{LA}(\vec{v}) = \begin{bmatrix} \gamma & -\gamma\vec{\beta}^T \\ -\gamma\vec{\beta} & \overline{\overline{\gamma}} \end{bmatrix} \tag{96}$$

where:

$$\vec{\beta} = \frac{\vec{v}}{c_0}; \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}; \quad \overline{\overline{\gamma}} = \overline{\overline{I}} + (\gamma - 1)[\overline{\overline{\vec{\beta}\vec{\beta}^T}}]/\beta^2 \tag{97}$$

The EM tensor $\overline{\overline{\mathcal{F}}}_L$ in L can be so calculated as:

$$\overline{\overline{\mathcal{F}}}_L = \overline{\overline{\Lambda}}_{LA} \cdot \overline{\overline{\mathcal{F}}}_A \cdot \overline{\overline{\Lambda}}_{LA}^T \tag{98}$$

The same equation can be rewritten with a notation more common in Relativity:

$$F'^{\alpha\beta} = \Lambda^\alpha{}_\mu F^{\mu\nu} \Lambda^\beta{}_\nu \tag{99}$$

where $F'^{\alpha\beta} = \overline{\overline{\mathcal{F}}}_L$ and $F^{\mu\nu} = \overline{\overline{\mathcal{F}}}_A$. Once the EM tensor $\overline{\overline{\mathcal{F}}}_L$ has been calculated, the electric $\vec{E}_L$ and magnetic $\overline{\overline{B}}_L$ fields are known.

$$\begin{cases} \vec{E}_L = \gamma \left(\vec{E}_A - \vec{B}_A \times \vec{v} \right) & -(\gamma - 1) \dfrac{1}{v^2} \left(\vec{v}^T \cdot \vec{E}_A \right) \vec{v} \\ \vec{B}_L = \gamma \left(\vec{B}_A + \dfrac{1}{c_0^2} \vec{E}_A \times \vec{v} \right) & -(\gamma - 1) \dfrac{1}{v^2} \left(\vec{v}^T \cdot \vec{B}_A \right) \vec{v} \end{cases} \tag{100}$$

We underline that for (100) the usual 3D notation has been adopted, but since $\vec{B}$ is a *pseudovector* [21], it would be preferable to continue to express it as a tensor $\overline{\overline{B}}$. In a n-dimensional (ND) notation [13, 22] the same transformation can be rephrased as:

$$\begin{cases} \vec{E}_L = \gamma \left(\vec{E}_A - \overline{\overline{B}}_A \vec{v} \right) - (\gamma - 1) \dfrac{1}{v^2} [\overline{\overline{vv^T}}] \cdot \vec{E}_A \\ \overline{\overline{B}}_L = \overline{\overline{\gamma}}\, \overline{\overline{B}}_A \overline{\overline{\gamma}} + \dfrac{\gamma}{c_0^2} \left(\vec{E}_A \overset{\wedge}{\wedge} \vec{v} \right) \end{cases} \tag{101}$$

For sake of simplicity, in the following we are going to use the 3D notation.

7.3.2. Transformation for P and M

The EM tensor $\overline{\overline{\mathcal{F}}}$ can be transformed from one ref. frame to another, and it contains the *global* electric and magnetic fields.

Thanks to Maxwell's Equations linearity, we can decompose it in the sum of two tensors $\overline{\overline{\mathcal{F}}}_f$ and $\overline{\overline{\mathcal{F}}}_b$, the former associated to free charges, the latter to bound ones.

$$\overline{\overline{\mathcal{F}}} = \overline{\overline{\mathcal{F}}}_f + \overline{\overline{\mathcal{F}}}_b \tag{102}$$

More explicitly:

$$\overline{\overline{\mathcal{F}}}_f = \begin{bmatrix} 0 & -\vec{E}_f^T / c_0 \\ \vec{E}_f / c_0 & \overline{\overline{B}}_f \end{bmatrix}; \qquad \overline{\overline{\mathcal{F}}}_b = \begin{bmatrix} 0 & -\vec{E}_b^T / c_0 \\ \vec{E}_b / c_0 & \overline{\overline{B}}_b \end{bmatrix} \tag{103}$$

Like the global EM tensor $\overline{\overline{\mathcal{F}}}$, both $\overline{\overline{\mathcal{F}}}_f$ and $\overline{\overline{\mathcal{F}}}_b$ transform according to Lorentz. We can so define a *magnetization-polarization tensor* $\overline{\overline{\mathcal{M}}}$ [8, 12, 20] equal to:

$$\overline{\overline{\mathcal{M}}} = \frac{1}{\mu_0} \in_\Omega \overline{\overline{\mathcal{F}}}_b \tag{104}$$

$$\overline{\overline{\mathcal{M}}} = \frac{1}{\mu_0} \in_\Omega \begin{bmatrix} 0 & -\vec{E}_b^T / c_0 \\ \vec{E}_b / c_0 & \overline{\overline{B}}_b \end{bmatrix} \tag{105}$$

Recalling the definition (76) for $\vec{P}$ and $\vec{M}$, we can express the magnetization-polarization tensor $\overline{\overline{\mathcal{M}}}$ in function of them:

$$\overline{\overline{\mathcal{M}}} = \begin{bmatrix} 0 & c_0 \vec{P}^T \\ -c_0 \vec{P} & \overline{\overline{M}} \end{bmatrix} \tag{106}$$

The tensor $\overline{\overline{\mathcal{M}}}$ transforms like $\overline{\overline{\mathcal{F}}}$, since $\in_\Omega$ is an *invariant scalar field*. In fact, an event $(\vec{x}, t)$ can belong or not to a space-time domain $\Omega(t)$ independently from the chosen ref. frame.

$$\in_{\Omega\, A} (\vec{x}_A, t_A) = \in_{\Omega\, L} (\vec{x}_L, t_L) = \in_\Omega (x^\mu) \quad \text{invariant scalar field, } \forall A, L \tag{107}$$

Finally, the magnetization-polarization tensor $\overline{\overline{\mathcal{M}}}$ in a ref. frame L can be calculated as:

$$\overline{\overline{\mathcal{M}}}_L = \overline{\overline{\Lambda}}_{LA} \cdot \overline{\overline{\mathcal{M}}}_A \cdot \overline{\overline{\Lambda}}_{LA}^T \tag{108}$$

Analogously to the transformation (100) for $\vec{E}$ and $\vec{B}$, we can retrieve the relation for $\vec{P}$ and $\vec{M}$:

$$\begin{cases} \vec{P}_L = \gamma \left(\vec{P}_A + \dfrac{1}{c_0^2} \vec{M}_A \times \vec{v} \right) & -(\gamma - 1) \frac{1}{v^2} \left(\vec{v}^T \cdot \vec{P}_A \right) \vec{v} \\ \vec{M}_L = \gamma \left(\vec{M}_A - \vec{P}_A \times \vec{v} \right) & -(\gamma - 1) \dfrac{1}{v^2} \left(\vec{v}^T \cdot \vec{M}_A \right) \vec{v} \end{cases} \tag{109}$$

So, if $\vec{P}$ and $\vec{M}$ are known in a ref. frame A, they can be rigorously determined in another frame L.

7.3.3. Transformation for D and H

Following a similar procedure, the displacement tensor $\overline{\overline{\mathcal{D}}}$ can be constructed as:

$$\overline{\overline{\mathcal{D}}} = \frac{1}{\mu_0}\overline{\overline{\mathcal{F}}} - \overline{\overline{\mathcal{M}}} \tag{110}$$

$$\overline{\overline{\mathcal{D}}} = \begin{bmatrix} 0 & -c_0\vec{D}^T \\ c_0\vec{D} & \overline{\overline{H}} \end{bmatrix} \tag{111}$$

In an another notation:

$$\mathcal{D}^{\mu\nu} = \frac{1}{\mu_0}\mathcal{F}^{\mu\nu} - \mathcal{M}^{\mu\nu} \tag{112}$$

Thus the transformations for the electric displacement $\vec{D}$ and the magnetic field $\vec{H}$ will be:

$$\begin{cases} \vec{D}_L = \gamma\left(\vec{D}_A - \frac{1}{c_0^2}\vec{H}_A \times \vec{v}\right) & -(\gamma-1)\frac{1}{v^2}\left(\vec{v}^T \cdot \vec{D}_A\right)\vec{v} \\ \vec{H}_L = \gamma\left(\vec{H}_A + \vec{D}_A \times \vec{v}\right) & -(\gamma-1)\frac{1}{v^2}\left(\vec{v}^T \cdot \vec{H}_A\right)\vec{v} \end{cases} \tag{113}$$

In principle, for every material, the constitutive relation linking $\vec{D}$ and $\vec{H}$ to $\vec{E}$ and $\vec{B}$ should be *invariant in form*, even if the body is observed from different reference frames [16, 17].

8. CONCLUSIONS AND FUTURE OUTCOMES

In this work the concepts of polarization P and magnetization M have been analysed, highlighting the limits of their classical definitions. We proposed a global definition for P and M with these features:

- it is fully consistent with the Maxwell's equations, instead of the classical definition which in some cases does not work (as explained in Sections 5.2, 5.3).
- fields P and M depend on the space occupied by the considered system, and thus on its *scale*.
- the definition for P and M is valid for any kind of material, no matter if non-linear or hysteretic. In fact, the material's constitutive relation should be assigned independently.
- fields P and M can be calculated also for multi-body systems, taking into account the mutual interactions.
- the proposed definition for P and M is consistent with the Special Relativity Theory.

Here we have presented the preliminary part of a larger work: our definition can be tested in many other contexts and problems. For example, the concepts of local and global magnetization can be applied to the analysis of Weiss domains in ferromagnetic materials. They can be also suitable for some lumped-parameter models for electrical machines (e.g., transformers).

As a matter of fact, this study on P and M definitions started from the problem of modelling metamaterials at different scales: from microscopic to macroscopic ones and vice-versa. This is ongoing research and further results will be presented in future publications.

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