Progress In Electromagnetics Research B, Vol. 64, 119–143, 2015

Resonant States in Waveguide Transmission Problems

Yury Shestopalov*

Abstract—We prove the existence of complex eigenfrequencies of open waveguide resonators in the form of parallel-plate waveguides and waveguides of rectangular crosssection containing layered dielectric inclusions. It is shown that complex eigenfrequencies are finite-multiplicity poles of the analytical continuation of the operator of the initial diffraction problem and its Green's function to a multi-sheet Riemann surface, and also of the transmission coefficient extended to the complex plane of some of the problem parameters. The eigenfrequencies are associated with resonant states (RSs) and eigenvalues of distinct families of Sturm-Liouville problems on the line; they form countable sets of points in the complex plane with the only accumulation point at infinity and depend continuously on the problem parameters. The set of complex eigenfrequencies is similar in its structure to the set of eigenvalues of a Laplacian in a rectangle. The presence of a resonance domain in the form of a parallel-plane layered dielectric insert removes the continuous frequency spectrum and gives rise to a discrete set of points shifted to (upper half of) the complex plane.

1. INTRODUCTION

Spectral problems in the mathematical theory of diffraction have been attracting significant attention of researchers since the development in [5] of the spectral theory of open structures. From the mathematical viewpoint, these problems describe singularities of the analytical continuation of the solution to diffraction problems (with sources) to the domain of complex values of parameters (frequency, energy) that sometimes have no direct physical meaning. In order to formulate a correct mathematical statement of a spectral problem in an unbounded spatial domain (e.g., a waveguide), it is necessary to construct an appropriate analytical continuation to the complex domain (usually, to a well-defined multi-sheet Riemann surface) of the operator of the initial diffraction problem and its Green's function, and first of all, of the conditions at infinity. For example, spectral problems in waveguides are connected with the analysis of either complex waves or eigenoscillations of open waveguide resonators when frequency is taken as a (complex) spectral parameter. In the latter case, the diffraction problem considered initially at real frequencies is continued (extended) analytically to a certain multi-sheet Riemann surface H where the spectral parameter is varied; it turns out [5] that (nonhomogeneous) diffraction problem is uniquely solvable everywhere in H except for a discrete set of eigenfrequencies (eigenvalues of the corresponding homogeneous (spectral) problem) forming a countable set of complex points with the only accumulation point at infinity; these points are finite-multiplicity poles of the analytical continuation of the operator of the initial diffraction problem and its Green's function. A central question that governs further development of the spectral theory of open structures in particular and mathematical theory of diffraction in general is rigorous proofs of *existence* of complex eigenvalues of the spectral problems and description of their distribution in the complex domain (including analysis of the structure of multi-sheet Riemann surface H), dependence on (nonspectral) parameters and other important properties.

Received 30 August 2015, Accepted 29 October 2015, Scheduled 18 November 2015
* Corresponding author: Yury V. Shestopalov (shestop@hotmail.com).
The author is with the University of Gävle, Gävle, Sweden.

In this work we prove the existence of complex eigenvalues for a family of waveguide spectral problems associated with eigensolutions (eigenfrequencies) of open waveguide resonators and use this information to identify and classify resonant states (RSs) of various nature. We start the analysis by considering the problems of scattering of a normal waveguide mode by layered parallel-plane dielectric inclusions in a planar waveguide and parallel-plane dielectric diaphragms in a waveguide of rectangular cross section [1]. The closed-form solution to these problems are well known [1, 2]. However, some of its very important features remained untouched, to the best of our knowledge; namely, a fact (known in quantum scattering theory) that the transmission and reflection coefficients have singularities in the complex domain of the problem parameters (frequency, longitudinal wavenumbers, permittivity) and that the scattering problem is not solvable at these singular values of parameters. We show that these singularities, referred to as RSs, correspond to poles of the transmission coefficient function (at which the forward scattering problem is not solvable) considered as an extension to complex domain of one of the problem parameters and to eigensolutions in open waveguide resonators. We propose classification and interpretation of the corresponding solutions. We show that the singularities can be calculated with prescribed accuracy by explicit formulas using the (numerical) solution to simple transcendental equations and that the corresponding eigensolutions have a form of standing waves.

RSs were first reported in [3] and considered by many authors (see [4] and references therein) in different problem statements mainly in quantum scattering theory. A correct mathematical approach for the analysis of RSs in electromagnetics associated with eigenoscillations of cylindrical resonators is developed in [5, Ch. 4], as part of the spectral theory of open structures.

In Section 1 statements are given of the probelms of diffraction from a dielectric obstacle in a 3D- and 2D-waveguides and the corresponding spectral problems. The study of RSs of open waveguide resonators formed by parallel-plate waveguides and waveguides of rectangular cross-section containing layered dielectric inclusions (multi-section diaphragms) is performed in terms of analysis of the transmission coefficients as functions of (nonspectral) parameters. Section 2 is devoted to the determination of eigenvalues of the families of Sturm-Liouville problems on the line associated with the considered waveguide spectral problems. In Section 3 the existence of eigenvalues is proved. In Section 4 the results are presented of the calculation of complex singularities of the transmission coefficient with respect to different problem parameters revealing the distribution of RSs on the complex plane and illustrating some of their properties.

2. STATEMENT OF DIFFRACTION AND SPECTRAL PROBLEMS

2.1. Diffraction from a Dielectric Obstacle in a 3D-Guide

Assume that a waveguide

$$P := \{x : 0 < x_1 < a, \quad 0 < x_2 < b, \quad -\infty < x_3 < \infty\}$$

with the perfectly conducting boundary surface ∂P is given in the cartesian coordinate system. A three-dimensional body Q ($Q \subset P$ is a domain) with a constant magnetic permeability μ_0 and permittivity $\varepsilon(x)$ (a bounded function in $\bar{Q}$) is placed in the waveguide (Fig. 1). The boundary ∂Q of Q is piecewise smooth, and Q does not touch the walls of the waveguide.

We look for electromagnetic field $\mathbf{E}, \mathbf{H} \in L_2^{loc}(P)$ excited in the waveguide by an external field with the time dependence $e^{-i\omega t}$ induced by electric current $\mathbf{j}_E^0 \in L_2^{loc}(P)$. The differential operators grad, div, and rot are interpreted in the sense of distributions. We seek weak (generalized) solutions to

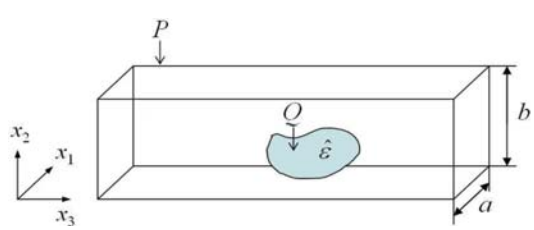


Figure 1. A dielectric obstacle in a waveguide.

Maxwell's system of equations

$$\begin{aligned} \operatorname{rot}\mathbf{H} &= -i\omega\varepsilon\mathbf{E} + \mathbf{j}_E^0, \\ \operatorname{rot}\mathbf{E} &= i\omega\mu_0\mathbf{H}. \end{aligned} \tag{1}$$

E and **H** satisfy the boundary conditions

$$\mathbf{E}_\tau|_{\partial P} = 0, \quad \mathbf{H}_\nu|_{\partial P} = 0, \tag{2}$$

and the conditions at infinity: **E** and **H** admit for $|x_3| > C$ and sufficiently large $C > 0$ the representations ($\pm$ correspond to $\pm\infty$)

$$(\mathbf{E}\;\;\mathbf{H})^T = \sum_p R_p^{(\pm)} e^{-i\gamma_p^{(1)}|x_3|} \begin{pmatrix} \lambda_p^{(1)}\Pi_p e_3 - i\gamma_p^{(1)}\nabla_2\Pi_p \\ -i\omega\varepsilon_0(\nabla_2\Pi_p)\times e_3 \end{pmatrix} + \sum_p Q_p^{(\pm)} e^{-i\gamma_p^{(2)}|x_3|} \begin{pmatrix} i\omega\mu_0(\nabla_2\Psi_p)\times e_3 \\ \lambda_p^{(2)}\Psi_p e_3 - i\gamma_p^{(2)}\nabla_2\Psi_p \end{pmatrix}, \tag{3}$$

$\gamma_p^{(j)} = \sqrt{k_0^2 - \lambda_p^{(j)}}$, $\operatorname{Im}\gamma_p^{(j)} < 0$ or $\operatorname{Im}\gamma_p^{(j)} = 0$, $k_0\gamma_p^{(j)} \geq 0$, and $\lambda_p^{(1)}$, $\Pi_p(x_1, x_2)$ and $\lambda_p^{(2)}$, $\Psi_p(x_1, x_2)$ ($k_0^2 = \omega^2\varepsilon_0\mu_0$) are the complete system of eigenvalues and orthogonal and normalized in $L_2(\Pi)$ eigenfunctions of the two-dimensional Laplace operator $-\Delta$ in the rectangle $\Pi := \{x' = (x_1, x_2) : 0 < x_1 < a, 0 < x_2 < b\}$ with the Dirichlet and the Neumann conditions, respectively. The complete statement of the conditions at infinity is given in Section 2.5.

2.2. Diffraction Problem in a Parallel-plate Waveguide

Consider a parallel-plate waveguide $S = \{x = (x_1, x_2, x_3) : \; 0 < x_1 < a, -\infty < x_2, x_3 < \infty\}$, containing a nonmagnetic, isotropic, and inhomogeneous dielectric inclusion having the cross section

$$D \subset \mathcal{Q} = \left\{x' = (x_1, x_3) : \; 0 < x_1 < a, \; -d < x_3 < d\right\}$$

bounded by a piecewise smooth closed contour ∂D where $\mathcal{Q} \subset S$ denotes the so-called transition domain. The permittivity $\varepsilon = \varepsilon(x')$ is assumed to be a complex-valued function piecewise continuously differentiable and bounded in S such that $\operatorname{supp} m(x') \subset \mathcal{Q}$, where $m(x') = 1 - \varepsilon(x')$.

The problem of diffraction of the TE mode by a dielectric inclusion D, when the solution to BVP (1)–(3) for Maxwell's equations has the form

$$\mathbf{E} = (0, E_2, 0), \quad \mathbf{H} = (H_1, 0, H_3), \tag{4}$$

is reduced to the boundary-value problem (BVP) [8]

$$[\Delta + \lambda\varepsilon(x')]u(x') \;=\; 0 \;\text{ in }\; S, \quad u(0, x_3) = u(a, x_3) = 0, \tag{5}$$

$$u(x') \;=\; u^0 + u^s, \quad u^s = \sum_{n=-\infty}^{\infty} a_n^{\pm} P_n(x'). \tag{6}$$

Here $u(x') = E_2(x') = E_2^0(x') + E_2^{scat}(x') = u^0(x') + u^s(x')$ is the longitudinal component of the total field of diffraction by D of the unit-magnitude TE-wave with the only nonzero component E_2, $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_3^2}$ is the Laplace operator, superscripts $+$ and $-$ correspond, respectively, to the domains $x_3 > 2\pi\delta$ and $x_3 < -2\pi\delta$, $\lambda = k_0^2$, $\omega = k_0 c$ is the dimensionless circular frequency, $c = (\varepsilon_0\,\mu_0)^{-1/2}$ is the speed of light in vacuum, and $\gamma_n = \sqrt{\lambda - \left(\frac{\pi n}{a}\right)^2}$ is the transverse wavenumber satisfying, for real λ, the conditions

$$\Im\gamma_n \geq 0, \qquad \gamma_n \;= i|\gamma_n|, \qquad |\gamma_n| = \;\Im\gamma_n \;= \sqrt{\left(\frac{\pi n}{a}\right)^2 - \lambda}, \quad \frac{\pi n}{a} > \lambda; \tag{7}$$

functions

$$P_n(\mathbf{r}') = \exp(i\gamma_n x_3) f_{0n}(x_1), \quad f_{0n}(x_1) = \sin\frac{\pi n x_1}{a}, \quad n = 1, 2, \ldots, \tag{8}$$

solve homogeneous BVP (5)–(7) in a parallel-plate waveguide S without inclusion ($\{(\frac{\pi n}{a})^2\}$, $\{f_{0n}\}$ is the complete system of eigenvalues and orthogonal and normalized in $L_2(0, a)$ eigenfunctions of the one-dimensional Laplace operator $-\frac{d^2}{dx^2}$ in the interval $0 < x_1 < a$ with the Dirichlet condition).

It is assumed that series in (3) and (6) converge absolutely and uniformly and allows for double differentiation. If $\varepsilon(\mathbf{r}')$ is a piecewise continuous function, the continuity and transmission conditions [7, Ch. 2], should be added on the discontinuity lines. Note that $u^0(x')$ satisfies (5) in S, the boundary condition, and radiation condition (6), (7) only in the positive direction, so that the electromagnetic field with the x_2-component $u^0(x')$ may be interpreted as *a mode* coming from the domain $x_3 < -d$.

2.3. Spectral Problems

(1)–(3) and (5)–(7) constitute BVPs for Maxwell's equations or the Helmholtz equation with variable (piece-wise constant) coefficient in an unbounded cylindrical domain. The corresponding homogeneous BVPs may have nontrivial solutions giving rise to eigenvalues associated [5] with singular points of Green's function of the BVPs in question and resonances of cylindrical inhomogeneously filled open waveguide resonators formed by the unbounded cylindrical domain. The nature of these resonances and eigenvalues may be different depending on which of the problem parameters (frequency, longitudinal wavenumber, or other quantity) is chosen as a spectral one. When frequency ($\kappa = k_0$) is taken as a (complex) spectral parameter, the diffraction problem considered initially at real frequencies is continued analytically to a multi-sheet Riemann surface where the spectral parameter is varied. In [5] it is proved that the (nonhomogeneous) diffraction problem is uniquely solvable everywhere on the Riemann surface except for a discrete set $\mathcal{K} = \{\kappa_n\}$ of eigenfrequencies of the considered open waveguide resonator (eigenvalues of the corresponding homogeneous (spectral) problem). The spectrum of eigenfrequencies forms a countable set of complex points with the only accumulation point at infinity; these points are finite-multiplicity poles of the analytical continuation of the operator of the initial diffraction problem and its Green's function.

The existence of such eigenvalues, referred to as RSs, has never been rigorously proved, to the best of our knowledge. In this work we confirm the general result concerning the discreteness of the eigenfrequency set and prove that RSs are associated with violation of the solvability of the diffraction problem. What is more, we prove that RSs *exist* for open waveguide resonators formed by parallel-plate waveguides or waveguides of rectangular cross-section containing inclusions in the form of multi-sectional diaphragms. The occurence of RSs is shown when either frequency, permittivity or longitudinal wavenumber of any of the section is taken as a spectral parameter. Main properties of RSs are investigated by the reduction to determination of zeros of well-defined families of entire functions of the spectral parameter.

2.4. Layered Dielecric Diaphragms

Consider first parallel-plate waveguide S and assume that inclusion D is a rectangle, $D = \mathcal{Q} = \{x' : 0 < x_1 < a, 0 < x_3 < l\}$ adjacent to the walls of S and separated into n rectangles $\mathcal{Q}_0 = \{-\infty < x_3 < 0\}$, $\mathcal{Q}_j = \{l_{j-1} < x_3 < l_j\}$, $\mathcal{Q}_{n+1} = \{l_n < x_3 < +\infty\}$ $(0 < x_1 < a,\ l_0 = 0)$, and permittivity assumes constant values ε_j in $\mathcal{Q}_j$ $(j = 1, \ldots, n)$. Domain $S\backslash\bar{\mathcal{Q}}$ is filled with an isotropic and homogeneous layered medium having constant permeability $(\mu_0 > 0)$ in P (Fig. 2).

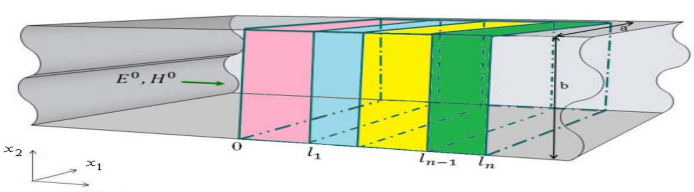


Figure 2. Multisectional diaphragm in a waveguide.

Solving BVP (5)–(7) in S with $u^0(x') = Ae^{-i\gamma_{0m}x_3}$ we obtain explicit expressions for $u(x')$ inside every $\mathcal{Q}_j$ $(j = 1, \ldots, n)$ and outside $\mathcal{Q}$

$$\begin{aligned} u(x') &= u_{(0m)}(x') = f_{0m}(x_1)(Ae^{-i\gamma_{0m}x_3} + B_{jm}e^{i\gamma_{0m}x_3}), \quad x \in \mathcal{Q}_0, \\ u(x') &= u_{(jm)}(x') = f_{0m}(x_1)(C_{jm}e^{-i\gamma_{jm}x_3} + D_{jm}e^{i\gamma_{jm}x_3}), \quad x \in \mathcal{Q}_j, \end{aligned} \tag{9}$$

where $m = 1, 2, \ldots$, $\gamma_{0m} = \sqrt{k_0^2 - \frac{\pi^2 m^2}{a^2}}$, $\gamma_{jm} = \sqrt{k_j^2 - \frac{\pi^2 m^2}{a^2}}$, $j = 1, \ldots, n+1$, $D_{n+1\,m} = 0$, $C_{n+1m} = F_{nm}$, and $k_j^2 = k_0^2 \varepsilon_j$. From the transmission conditions on the lines $x_3 = l_j$, $j = 0, 1, 2, \ldots, n$, where permittivity undergoes breaks

$$[u_{(j-1\,m)}] = [u_{(j\,m)}] = 0, \quad \left[\frac{\partial u_{(j-1\,m)}}{\partial x_3}\right] = \left[\frac{\partial u_{(j\,m)}}{\partial x_3}\right] = 0, \; j = 1, \ldots, n, \tag{10}$$

we obtain for every $m = 1, 2, \ldots$ a system of equations for the unknown (complex) coefficients and finally a recurrent formula that couples A and F_{nm},

$$A = g_{nm} F_{nm} e^{-i\gamma_0 l_n}, \tag{11}$$

where

$$g_{nm} = \frac{1}{2\prod\limits_{j=0}^{n} \gamma_{jm}} (\gamma_{nm} p_{n+1} + \gamma_0 q_{n+1}), \tag{12}$$

$$\begin{aligned} p_{j+1} &= \gamma_{j-1\,m} p_j \cos\alpha_{jm} + \gamma_{jm} q_j i \sin\alpha_{jm}, \quad p_1 := 1, \\ q_{j+1} &= \gamma_{j-1\,m} p_j i \sin\alpha_{jm} + \gamma_{jm} q_j \cos\alpha_{jm}, \quad q_1 := 1, \end{aligned}$$

and $\alpha_{jm} = \gamma_{jm}(l_j - l_{j-1})$; $j = 2, \ldots, n$.

Consider now waveguide P of rectangular cross section where body Q is a parallelepiped, $Q = \{x : 0 < x_1 < a, 0 < x_2 < b, 0 < x_3 < l\}$ adjacent to the waveguide walls and separated into n sections $Q_0\,\{-\infty < x_3 < 0\}$, $Q_j\,\{l_{j-1} < x_3 < l_j\}$ $(j = 1, \ldots, n)$, $Q_{n+1}\,\{l_n < x_3 < +\infty\}$ $(0 < x_1 < a, 0 < x_2 < b)$, filled each with a medium having constant permittivity $\varepsilon_j > 0$. Domain $P\backslash\bar{Q}$ is filled with an isotropic and homogeneous layered medium having constant permeability ($\mu_0 > 0$) in P. Assume that the solution to BVP (1)–(3) for Maxwell's equations has the form (4), $\pi/a < k_0 < \pi/b$, and the incident electrical field is

$$\mathbf{E}^0 = \mathbf{e}_2 A f_{01}(x_1) e^{-i\gamma_0 x_3}, \quad \gamma_{01} = \sqrt{k_0^2 - \frac{\pi^2}{a^2}}. \tag{13}$$

Solving BVP (1)–(3) for TE-modes by applying transmission conditions similar to (10) on the boundary surfaces of the diaphragm sections $x_3 = l_j$ $(j = 0, 1, 2, \ldots, n)$ where permittivity undergoes breaks

$$[E_{(j-1)}] = [E_{(j)}] = 0, \quad \left[\frac{\partial E_{(j-1)}}{\partial x_3}\right] = \left[\frac{\partial E_{(j)}}{\partial x_3}\right] = 0, \quad j = 1, \ldots, n, \tag{14}$$

we obtain explicit expressions in the form (9) at $m = 1$

$$\begin{aligned} E_{(0)} &= f_{01}(x_1)(A e^{-i\gamma_0 x_3} + B_{jm} e^{i\gamma_0 x_3}), \quad x \in Q_0, \\ E_{(j)} &= f_{01}(x_1)(C_{jm} e^{-i\gamma_{j1} x_3} + D_{jm} e^{i\gamma_{j1} x_3}), \quad x \in Q_j, \end{aligned} \tag{15}$$

for the field components inside every section of diaphragm Q and outside the diaphragm and then the same recurrent formula (11) and expressions (12) with $m = 1$ that couple A and F_{n1}.

2.5. Conditions at Infinity for Waveguide Spectral Problems

The medium in a waveguide with a piecewise constant permittivity is a particular case of a more general problem with the permittivity function depending on the longitudinal coordinate, $\varepsilon = \varepsilon(x_3)$; this setting is considered in [6] when planar dielectric layers are situated in free space. In this paper, we will show that the former problem admits a closed-form solution and obtain complete information about its spectrum. Therefore, the case with piecewise constant permittivity may serve as the first necessary step in the analysis of more general problems of waveguides filled with arbitrary inhomogeneous medium.

Inhomogeneous (diffraction) BVPs (1)–(3) or (5)–(7) are considered, respectively, for real values of the frequency parameter $\kappa = k_0$ satisfying $\frac{\pi}{a} < \kappa < \frac{\pi}{b}$ or $\kappa > \frac{\pi m}{a}$. The corresponding homogeneous (spectral) BVPs are obtained from the inhomogeneous BVPs by setting $A = 0$ in (15) and (9) or $\mathbf{j}_E^0 = 0$ in (1) and are considered, according to [5], as analytical continuation with respect to, e.g.,

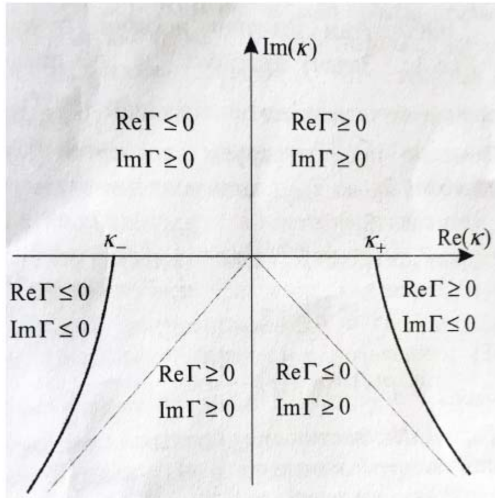


Figure 3. Riemann surface H_2 of the frequency parameter $\kappa = k_0$.

complex parameter κ on a Riemann surface. For BVP (1)–(3) it will be a two-sheet Riemann surface H_2 defined [5] as follows: the sheets of H_2 are complex κ-planes cut along the curves

$$d_1(\kappa) = (\Re\kappa)^2 - (\Im\kappa)^2 - \left(\frac{\pi}{a}\right)^2, \quad \Im\kappa \le 0 \tag{16}$$

starting at the points $\kappa_\pm = \pm\frac{\pi}{a}$ (Fig. 3). On the first sheet the quantity $\Gamma = \gamma_{01}(\kappa) = \sqrt{\kappa^2 - \frac{\pi^2}{a^2}} = \Re\Gamma + i\Im\Gamma$ is defined as follows: in the upper half-plane $0 < \arg\kappa < \pi$ $\Im\Gamma > 0$ and $\Re\Gamma \ge 0$ for $0 \le \arg\kappa \le \pi/2$ and $\Re\Gamma \le 0$ for $\pi/2 < \arg\kappa < \pi$; for κ in the quadrant $3\pi/2 \le \arg\kappa \le 2\pi$ such that $(\Re\kappa)^2 - (\Im\kappa)^2 - (\frac{\pi}{a})^2 > 0$ $\Re\Gamma > 0$ and $\Im\Gamma < 0$, and for $(\Re\kappa)^2 - (\Im\kappa)^2 - (\frac{\pi}{a})^2 < 0$ $\Re\Gamma < 0$ and $\Im\Gamma > 0$; for κ in the quadrant $\pi \le \arg\kappa \le 3\pi/2$ the conditions are the same but the sign of $\Re\Gamma$ must be changed to the opposite. On the second sheet $\Gamma = -\Re\Gamma - i\Im\Gamma$ is defined in the same manner.

For BVP (5)–(7), it will be an infinite-sheet Riemann surface H defined [5] as follows: the sheets of H are complex κ-planes cut along the curves

$$d_m(\kappa) = (\Re\kappa)^2 - (\Im\kappa)^2 - \left(\frac{\pi m}{a}\right)^2, \quad \Im\kappa \le 0 \quad (m = 1, 2, \ldots). \tag{17}$$

The structure of H governed by the necessity to take into account conditions at infinity (7) or (3) involving complex quantities $\Gamma_m = \gamma_{0m} = \gamma_{0m}(\kappa) = \sqrt{\kappa^2 - \frac{\pi^2 m^2}{a^2}}$ $(m = 1, 2, \ldots)$ is described in [5, Ch. 4].

A reason that explains the necessity of extending the analysis of spectral problems to the complex domain is in the following fact which we verify in this study: a continuation of the transmission coefficient (11) $F = F_{nm}$ to the complex domain of some of its parameters (e.g., as $F = F(z)$, with respect to the variable $z = \sqrt{\epsilon_1 - \pi^2/(k_0 a)^2}$) may have singularities because $g_{nm}(z)$ may have zeros. In the next section we will show that functions $g_{nm}(z)$ do have infinite families of complex zeros depending continuously of nonspectral parameters. These zeros are actually associated with (complex) eigenvalues of the families of Sturm-Liouville problems on the line considered with respect to different spectral parameters.

2.6. Transmission Coefficient as Function of Parameters

The aim of our study is to find singularities (poles) of the transmission coefficient, that is, zeros of g_{nm} in (11) considered generally with respect to any of its complex parameters. To do this in a mahematically correct way it is necessary (a) to consider $g_{nm} = g_{nm}(z, \overline{\beta})$ as a function of the chosen spectral parameter (sought-for quantity) z and a vector of (remaining) real and complex nonspectral parameters $\overline{\beta} = \{a, l_j, \varepsilon_j\}$; (b) to prove, using the theorem of existence of implicit function of several variables, that there exists an implicit function $z = z^*(\overline{\beta})$ specified by the equation $g_{nm}(z, \overline{\beta}) = 0$; the latter may be called generalized dispersion equation (GDE) and the former generalized dispersion curve (a hypersurface in the space of parameters associated with vector $\overline{\beta}$).

In the case of a one-sectional diaphragm, which is the basic problem of analysis, the transmission coefficient

$$F = F_1 = A\frac{e^{i\gamma_0 l_s}}{g_1(z)}, \quad g_1(z) = \cos tz + iZ(z, c)\sin tz, \tag{18}$$

where $t = k_0 l_1$, $c = \sqrt{1 - \pi^2/(k_0 a)^2}$, and

$$h(s) = \frac{1}{2}\left(s + \frac{1}{s}\right), \quad Z(s, c) = h\left(\frac{s}{c}\right) = \frac{1}{2}\left(\frac{s}{c} + \frac{c}{s}\right) \qquad (c \neq 0); \tag{19}$$

$h(s)$ and $Z(s, c)$ are, respectively, the Zhukovsky function and the modified Zhukovsky function. For g_1 in the form (18) the spectral parameter $z = c_1 = \sqrt{\epsilon_1 - \pi^2/(k_0 a)^2}$ and the parameter vector $\overline{\beta} = \{a, l_1, \varepsilon_1, k_0\}$ or $\overline{\beta} = \{\hat{a}, \hat{l_1}, \varepsilon_1\}$ with $\hat{a} = k_0 a$, $\hat{l_1} = k_0 l_1$ being dimensionless variables; if $z = \kappa = k_0$ then $\overline{\beta} = \{a, l_1, \varepsilon_1\}$ and

$$g_1(\zeta) = \cos l_1\zeta + iZ(\zeta, c)\sin l_1\zeta, \quad \zeta = \sqrt{\kappa\epsilon_1 - \pi^2/a^2}. \tag{20}$$

Consider in view of (18) the functions

$$g(z) = \cos z + \frac{i}{2}\left(\frac{z}{C} + \frac{C}{z}\right) = \cos z + iZ(s, C)\sin z \tag{21}$$

and

$$\hat{g}_\pm(z) = \cos(tz) \pm \frac{i}{2}\left(\frac{z}{C} + \frac{C}{z}\right)\sin(tz) = \cos z' \pm iZ(ts, tC)\sin z', \quad z' = tz, \tag{22}$$

where C and t are real constants. It is easy to check that $Z(s, C) = Z(ts, tC)$; $Z(s, C)\sin tz$, $g(z)$ and $\hat{g}_\pm(z)$ are entire even (depending on z^2) functions for every real C and t; $g(z)$ and $\hat{g}_\pm(z)$ have no real zeros; and $|g(z)| \geq 1$ and $|\hat{g}_\pm(z)| \geq 1$ (and $|F| \leq 1$) for real z.

In order to reveal the structure and properties of the functions representing the transmission coefficient, write explicit formulas (12) for the transmission coefficient in the cases of one-, two-, and three-sectional diaphragms (omitting index m):

$$F = F_s = A\frac{e^{i\gamma_0 l_s}}{g_s(z)}, \quad s = 1, 2, 3, \tag{23}$$

$$g_1(z) = \cos tz + \frac{i}{2}\left(\frac{z}{c} + \frac{c}{z}\right)\sin tz = \cos tz + iZ(s, c)\sin tz, \tag{24}$$

$$\begin{aligned} g_2(z) &= \cos\alpha_2 \times \left\{\cos tz + \frac{i}{2}\left(\frac{z}{c} + \frac{c}{z}\right)\sin tz + \frac{\tan\alpha_2}{2}\left[-\left(\frac{z}{c_2} + \frac{c_2}{z}\right)\sin tz + i\left(\frac{c}{c_2} + \frac{c_2}{c}\right)\cos tz\right]\right\} \\ &= \cos\alpha_2\left\{\cos tz + iZ(s, c)\sin tz + \tan\alpha_2\left[-Z(s, c_2)\sin tz + iZ(c, c_2)\cos tz\right]\right\} \\ &= \cos\alpha_2\Phi_2(z, \alpha_2), \quad \Phi_2(z, \alpha_2) = g_1(z) + \tan\alpha_2 g_{12}(z); \end{aligned} \tag{25}$$

$$\begin{aligned} g_3(z) &= \cos\alpha_2\cos\alpha_3 \times \left\{\begin{array}{l} \cos tz + \frac{i}{2}(\frac{z}{c} + \frac{c}{z})\sin tz - \tan\alpha_2\{\frac{1}{2}(\frac{z}{c_2} + \frac{c_2}{z})\sin tz - \frac{i}{2}(\frac{c}{c_2} + \frac{c_2}{c})\cos tz\} + \\ -\tan\alpha_2\tan\alpha_3\{\frac{1}{2}(\frac{c_3}{c_2} + \frac{c_2}{c_3})\cos tz + \frac{i}{2}(\frac{zc_3}{cc_2} + \frac{cc_2}{z_3 c})\sin tz\} + \\ -\tan\alpha_3\{\frac{i}{2}(\frac{c}{c_3} + \frac{c_3}{c})\cos tz - \frac{1}{2}(\frac{z}{c_3} + \frac{c_3}{z})\sin tz\} \end{array}\right\} \\ &= \left\{\begin{array}{l} g_2(z) - \tan\alpha_3\left[\tan\alpha_2\left(\frac{1}{2}(\frac{c_3}{c_2} + \frac{c_2}{c_3})\cos tz + \frac{i}{2}(\frac{zc_3}{cc_2} + \frac{cc_2}{zc_3})\sin tz\right) + \right. \\ \left. +\frac{i}{2}(\frac{c}{c_3} + \frac{c_3}{c})\cos tz - \frac{1}{2}(\frac{z}{c_3} + \frac{c_3}{z})\sin tz\right] \end{array}\right\} \\ &= \left\{\begin{array}{l} g_2(z) - \tan\alpha_3\left[\tan\alpha_2\left(Z(c_3, c_2)\cos tz + Z(zc_3, cc_2)\sin tz\right) + \right. \\ \left. +Z(c_3, c)\cos tz - Z(c_3, z)\sin tz\right] \end{array}\right\} \\ &= \cos\alpha_2\cos\alpha_3\Phi_3(z, \alpha_2), \quad \Phi_3(z, \alpha_2) = g_2(z) + \tan\alpha_3 g_{23}(z); \end{aligned} \tag{26}$$

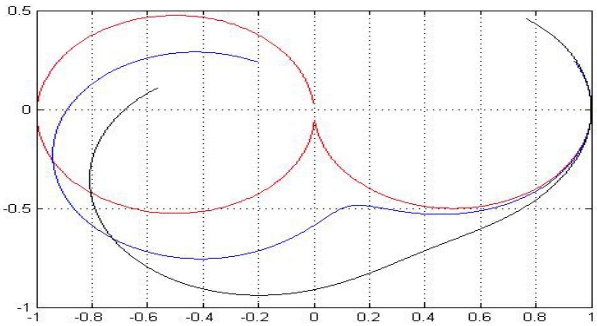


Figure 4. 'Signature curve' of the transmission coefficient F_1 on the complex F_1-plane parametrized by real z, $0 < z < 4$; $C = 0.05$ (red), 0.5 (blue), 1 (black).

here $t_2 = k_0 l_2$, $c_j = \sqrt{\epsilon_j - \pi^2/(k_0 a)^2}$ $(j = 2, 3)$, $\alpha_2 = c_2(t_2 - t)$, and $\alpha_3 = c_3(t_3 - t_2)$.

It is not difficulut to check using the analysis applied for Zhukovsky functions and functions (21) and (22) and explicit expressions (25) and (26) that $g_s(z)$ and $\Phi_s(z)$, $s = 2, 3$, are entire even (depending on z^2) functions for every real-parameter set $\{t, c, c_2, c_3\}$; have no real zeros; and $|g_s(z)| \geq 1$ (and $|F_s| \leq 1$) for real z. Fig. 4 presents a typical 'signature curve' of the transmission coefficient illustrating a one-to-one correspondence between a parameter set and the set of values of $F_1(z)$ on the complex plane.

The same conclusions apply for functions $g_n(z)$, $n > 3$ in the case of an a n-sectional diaphragm; they may be also verified using perturbation analysis with respect to small parameters α_n and the reasoning of the next paragraph.

We see that at $\alpha_2 = 0$ the second section vanish and a two-sectional diaphragm is transformed to a one-sectional and the equation $g_2(z) = 0$ equivalent to $\Phi_2(z, \alpha_2) = 0$ that governs possible singularities of the transmission coefficient in the two-sectional case takes the form $g_1(z) = 0$ valid in the one-sectional case $(g_2(z) = g_2(z, \alpha_2) = g_1(z)$ and $\Phi_2(z, \alpha_2) = g_1(z)$ at $\alpha_2 = 0$: $g_2(z, 0) = \Phi_2(z, 0) = g_1(z))$. Similarly, $g_3(z) = 0$ takes the form $g_2(z) = 0$ at $\alpha_3 = 0$, and so on.

Choosing α_2 as a small parameter, denoting $z^* = z^*(\alpha_2)$ a root of the equation $g_2(z, \alpha_2) = 0$, and taking into account that $g_1(z_0^*) = 0$ where $z_0^* = z^*(0)$, we can obtain from (25) a linear asymptotic expansion of $z^*(\alpha_2)$ in the form

$$\begin{aligned} z^*(\alpha_2) &= z^*(0) + \alpha_2 \left.\frac{dz^*}{d\alpha_2}\right|_{\alpha_2=0} + O(\alpha_2^2) = z_0^* - \alpha_2 \left.\frac{\partial \Phi_2/\partial \alpha_2}{\partial \Phi_2/\partial z}\right|_{z=z_0^*, \alpha_2=0} \\ &+ O(\alpha_2^2) = z_0^* - \alpha_2 \frac{g_{12}(z_0^*)}{g_1'(z_0^*)} + O(\alpha_2^2) = z_0^* + \alpha_2 \mathcal{A}_{\prime}^* + O(\alpha_2^2), \qquad (27) \\ \mathcal{A}_{\prime}^* &= \frac{Z(z_0^*, c_2) - Z(c, c_2) Z(z_0^*, c)}{t(Z^2(z_0^*, c) - 1) - i Z'(z_0^*, c)} = \frac{2c^2((z_0^*)^2 + c_2^2) - (c + c_2^2)((z_0^*)^2 + c^2)}{[c_2((z_0^*)^2 - c^2)]\,[t((z_0^*)^2 - c^2) - 2ic]}. \end{aligned}$$

In addition to these regular transformations, it is easy to check that zeros of $g_1(z)$ undergo regular perturbation with respect to (real) parameters α_2 and α_3, so that, e.g., if z^* is a zero of $g_1(z)$, then a continuation exists $z^*(\alpha_2)$ specified by the equations $g_2(z, \alpha_2) = 0$ or $\Phi_2(z, \alpha_2) = 0$; namely, $z^*(\alpha_2)$ is an implicit function defined at least locally, for sufficiently small $\alpha_2 \in (0, \alpha_2^*)$, as a (unique differentiable) solution to the Cauchy problem

$$\begin{aligned} \frac{dz}{d\alpha_2} &= -\frac{\partial \Phi_2/\partial \alpha_2}{\partial \Phi_2/\partial z} = -\frac{1}{\cos^2 \alpha_2} \frac{g_{12}(z)}{g_1'(z) + \tan \alpha_2 g_{12}'(z)}, \quad \alpha_2 \in (0, \alpha_2^*), \qquad (28) \\ z(0) &= z_0^*, \end{aligned}$$

because it is easy to check that

$$\left.\frac{\partial \Phi_2}{\partial z}\right|_{z=z^*, \alpha_2=0} \neq 0 \qquad (29)$$

for sufficiently small α_2 according to the explicit form of functions $g_1(z)$ and $g_{12}(z)$ and their derivatives.

In the case of an n-sectional diaphragm we have

$$\begin{aligned} p_{j+1} &= \gamma_{j-1}\cos\alpha_j\left(p_j + iq_j\frac{\gamma_j}{\gamma_{j-1}}\tan\alpha_j\right) \to \gamma_{j-1}p_j, \quad \alpha_j \to 0, \\ q_{j+1} &= \gamma_j\cos\alpha_j\left(q_j + ip_j\frac{\gamma_{j-1}}{\gamma_j}\tan\alpha_j\right) \to \gamma_j q_j, \quad \alpha_j \to 0, \\ g_j &\to \frac{1}{2\prod_{k=0}^{j}\gamma_k}\left(\gamma_j\gamma_{j-1}p_j + \gamma_0\gamma_j q_j\right) = g_{j-1}, \end{aligned} \tag{30}$$

so that at $\alpha_j = 0$ a j-sectional diaphragm is transformed continuously to a $(j-1)$-sectional $(j = 2, 3, \dots)$ in a sense that the equation $g_j(z) = 0$ that governs possible singularities of the transmission coefficient in the j-sectional case takes the form $g_{j-1}(z) = 0$. In general, make use of formulas (11) and (12) where index m (specifying the exciting mode) is omitted and form a parameter vector $\overline{\alpha} = \{\alpha_j\}_{j=1}^{n-1}$ $(n > 2)$. Next, we can check, using the theorem of existence of implicit function of several variables, that there exists an implicit function $z^*(\overline{\alpha})$ specified for sufficiently small $\|\overline{\alpha}\| \in (0, \alpha_2^*)$ by the equations $g_n(z, \overline{\alpha}) = 0$ or $\Phi_n(z, \overline{\alpha}) = 0$ with the appropriately defined function $\Phi_n(z, \overline{\alpha})$. Therefore zeros of $g_n(z)$ exist locally (for sufficiently small $\|\overline{\alpha}\|$) and undergo regular perturbation with respect to (real) parameters $\{\alpha_j\}_{j=1}^{n-1}$.

In view of further analysis of spectral problems with respect to different spectral parameters consider determination of poles of the transmission coefficient (zeros of g_{nm} in (11)) with respect to different complex variables. Take first the case of a two-sectional diaphragm and determination of zeros of function g_2 in (25). Let $z_2 = c_2 = \sqrt{\epsilon_2 - \pi^2/(k_0 a)^2}$ be a new spectral parameter, $c_1 = \sqrt{\epsilon_1 - \pi^2/(k_0 a)^2}$, $\alpha_1 = z_2 t'$, and $t' = t_2 - t$; rewrite expressions (25) in the form

$$\begin{aligned} g_2 &= \cos tc_1\widetilde{\Phi_2}(z_2, \alpha_1), \quad \widetilde{\Phi_2}(z_2, \alpha_1) = \widetilde{g_1}(z_2) + \tan tc_1\widetilde{g_{12}}(z_2), \\ \widetilde{g_1}(z_2) &= \cos t'z_2 + iZ(z_2, c)\sin t'z_2, \quad \widetilde{g_{12}}(z_2) = -Z(z_2, c_1)\sin t'z_2 + iZ(c, c_1)\cos t'z_2. \end{aligned} \tag{31}$$

We see again that at $t = 0$ when the first section vanishes, a two-sectional diaphragm is transformed to a one-sectional, and the equation $g_2(z_2) = 0$ equivalent to $\widetilde{\Phi_2}(z_2, \alpha_1) = 0$ that governs possible singularities of the transmission coefficient in the two-sectional case with respect to the spectral parameter z_2 takes the form $\widetilde{g_1}(z_2) = 0$ valid in the one-sectional case ($\widetilde{g_1}(z_2) = g_1(z_2)$ at $t' = t$). Next, assuming the existence of zeros of $g_1(z)$ which yields the existence of zeros of $\widetilde{g_1}(z_2)$, taking t as a small parameter, and applying the analysis based on reduction to a Cauchy problem as above, we can check that zeros of $\widetilde{g_1}(z_2)$ exist at least locally, for sufficiently small t.

Applying similar analysis based on the parameter-differentiation method and explicit recurrent formulas (30) and expressions (12) one can prove the existence of zeros of g_{nm} given by (11) considered with respect to every $z_j = c_j = \sqrt{\epsilon_2 - \pi^2/(k_0 a)^2}$ $(j = 2, 3, \dots)$ for every $n, m = 2, 3, \dots$.

Important conclusions can be made:

(i) the equation $g_1(z) = 0$ plays a fundamental role in the RS theory: if we prove the existence of roots of this equation then the existence of roots of all subsequent equations $g_n(z) = 0$, $n = 2, 3, \dots$, will be a result of regular perturbation of zeros of g_1; what is more, any root of $g_n(z) = 0$ will be a regular perturbation of a root of $g_{n-1}(z) = 0$, $n = 2, 3, \dots$;

(ii) RSs arise as poles of the transmission coefficient (zeros of g_{nm} in (11)) considered with respect to any of the spectral parameters $z_j = c_j = \sqrt{\epsilon_2 - \pi^2/(k_0 a)^2}$ $(j = 2, 3, \dots)$. Hence, any section of a multi-sectional diaphragm may produce its 'own' RSs and may be considered as a resonance (resonating) volume.

3. STURM-LIOUVILLE PROBLEMS ASSOCIATED WITH RESONANT STATES

The aim of this section is to establish connections between (i) singularities of the transmission coefficient in the complex domain, (ii) eigensolutions of Maxwell equations in waveguides with layered dielectric inclusions (parallel-plane diaphragms), and (iii) eigenvalues of certain families of Sturm-Liouville problems. We also reveal basic properties of the spectrum of eigensolutions (eigenvalues) including existence, discreteness and localization on the complex plane of the spectral parameter(s).

Set $\lambda = k_0^2$, $\Theta = m^2(\frac{\pi}{a})^2$, $\gamma_0 = \sqrt{\lambda - \Theta}$, introduce the piece-wise constant weight function

$$\rho = \rho(x_3) = \begin{cases} \varepsilon_0, & x_3 < 0, \\ \varepsilon_1, & 0 < x_3 < l_1 \\ \varepsilon_0, & x_3 > l_1, \end{cases} \tag{32}$$

and consider four Sturm–Liouville problems P_-^+, P_+^-, P_+^+, and P_-^- on the line with piece-wise constant coefficient and four different asymptotic conditions at infinity specified by operator $\mathcal{L}_\pm^\pm$

$$\mathcal{S}u \equiv -u'' + \Theta u = \rho\lambda u, \quad x_3 \in \mathrm{R}, \tag{33}$$

$$\mathcal{L}_\pm^\pm u = 0, \quad x_3 \to \pm\infty, \quad u(x_3) \in C^1(\mathrm{R}), \tag{34}$$

where

$$\mathcal{L}_\pm^\pm u \equiv u(x_3) - O(e^{\pm i\gamma_0 x_3}) = 0, \quad x_3 \to \pm\infty \quad (\mathrm{P}_+^+, \mathrm{P}_-^-), \tag{35}$$

$$\mathcal{L}_-^+ u \equiv u(x_3) - O(e^{i\gamma_0 x_3}) = 0, \quad x_3 \to -\infty, u(x_3) - O(e^{-i\gamma_0 x_3}) = 0, \quad x_3 \to \infty \quad (\mathrm{P}_-^+), \tag{36}$$

$$\mathcal{L}_+^- u \equiv u(x_3) - O(e^{-i\gamma_0 x_3}) = 0, \quad x_3 \to -\infty, u(x_3) - O(e^{i\gamma_0 x_3}) = 0, \quad x_3 \to \infty \quad (\mathrm{P}_+^-). \tag{37}$$

In what follows we assume that $\varepsilon_0 = 1$.

It is easy to see that

$$u(x_3) = \begin{cases} Be^{\pm i\gamma_0 x_3} & x_3 < 0, \\ C_1 e^{-i\gamma_1 x_3} + D_1 e^{i\gamma_1 x_3}, & 0 < x_3 < l_1, \\ Fe^{\pm i\gamma_0 x_3}, & x_3 > l_1, \end{cases} \tag{38}$$

where $\gamma_1 = \sqrt{\lambda\varepsilon_1 - \Theta}$ and B, C_1, D_1, F are arbitrary constants, solves (33) on the respective intervals.

In the particular case $\varepsilon_0 = \varepsilon_1 = 1$ the piece-wise constant weight function ρ is constant and

$$u(x_3) = Be^{\pm i\gamma_0 x_3}, \quad x_3 \in \mathrm{R}, \tag{39}$$

with an arbitrary constant B solves, for $\lambda > \Theta$, (33) and Sturm-Liouville problems P_+^+ and P_-^-. Hence the half-line

$$\Lambda(\Theta) = \{\lambda : \lambda > \Theta\} \tag{40}$$

constitutes the continuous spectrum of operator $\mathcal{S}$. We will show that introduction of a dielectric obstacle (inclusion) in the form of a parallel-plane dielectric section eliminates the continuous spectrum and gives rise to discrete spectrum of eigenwaves or eigenoscillations (depending on the choice of the spectral parameter).

Applying to (38) the continuity conditions in (34) we obtain (in the case of problem P_+^-) a system of four linear equations

$$\begin{cases} -B + C_1 + D_1 = 0, \\ \gamma_0 B + \gamma_1 (C_1 - D_1) = 0, \\ C_1 e^{-i\gamma_1 l_1} + D_1 e^{i\gamma_1 l_1} - Fe^{-i\gamma_0 l_1} = 0, \\ \gamma_1(-C_1 e^{-i\gamma_1 l_1} + D_1 e^{i\gamma_1 l_1}) + \gamma_0 Fe^{-i\gamma_0 l_1} = 0, \end{cases} \tag{41}$$

which can be written in the matrix form as

$$\mathcal{A}_+^- \mathbf{S} = \mathbf{0}, \quad \mathcal{A}_+^- \mathbf{0} = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & \gamma_1/\gamma_0 & -\gamma_1/\gamma_0 & 0 & 0 \\ 0 & 1 & e^{2i\gamma_1 l_1} & -e^{il_1(\gamma_1 - \gamma_0)} & 0 \\ 0 & -1 & e^{2i\gamma_1 l_1} & (\gamma_0/\gamma_1)e^{il_1(\gamma_1-\gamma_0)} & 0 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} B \\ C_1 \\ D_1 \\ F \end{pmatrix}. \tag{42}$$

We see that matrix $\mathcal{A}_+^-$ has a typical block structure, consisting of $n = 2$ rectangular blocks of the size 2×3 each produced by the continuity condition on the permittivity break line and shifted $n = 1$ time to the right by inserting a column of two zeros after the first two rows. This result will be generalized below for the case of an n-sectional diaphragm with arbitrary number of sections n.

Introduce the quantities

$$T_{\mp} = T_{\mp}(\lambda, a, l_1, \epsilon_1) = \frac{1}{p_{\mp}}\left[e^{\mp i\gamma_1 l_1}(\gamma_1+\gamma_0)^2 - e^{\pm i\gamma_1 l_1}(\gamma_0-\gamma_1)^2\right] = \frac{4\gamma_0\gamma_1}{p_{\mp}} g_{\mp}(\lambda, a, l_1, \epsilon_1), \quad (43)$$

$$\begin{aligned} g_{\mp}(\lambda, a, l_1, \epsilon_1) &= \cos(l_1\gamma_1) \mp \frac{i}{2}\left(\frac{\gamma_1}{\gamma_0} + \frac{\gamma_0}{\gamma_1}\right)\sin(l_1\gamma_1) \\ &= \cos l_1 z \mp \frac{i}{2}\left(\sqrt{\varepsilon_1}\frac{z}{\sqrt{z^2-q}} + \frac{1}{\sqrt{\varepsilon_1}}\frac{\sqrt{z^2-q}}{z}\right)\sin l_1 z, \end{aligned} \quad (44)$$

$$T_0 = T_0(\lambda, a, l_1, \epsilon_1) = \frac{1}{p_{\mp}}\left[(\gamma_1^2-\gamma_0^2)(e^{i\gamma_1 l_1} - e^{-i\gamma_1 l_1})\right] = \frac{2i(\gamma_1^2-\gamma_0^2)}{p_{\mp}} g_0(\lambda, a, l_1, \epsilon_1), \quad (45)$$

$$\begin{aligned} g_0(\lambda, a, l_1, \epsilon_1) &= \sin(l_1\gamma_1), \\ p_{\mp} &= p_{\mp}(\lambda, a, l_1, \epsilon_1) = (\gamma_0+\gamma_1)e^{-i\gamma_1 l_1} - (\gamma_0-\gamma_1)e^{i\gamma_1 l_1}, \\ p_{\pm} &= p_{\pm}(\lambda, a, l_1, \epsilon_1) = (\gamma_0+\gamma_1)e^{i\gamma_1 l_1} - (\gamma_0-\gamma_1)e^{-i\gamma_1 l_1}, \end{aligned}$$

where $z = \gamma_1$ and $q = \Theta(\varepsilon_1 - 1)$; $T_{\mp}$ correspond, respectively to problems P_{+}^{-} and P_{-}^{+}, and $T_0^{\pm}$ to problems P_{+}^{+} and P_{-}^{-}.

Applying the Gauss elimination and reducing the system matrix to an upper-diagonal form (or calculating its determinant) we conclude that homogeneous systems similar to (42) obtained for all four problems P_{+}^{-}, P_{-}^{+}, P_{+}^{+}, and P_{-}^{-} are uniquely solvable if

$$T_{\mp} \neq 0 \quad \text{or} \quad T_0 \neq 0. \quad (46)$$

These cases correspond to certain sets of parameters

$$\Omega_{reg}^{\mp} = \{(\lambda, a, l_1, \varepsilon_1) : T_{\mp}(\lambda, a, l_1, \varepsilon_1) \neq 0\} \quad \text{or} \quad \Omega_{reg}^{0} = \{(\lambda, a, l_1, \varepsilon_1) : T_0(\lambda, a, l_1, \varepsilon_1) \neq 0\} \quad (47)$$

and may be called regular (or nonresonant).

Note that under the conditions

$$\gamma_0 \neq 0, \quad \gamma_1 \neq 0, \quad (48)$$

$p_{\pm} \neq 0$ and $p_{\mp} \neq 0$ if $T_{\mp} = 0$ or $T_0 = 0$; $p_{\pm} = 0$ implies $T_{\mp} \neq 0$ and correspond thus to the regular case.

If

$$g_{\mp}(\lambda, a, l_1, \varepsilon_1) = 0, \quad (49)$$

we have $T_{\mp} = 0$ and the resonant case corresponding to the set of parameters

$$\Omega_{res}^{\mp} = \{(\lambda, a, l_1, \varepsilon_1) : g_{\mp}(\lambda, a, l_1, \varepsilon_1) = 0\}. \quad (50)$$

It is easy to see that (49) yields the equation

$$e^{\pm 2i\gamma_1 l_1} = \frac{(\gamma_1-\gamma_0)^2}{(\gamma_1+\gamma_0)^2} \quad (51)$$

which has no solutions given by the sets of real numbers $\{(\lambda, a, l_1, \varepsilon_1)\}$ satisfying (48). This means that on the nonresonant set Ω_{reg} (when condition (46) is fulfilled) the problem (P) of the scattering of the normal waveguide mode in a single-mode rectangular waveguide with a lossless dielectric diaphragm is uniquely solvable for real $\omega, a, l_1, \varepsilon_1$ satisfying (48).

If

$$g_0(\lambda, a, l_1, \varepsilon_1) = 0, \quad (52)$$

we have $T_0 = 0$ and the resonant case corresponding to the set of parameters

$$\Omega_{res}^{0} = \{(\lambda, a, l_1, \varepsilon_1) : \gamma_1 l_1 = \pi r,\ r = 1, 2, \ldots\}. \quad (53)$$

In the resonant cases system (42) or homogeneous systems similar to (42) obtained for each of four problems P_{+}^{-}, P_{-}^{+}, P_{+}^{+}, and P_{-}^{-} has a nontrivial solution given by

$$\begin{cases} B = G\frac{2\gamma_1}{\gamma_1+\gamma_0}F, \\ C_1 = G\frac{\gamma_1-\gamma_0}{\gamma_1+\gamma_0}F, \quad G = \dfrac{e^{il_1(\gamma_1-\gamma_0)}}{p}, \quad p = p_{\pm},\ p_{\mp}, \\ D_1 = GF, \end{cases} \quad (54)$$

where F is an arbitrary complex number; the corresponding nontrivial solution to the Sturm-Liouville problem (33), (34) may be taken in the form (for $F = 1$)

$$u(x_3)=\begin{cases} \frac{2G\gamma_1}{\gamma_1+\gamma_0}e^{-i\gamma_0 x_3}, \quad x_3<0, \\ ((\gamma_1-\gamma_0)e^{-i\gamma_1 x_3}+(\gamma_1+\gamma_0)e^{i\gamma_1 x_3}) = \frac{2G}{\gamma_1+\gamma_0}(\gamma_1\cos(\gamma_1 x_3)+i\gamma_0\sin(\gamma_1 x_3)), \quad 0<x_3<l_1, \\ e^{i\gamma_0 x_3}, \quad x_3>l_1. \end{cases} \tag{55}$$

As *standing-wave* we will call eigensolutions (55) to Sturm-Liouville problems (33), P_+^+ and P_-^-, with real γ_0 that have real eigenvalues

$$\lambda_{1,rm} = \frac{1}{\varepsilon_1}\left[\left(\frac{\pi r}{l_1}\right)^2 + \left(\frac{\pi m}{a}\right)^2\right], \quad r,m=1,2,\dots \tag{56}$$

corresponding to the resonant case (53).

As *proper* we will call eigensolutions (55) to Sturm-Liouville problems (33) with complex γ_0 ($\Im\gamma_0 \neq 0$) that decay exponentially as $x_3 \to \pm\infty$. For problems P_+^- or P_-^+ the proper eigensolutions are specified by the conditions $\Im\gamma_0 > 0$ (upper half-plane of the first sheet of the Riemann surface of parameter κ) or $\Im\gamma_0 < 0$ (for $m = 1$, the part of the lower half-plane on the first sheet of the Riemann surface H_2 of κ bounded by the curves $d_1(\kappa)$ defined by (16)), respectively.

If, for a fixed parameter set $\{a, l_1, \varepsilon_1\}$ $z = z^*_\mp$ is a zero of the function (44)

$$g_\mp = g_\mp(z) = \cos l_1 z \mp \frac{i}{2}\left(\sqrt{\varepsilon_1}\frac{z}{\sqrt{z^2-q}} + \frac{1}{\sqrt{\varepsilon_1}}\frac{\sqrt{z^2-q}}{z}\right)\sin l_1 z, \tag{57}$$

then

$$\lambda_\mp = \frac{(z^*_\mp)^2+\Theta}{\varepsilon_1} \tag{58}$$

is an eigenvalue of the Sturm-Liouville problems P_+^- and P_-^+ (33)–(34) with the eigenfunction (55).

In the next section we prove that under certain conditions imposed on the parameter set $\{a, l_1, \varepsilon_1\}$ functions (57) have infinitely many complex zeros forming countable sets $\{z^*_{\mp,rm}\}$ ($r, m = 1, 2, \dots$) in the complex plane z with an accumulation point at infinity† so that we can naturally number these zeros and the corresponding eigenvalues, as in the case (56), by a double index rm. Thus the Sturm-Liouville problems P_+^- and P_-^+ (33), (34) have countable sets of (complex) eigenvalues

$$\lambda_{\mp,rm} = \frac{1}{\varepsilon_1}\left[(z^*_{\mp,rm})^2 + \left(\frac{\pi m}{a}\right)^2\right], \quad r,m=1,2,\dots. \tag{59}$$

The corresponding eigenfrequencies of the considered open waveguide resonators form a countable set of complex points

$$\kappa_{\mp,rm} = \sqrt{\lambda_{\mp,rm}}, \quad r,m=1,2,\dots, \tag{60}$$

with the only accumulation point at infinity. These points are finite-multiplicity poles of the analytical continuation of the operator of the initial diffraction problem and its Green's function to a two-sheet or a multi-sheet Riemann surface H_2 or H, respectively, of spectral parameter κ defined [5] using (16) or (17).

In the case of an n-sectional diaphragm, specifically for $n = 2, 3$, applying to the expressions for u similar to (38) that follow from (9) and (15) the transmission conditions on the lines $x_3 = l_j$, $j = 0, 1, 2, \dots, n$, where permittivity undergoes breaks (the continuity conditions in (33)) we obtain a homogeneous linear equation system $\mathcal{A}_n\mathbf{S_n} = \mathbf{0}$ of $2(n+1)$ linear equations for the unknown coefficients. The system resembles the form of (42) and its matrix $\mathcal{A}_n$ has a typical block structure: it consists of n 2×3 rectangular blocks shifted each $n-1$ times to the right by inserting a column of two zeros after every two subsequent rows. Then by a recurrent procedure similar to that leading to the expressions (11) and (12), we show that unique solvability of the nonhomogeneous coefficient systems $\mathcal{A}_n\mathbf{S_n} = \mathbf{G_n}$ is violated when g_{nm} vanishes. Hence zeros of g_{nm} in (12) with respect to a chosen spectral parameter are associated with eigenvalues of the Sturm-Liouville problems P_+^- and P_-^+ (33), (34).

† For every $m = 1, 2, \dots$ functions (57) have infinitely many complex zeros $\{z^*_{\mp,rm}\}$, $r = 1, 2, \dots$.

3.1. Real Eigenvalues, Extrema, and Eigenfrequencies

It is clear that for any complex ζ there is a complex ε_1 such that $\Im w = 0$, where

$$\lambda_{\mp} = w = w(\zeta) = \frac{\zeta^2 + \Theta}{\varepsilon_1} \quad (\Im\Theta = 0). \tag{61}$$

Indeed, denoting $\zeta = u + iv$ and $\varepsilon_1 = h + ig$ we have

$$w = \frac{u^2 - v^2 + \Theta + 2iuv}{h + ig} = \frac{h(u^2 - v^2 + \Theta) + 2uvg - i\left[-g(u^2 - v^2 + \Theta) + 2huv\right]}{h^2 + g^2}, \tag{62}$$

so that $\Im w = 0$ at

$$g(u^2 - v^2 + \Theta) = 2huv \quad \text{or} \quad \frac{g}{h} = R(\zeta),\ R(\zeta) = 2\frac{uv}{u^2 - v^2 + \Theta}, \tag{63}$$

which yields

$$\varepsilon_1 = \tilde{\varepsilon}_1(\zeta) = \epsilon_1(1 + iR(\zeta)), \tag{64}$$

and

$$w = \Re w = \frac{u^2 - v^2 + \Theta}{\epsilon_1} \tag{65}$$

where $\epsilon_1 \neq 0$ is an arbitrary real number. Thus

$$\lambda^r_{\mp,rm} = \frac{(z^*_{\mp,rm})^2 + \left(\frac{\pi m}{a}\right)^2}{\tilde{\varepsilon}_1(z^*_{\mp,rm})} = \frac{(z^*_{\mp,rm})^2 + \left(\frac{\pi m}{a}\right)^2}{\epsilon_1(1 + iR(z^*_{\mp,rm}))} \quad r, m = 1, 2, \ldots, \tag{66}$$

are real eigenvalues of the Sturm-Liouville problems P^-_+ and P^+_- (33), (34) with ε_1 in (32) given by (62)–(64). A conclusion is that at real ε_1 these problems have no real eigenvalues. The corresponding real eigenfrequencies of the considered open waveguide resonators are determined from (60); the positive sign in (65) is obtained by setting $\epsilon_1 = \tilde{\epsilon_1}\,\mathrm{sgn}(u^2 - v^2 + \Theta)$ where $\tilde{\epsilon_1} > 0$.

Recall that zeros of function $g_1(z)$ in (21) determine RSs of a one-sectional diaphragm. It is easy to call that $|g_1(z)| = 1$ at $z = z^{ex}_r(t) = \frac{\pi r}{t}$, $r = 1, 2, \ldots$, i.e., for $z = \gamma_1 = \sqrt{\lambda\varepsilon_1 - \Theta}$, at

$$\lambda = \lambda_{1\,mr} = \frac{1}{\varepsilon_1}\left[\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi r}{t}\right)^2\right]. \tag{67}$$

so that $z^{ex}_r(t)$ and $\lambda_{1\,mr}$ are points of extremum of the transmission coefficient $F = F_1(z)$ given by (18), as illustrated by Figs. 5, A1 and A2. Detailed proofs are given in Appendix A.

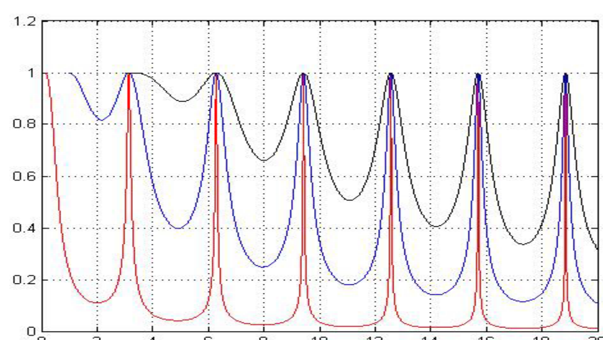


Figure 5. Modulus of the transmission coefficient $F_1 = F_1(z)$ given by (18) at $t = 1$ with respect to real z at $C = 0.1$ (red), $C = 1$ (blue), and $C = 3$ (black); local minima at the 'resonance' points $z = x^*_n \in (\pi/2(2n - 1), \pi n)$, maxima $T = 1$ are at $z = \pi n$ $(n = 1, 2, \ldots)$.

$\lambda_{1\,mr}$ are 'weighted' eigenvalues of the Laplacian in a rectangle $\Pi = \{(x_1, x_3) : \, 0 < x_1 < a, \, 0 < x_3 < l_1\}$, that is, in the (first) section of the waveguide, $\mathcal{Q}_1$, $Q_1 = \{(x_1, x_3) : \, l_0 < x_3 < l_1\}$ and coincide with real eigenvalues (56) corresponding to standing-wave eigensolutions (55) of Sturm-Liouville problems (33), P_+^+ and P_-^-, and resonant case (53). We see that for every given a or l_1 there is a (real) ε_1 (or, more precisely, there are parameter triples governed by (67) such that the transmission coefficient (23) with $s = 1$ attains unity.

In the case of an n-sectional diaphragm, specifically for $n = 2, 3$, the points of extremum $z_{r,n}^{ex} = z_{r,n}^{ex}(t, \overline{\alpha})$ and $\lambda_{n\,mr} = \lambda_{n\,mr}(t, \overline{\alpha})$ of the transmission coefficient $F = F_n(z)$ given by (23)–(26) can be determined as regular perturbations of the extrema $z_r^{ex}(t)$ and $\lambda_{1\,mr}$ of $F_1(z)$ with respect to small parameter α_2 or α_3 (or the whole vector of nonspectral parameters $\overline{\alpha}$) using the parameter-differentiation method and reduction to Cauchy problems (28), (29). One can obtain their linear asymptotic expansions in the form similar to (27).

4. EXISTENCE AND CLASSIFICATION

In this section, we will prove the existence of eigenvalues of the Sturm-Liouville problems P_+^- and P_-^+ (33), (34) by showing that functions (57) have infinitely many complex zeros. To this end, prove first the existence of complex zeros of the auxiliary function

$$g(z) = \cos z + \frac{i}{2}\left(\frac{z}{C} + \frac{C}{z}\right)\sin z, \tag{68}$$

where C is a real constant. $g(z)$ does not vanish at zeros of $\cos z$ (points $\frac{\pi}{2} + \pi k, \; k = 0, 1, 2, ...$); therefore these points may be excluded from the analysis of zeros of $g(z)$ and we can write the equation $g(z) = 0$ in the equivalent form

$$g_0(z) \equiv g_1(z) + g_2(z) = 0, \tag{69}$$

where

$$g_1(z) = \tan z, \qquad g_2(z) = -\frac{2izc}{C^2 + z^2}. \tag{70}$$

Since $g_0(z)$ is an odd function, we will consider it, for the sake of definiteness, in the right half-plane $\mathrm{Re} z \geq 0$.

Let $z_k = \pi k, \; k \geq 0$ and $\Gamma_k = \{z : \, |z - z_k| = r\}$, $r < \pi$. Inside every circle Γ_k function $g_1(z)$ has exactly one zero of multiplicity one at the point $z_k = \pi k, \; k \geq 0$. According to the condition $r < \pi$ the poles of $g_1(z)$ (at the points $\frac{1}{2}\pi + \pi k, \; k \geq 0$) are situated outside the circles. Since $\tan(z + \pi k) = \tan(z)$ we have

$$M_k = \min_{z \in \Gamma_k} |g_1(z)| = \min_{z \in \Gamma_k} |\tan z| = \min_{z \in \Gamma_k} |\tan(z - \pi k)| = \min_{z \in \Gamma_0} |\tan z| = M_0.$$

Function $g_1(z)$ is continuous on Γ_k; therefore the constant $M_0 > 0$ exists and does not depend on k. For function $g_2(z)$ the following estimate is valid

$$|g_2(z)| \leq \frac{2|C||z|}{|z|^2 - |C|^2} \leq \frac{2|C|}{|z| - |C|} \tag{71}$$

at $|z| < |C|$. Then for $k > \frac{|C| + r}{\pi}$ we have

$$\max_{z \in \Gamma_k} |g_2(z)| \leq \frac{2|C|}{\pi k - r - |C|} \equiv N_k.$$

Thus is $k > \frac{|C|+r}{\pi}$ and $N_k < M_0$, all conditions of the Rouche's theorem are fulfilled, so that function $g_0(z)$ has exactly one zero of multiplicity one at a certain point z_k^* inside Γ_k. Since $N_k \to 0, \; k \to \infty$, then $N_k < M_0$ beginning from a certain k_0. Therefore there are infinitely many zeros of $g_0(z)$ in the right half-plane $\mathrm{Re} z \geq 0$. Moreover, radius r of the circles may be decreased by choosing e.g., $r = \frac{1}{n}, \; n \to \infty$. Then for every n there is a k_n such that for $k > k_n$ the zeros of function $g_0(z)$ will be inside the circles Γ_k of the radius $r = \frac{1}{n}$. The latter yields the asymptotics for these zeros of function $g_0(z)$ (see Fig. 6):

$$g_0(z_k^*) = 0, \qquad |z_k^* - z_k| \to 0, \; k \to \infty.$$

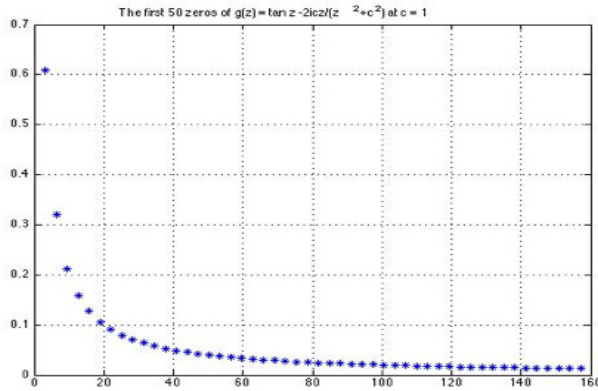


Figure 6. The first 50 zeros z_k^* of $g_0(z)$ located in the vicinities of $z_k = \pi k,\ k = 1, 2, \dots, 50$, calculated by the Newton method in the complex domain at $c = 1$.

Function $g(z)$ has another family of zeros

$$\begin{aligned} z_n &= x_n + iy_n, \\ x_n &= \pi/2 + n\pi, \quad y_n = \sqrt{C_n^2 - x_n^2},, \quad n = 0, 1, 2, \dots, \end{aligned} \tag{72}$$

where C_n is the (real) root of the equation

$$h_n(C) \equiv e^{2\sqrt{C^2 - x_n^2}} - \frac{C + x_n}{C - x_n} = 0, \quad n = 0, 1, 2, \dots. \tag{73}$$

The proof and analysis of properties of this family of zeros are in Appendix A.

Using a similar reasoning, we prove the existence of complex zeros of the functions

$$\hat{g}_\pm(z) = \cos(tz) \pm \frac{i}{2}\left(\frac{z}{C} + \frac{C}{z}\right)\sin(tz), \tag{74}$$

where C and t are real constants, in the vicinities of the points $\tilde{z}_{kt}^* = \frac{\pi r}{t}$, $r = 1, 2, \dots$.

Zeros of functions (68) and (74) in the complex domain of variable z correspond to the resonant case when (real) frequency ω (parameter k_0) and geometric quantities l_1 and a of a one-sectional diaphragm in a rectangular waveguide are taken as nonspectral parameters (fixed). Here, in view of the relation $z = \gamma_1 = \sqrt{k_0^2\varepsilon_1 - \Theta}$, only permittivity ε_1 may be the spectral parameter (a complex variable quantity). Thus complex zeros of (68) and (74) give rise to complex RSs that are complex values of permittivity ε_1 of the medium filling the diaphragm (section).

Now we can extend this result and show that functions $g_\mp(z)$ given by (57) have infinitely many complex zeros. To this end write the equation $g_\mp(z) = 0$ in the equivalent form

$$g_0^\mp(z) \equiv g_1^\mp(z) + g_{2p}(z) = 0, \tag{75}$$

where

$$g_1^\mp(z) = \mp \tan l_1 z, \qquad g_{2p}(z) = \frac{2i\sqrt{\varepsilon_1}}{P}\frac{z\sqrt{z^2 - q}}{z^2 - Q}, \tag{76}$$

where

$$P = \varepsilon_1 + 1, \quad Q = \frac{q}{P} = \left(\frac{\pi}{a}\right)^2 \frac{\varepsilon_1 - 1}{\varepsilon_1 + 1}. \tag{77}$$

Next, we apply the same scheme of the proof as for functions $g_0(z)$ and $g\tilde{(z)}$ above. For function $g_{2p}(z)$ the following estimate is valid

$$|g_{2p}(z)| \le \frac{2|\sqrt{\varepsilon_1}||z|\sqrt{|z|^2 + |q|}}{|P|(|z|^2 - |Q|)} \le \frac{2|\sqrt{\varepsilon_1}|\sqrt{|z|^2 + |q|}}{|P|(|z| - \sqrt{|Q|})} \le \frac{2\sqrt{2}|\sqrt{\varepsilon_1}||z|}{|P|(|z| - |Q|)} \le \frac{4\sqrt{2}|\sqrt{\varepsilon_1}|}{|P|} \tag{78}$$

at $|z| > 2\sqrt{|Q|}$. Applying the same construction of circles Γ_k and the notation for M_0 we obtain

$$\max_{z\in\Gamma_k} |g_{2p}(z)| \leq \frac{4\sqrt{2}|\sqrt{\varepsilon_1}|}{|\varepsilon_1 + 1|} \equiv N(|\varepsilon_1|) < M_0$$

if $|\varepsilon_1|$ is sufficiently large, namely, if

$$|\varepsilon_1| > \left(\frac{2\sqrt{2} + \sqrt{8 - M_0^2}}{M_0}\right)^2. \tag{79}$$

Then we conclude that all conditions of the Rouche's theorem are fulfilled, so that function $g_0^{\mp}(z)$ has exactly one zero $z_k^{*\mp}$ of multiplicity one inside Γ_k.

Note that (79) is a sufficient condition providing the existence of a zero inside every circle Γ_k. It is easy to check (using e.g., a parameter-differentiation method) that the zeros $z_k^{*\mp} = z_k^{*\mp}(\varepsilon_1)$ depend continuously on ε_1, so that actually they exist for smaller $|\varepsilon_1|$.

Zeros of functions (57) in the complex domain of variable z correspond to the resonant case when geometric quantities l_1 and a of a one-sectional diaphragm in a rectangular waveguide are taken as nonspectral parameters (fixed) and $\kappa = k_0$ (or frequency ω) is the spectral parameter (a complex variable quantity). Thus complex zeros of (57) give rise to complex RSs that are complex resonance frequencies of a rectangular waveguide containing a one-sectional dielectric diaphragm obtained from (58).

The result may be generalized to the case when the diaphragm has several section of different lengths l_j, $j = 1, 2, \ldots, N$. In fact, the resulting representation for the transmission coefficient may be considered as a function of complex variable z and the equation that governs its singularities can be written in the form (69) where function $g_1(z)$ will be the same as in (69) and $g_2(z)$ will be a certain rational function for which estimate (71) remains valid.

5. ANALYSIS OF SIMULATION RESULTS AND PROPERTIES OF RESONANT STATES

In this section we present results of calculation of complex singularities of the transmission coefficient performed with respect to different problem parameters. The calculated data reveal distribution of RSs on the complex plane and illustrate several their important properties.

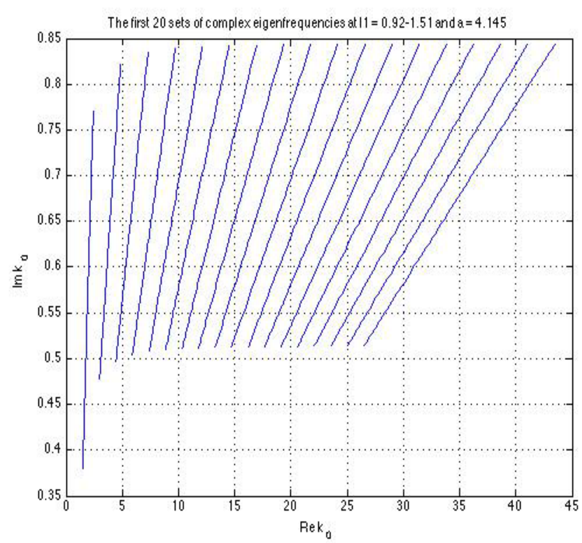


Figure 7. Eigenfrequencies k_0 of a one-sectional diaphragm.

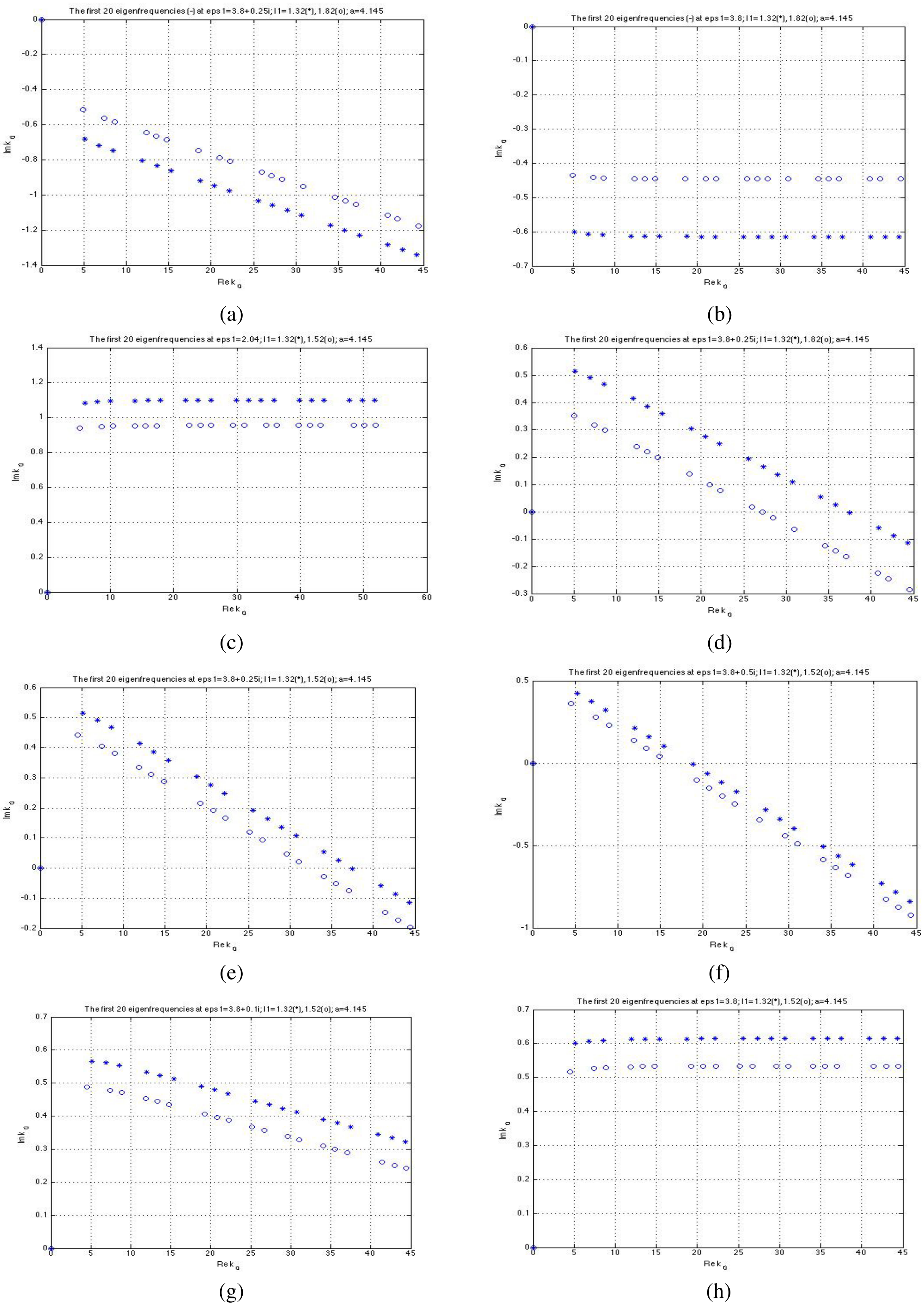


Figure 8. The first 20 eigenfreqiencies of a one-sectional diaphragm at $\varepsilon_1 = 3.8 + 0.25i$ and $3.8 + 0.5i$.

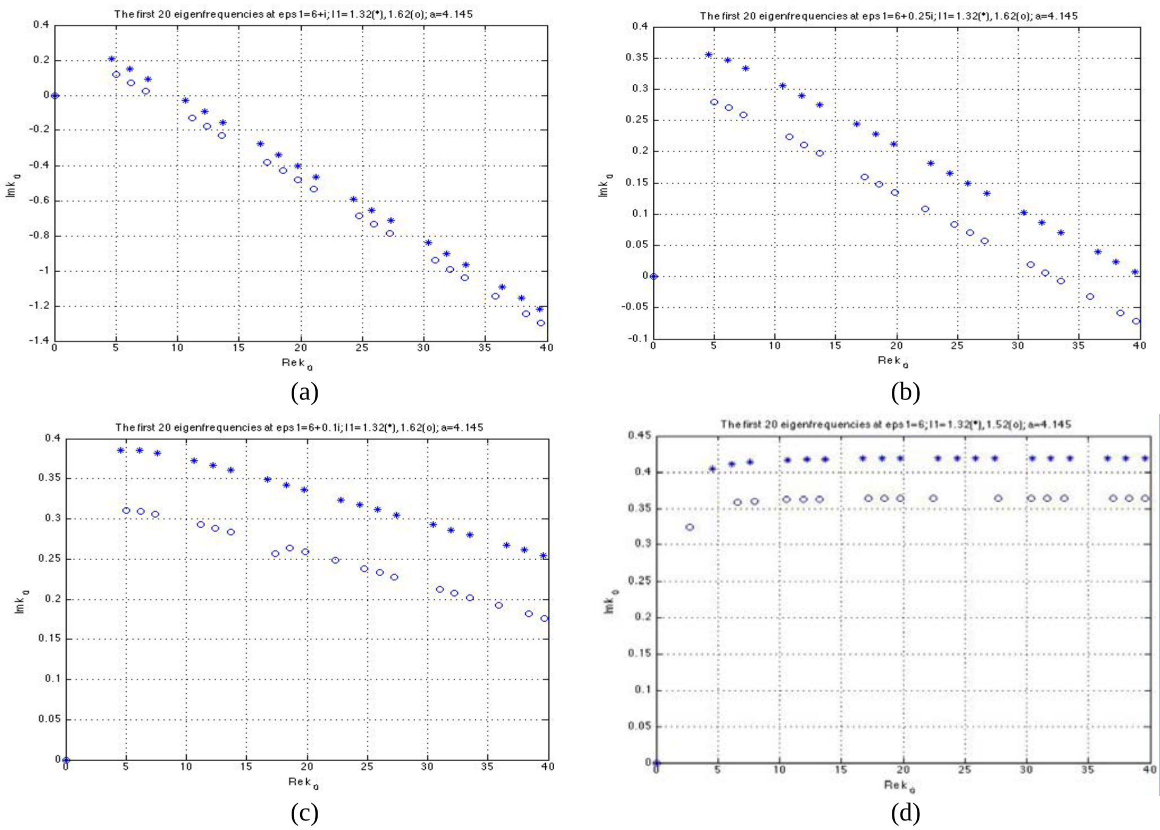


Figure 9. The first 20 eigenfreqiencies of a one-sectional diaphragm at $\varepsilon_1 = 6$, $6 + 0.1i$, $6 + 0.25i$, and $6 + i$.

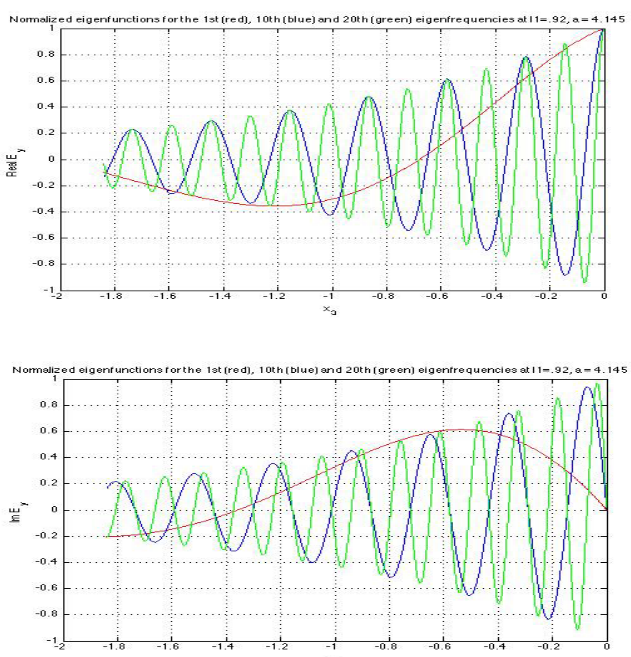


Figure 10. Eigenfields of a one-sectional diaphragm.

We determine numerically subsets of the resonant set $\Omega_{res}^{\mp}$ given by (50) in the form of curves in the complex planes of spectral parameters λ, k_0, or z; the curves are parametrized by some of the nonspectral (real) parameters. The main attention is paid to a one-sectional dielectric diaphragm and calculation of complex roots of the equation $g_1(z) = 0$ which plays a fundamental role in the RS theory. In this case the curves $\lambda = \lambda(l_1)$, $k_0 = k_0(l_1)$, or $z = z(l_1)$ are parametrized by the real parameter l_1.

Figure 7 shows the curves of first 20 eigenfrequencies k_0 of a one-sectional diaphragm in the complex plane calculated from (58), exemplifying continuous dependence on parameter l_1, relative length of the section.

Figures 8 and 9 demonstrate one of the important features of RSs: eigenfrequency k_0 may become real at certain complex values of permittivity. Namely, Figs. 8(e), (d), and (f) show the first 20 eigenfrequencies calculated at $\varepsilon_1 = 3.8 + 0.25i$ and $\varepsilon_1 = 3.8 + 0.5i$: we see that higher-order eigenfrequencies become real, namely an increase in l_1 shifts the eigenfrequency set to lower values of the imaginary part, so that the eigenfrequency with index 13 ($\varepsilon_1 = 3.8 + 0.25i$) or index 6 ($\varepsilon_1 = 3.8 + 0.5i$) 'lands' to the real axis. For comparison, Figs. 8(c) and (h) and 9(c) and (d) demonstrate the same data revealing the dynamics of the frequency transformation to real values for real $\varepsilon_1 = 2.04$, 3.8, 6 and $6 + 0.1i$.

Figures 10 and 11 show the corresponding eigenfunctions (real and imaginary parts of the E_y eigenfield component) of a one-sectional diaphragm for three eigenfrequencies of increasing index 1, 10, and 20 (may be identified by the number of oscillations); decaying eigenfunctions in Fig. 10 are for standing waves in Fig. 11 correspond to real eigenfrequencies.

Figure 12 illustrates a phenomenon which we call 'tuning' based on the data calculated for a three-sectional dielectric diaphragm: modulus of the transmission coefficient increases in a vicinity of an RS

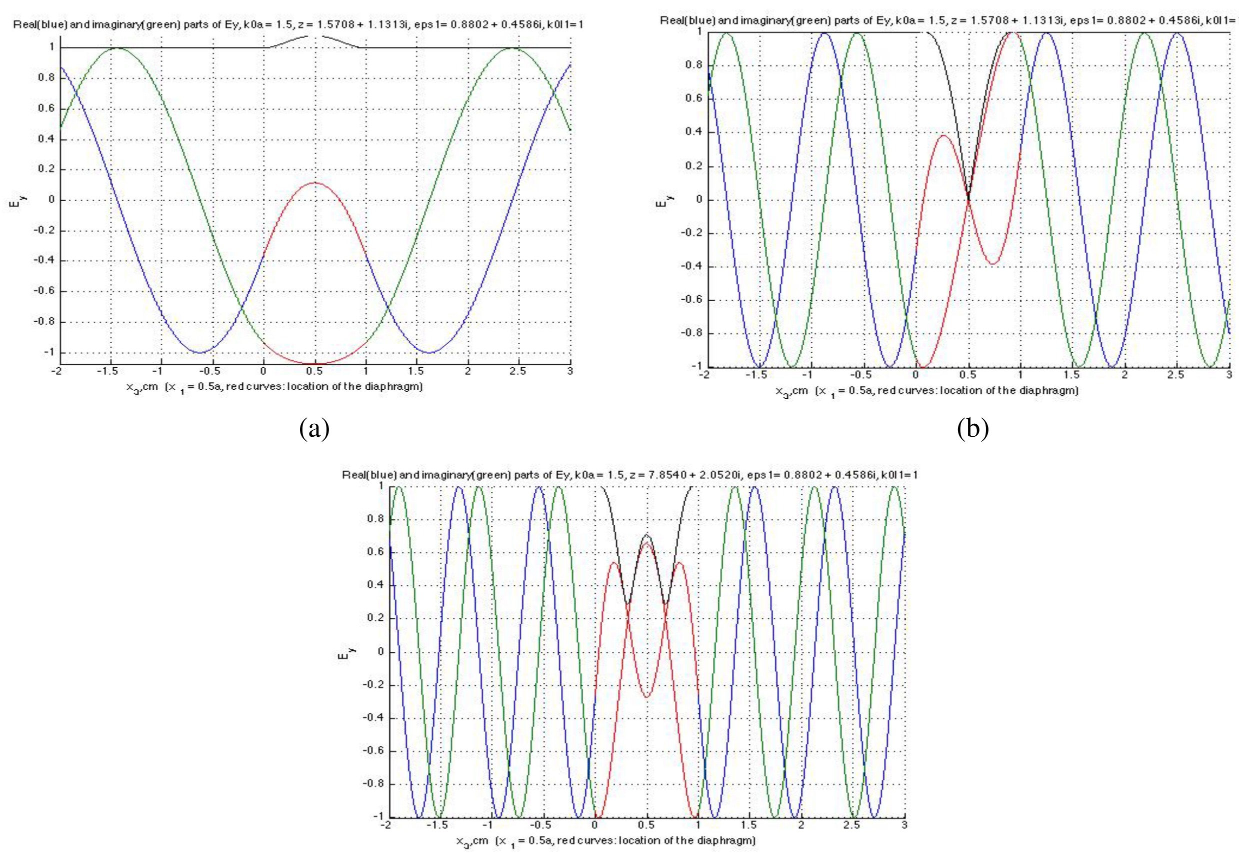
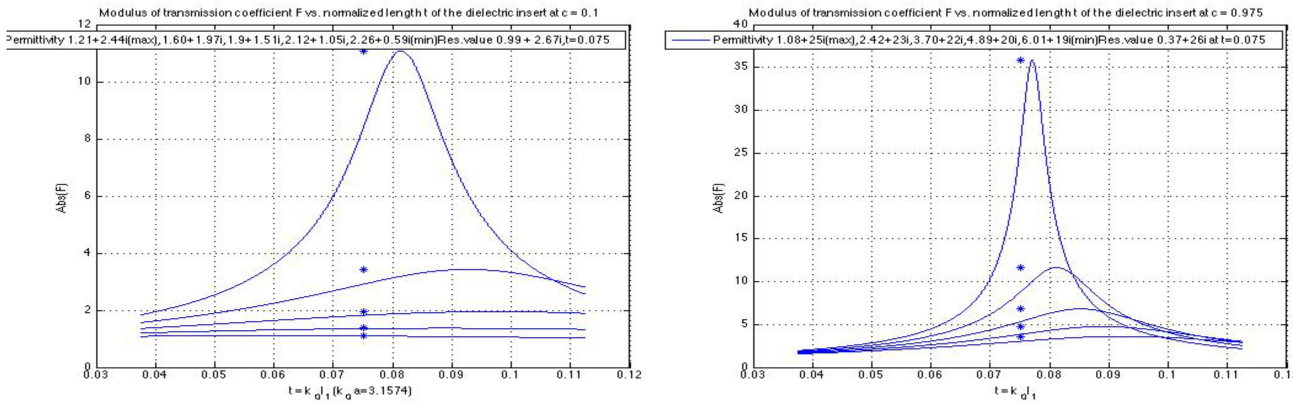


Figure 11. Eigenfields (standing waves: real and imaginary parts and modulus of the longitudinal field component E_y) of a one-sectional diaphragm.



(a) (b)

Figure 12. 'Tuning curves' for a one-sectional diaphragm.

when the parameter couple (t, ε_1) (a subset of the resonant set $\Omega^{\mp}_{res}$, (50)) approaches its resonant value (t^*, ε_1^*).

6. CONCLUSION

We have proved the existence of complex eigenvalues for a family of waveguide spectral problems associated with eigensolutions (eigenfrequencies) of open waveguide resonators formed by parallel-plate waveguides or waveguides of rectangular cross-section containing inclusions in the form of multi-sectional diaphragms. We have identified the RSs associated with these eigensolutions (eigenfrequencies) as singularities (finite-multiplicity poles) of the operators of the corresponding diffraction problems continued to the complex domain (where the sovability is violated); namely, as poles of the transmission coefficient. We have investigated main properties of RSs by reducing the problem to determination and analysis of zeros of well-defined families of entire functions of the spectral parameter. The occurrence of RSs has been shown when frequency, permittivity, or longitudinal wavenumber of any of the dielectric sections forming the inclusion is taken as a spectral parameter.

We have shown that RSs, eigensolutions, and eigenfrequencies are associated, for the considered family of open waveguide resonators, with eigenvalues of distinct families of Sturm-Liouville problems on the line. The eigenfrequencies form countable sets of points (60) with the only accumulation point at infinity; these points are finite-multiplicity poles of the analytical continuation of the operator of the initial diffraction problem and its Green's function to a multi-sheet Riemann surface.

In line with (56), (59), (66), and (60) the set of complex eigenfrequencies of the considered family of open waveguide resonators with parallel-plane layered dielectric inclusions is similar in its structure to the set of eigenvalues of a Laplacian in a rectangle, and may be conditionally described as eigenvalues of a Laplacian in a 'semi-infinite' rectangle $\Pi_\infty = \{0 < x_1 < a, -\infty < x_3 < \infty\}$. The presence of a resonance domain, an insert in the form of a diaphragm $D = \{0 < x_1 < a, 0 < x_3 < l\}$ where permittivity assumes piecewise-constant values, removes the continuous frequency spectrum and gives rise to a discrete set of points 'shifted' to (upper half of) the complex plane.

ACKNOWLEDGMENT

I gratefully acknowledge the contribution of Yury Smirnov and Martin Krepela to the mathematical proofs of the existence of zeros of the function that governs singularities of the transmission coefficient for a one-sectional diaphragm.

APPENDIX A.

A.1. Another Set of Poles of the Transmission Coefficient

The existence of (infinite sets of) poles of the transmission coefficient in the complex domain is proved in Section 4 by demonstrating that an entire function

$$g\left(z\right)=\cos z+\frac{i}{2}\left(\frac{z}{C}+\frac{C}{z}\right)\sin z \tag{A1}$$

where a constant $C>0$ that enters expression (18) for the transmission coefficient has infinite sets of complex zeros. Here we will prove the existence of the family of zeros (72) of function (A1) employing a different technique. To this end consider continuation of $g(z)$ to the complex domain of variable z (longitudinal wavenumber of the wave in the dielectric section) and then of variable ε_1 (permittivity of the dielectric section).

The real and imaginary parts of $g(z)=g_1(x,y)+ig_2(x,y)$ with $z=x+iy$ are

$$\begin{aligned}
g_1 &= \cos x\cosh y-h_1\cos x\sinh y-h_2\sin x\cosh y,\\
g_2 &= -\sin x\sinh y+h_1\sin x\cosh y-h_2\cos x\sinh y,\\
h_1 &= \frac{x}{2}\left(\frac{1}{C}+\frac{C}{\rho^2}\right),\quad h_2=\frac{y}{2}\left(\frac{1}{C}-\frac{C}{\rho^2}\right), \qquad \text{(A2)}\\
\rho &= \left|z\right|,\quad \rho^2=x^2+y^2. \qquad \text{(A3)}
\end{aligned}$$

In order to prove that zeros of function $g(z)$ may exist and to determine the domain of their localization consider the case

$$\rho=C,\qquad C^2=x^2+y^2; \tag{A4}$$

then $h_2=0$, $h_1=x/C$, and

$$\begin{aligned}
g_1 &= \cos x\left[(C-x)e^y+(C+x)e^{-y}\right],\quad g_2=\sin x\left[-(C-x)e^y+(C+x)e^{-y}\right],\\
g_1 &= 0\Leftrightarrow\cos x=0\vee\left[(C-x)e^y+(C+x)e^{-y}\right]=0,\\
g_2 &= 0\Leftrightarrow\sin x=0\vee\left[-(C-x)e^y+(C+x)e^{-y}\right]=0.
\end{aligned}$$

There is no x which satisfies the equations $\cos x=\sin x=0$ and

$$\left[(C-x)e^y+(C+x)e^{-y}\right]=0\wedge\left[-(C-x)e^y+(C+x)e^{-y}\right]=0.$$

Therefore $g(z)=0$ under condition (A4) if x solves the following equation system

$$\left\{\cos x=0\wedge\left[-(C-x)e^y+(C+x)e^{-y}\right]=0\right\}\vee\left\{\sin x=0\wedge\left[(C-x)e^y+(C+x)e^{-y}\right]=0\right\}.$$

From the first equation we have

$$\cos x=0\Leftrightarrow x=x_m=\frac{\pi}{2}+m\pi,\quad m\in \mathrm{Z}, \tag{A5}$$

$$\begin{aligned}
\left[-(C-x)e^y+(C+x)e^{-y}\right]=0\Leftrightarrow e^{2y} &= \frac{C+x}{C-x},\\
e^{2\sqrt{C^2-x^2}} &= \frac{C+x}{C-x}, \qquad \text{(A6)}
\end{aligned}$$

where $0<x<C$.

The following statements hold.

Lemma 1. Equation (A6) has exactly one root $x_0 = x_0(C)$ for every $C > 0$; $x_0(C)$ is a continuous increasing function of parameter C.

Lemma 2. There is a $C = C_n > \pi/2+n\pi$ such that Equation (A6) has the root $x_n = x_n(C_n) = \pi/2+n\pi$ $(n = 0, 1, \ldots)$.

$$0 < C_n - x_n < x_{n+1} - x_n = \pi \tag{A7}$$

and

$$\lim_{n\to\infty} \alpha_n = 1, \quad \alpha_n = \frac{x_n}{C_n}. \tag{A8}$$

Considering $g(z)$ a function of the complex variable $\zeta = \epsilon_1 = u + iv$, $v \geq 0$, we can express, u and v via $x = \Re z$ and $y = \Im z$,

$$u = \frac{x^2 - y^2 + L}{l_1^2 k_0^2}, \quad v = \frac{2xy}{l_1^2 k_0^2}, \qquad L = \left(\frac{l_1 \pi}{a}\right)^2. \tag{A9}$$

From (A9), it follows

$$\text{sign}\,(\Im\epsilon_1) = \text{sign}\,(\Re z)\text{sign}\,(\Im z). \tag{A10}$$

Set

$$p_n = x_n^2 - y_n^2 = 2x_n^2 - C_n^2, \quad q_n = 2x_n y_n, \quad x_n = \pi/2 + n\pi, \qquad y_n = \sqrt{C_n^2 - x_n^2}, \tag{A11}$$

then

$$\epsilon_1 = \epsilon_{1,n} = u_n + iv_n, \qquad u_n = \frac{p_n + L}{C_n^2 + L}, \quad v_n = \frac{q_n}{C_n^2 + L}, \tag{A12}$$

where C_n is the root of Equation (A6)

$$h_n(C) \equiv e^{2\sqrt{C^2 - x_n^2}} - \frac{C + x_n}{C - x_n} = 0, \quad n = 0, 1, 2, \ldots. \tag{A13}$$

Next we prove the following statement using Lemmas 1 and 2.

Set x_n, y_n, p_n, q_n, and C_n according to (A11). Then $g(z_n) = g(z(\epsilon_{1,n})) = 0$, where

$$\begin{aligned} z_n &= x_n + iy_n = l_1\sqrt{k_0^2\epsilon_{1,n} - \left(\frac{\pi}{a}\right)^2}, \\ k_0^2 &= \left(\frac{\pi}{a}\right)^2 + \left(\frac{C_n}{l_1}\right)^2. \end{aligned} \tag{A14}$$

In particular, $x_0 = \pi/2$ is a root of Equation (A6) at $n = 0$ and $C = C_0$. For this value $g(z_0) = 0$; the calculated quantities are

$$\begin{aligned} g\,(z_0) &\equiv \cos z_0 + \frac{i}{2}\left(\frac{z_0}{C_0} + \frac{C_0}{z_0}\right)\sin z_0 = 0, \\ x_0 &= \pi/2, \quad y_0 = 1.1312839806580586, \quad C_0 = 1.9357697552048596, \\ z_0 &= x_0 + i\sqrt{C_0^2 - x_0^2} \quad (|z_0| = C_0). \end{aligned}$$

A.2. Relation between Poles of the Transmission Coefficient and RSs

Complex zeros z_n are poles of the transmission coefficient F_1; in line with the general scattering theory, z_n are poles of the scattering matrix of an inhomogeneity in the waveguide and can be referred to as resonances (RSs) of the inhomogeneity (in the form of a dielectric insert, a parallel-plane diaphragm). This conclusion explains the behavior of the absolute value $|F/A|$ of the normalized transmission coefficient described below: this quantity (considered as a function of real variable z) has local minima at certain 'resonance' points $z = x_n^*$ each associated with a particular $x_n = \Re z_n$ $(n = 0, 1, \ldots)$.

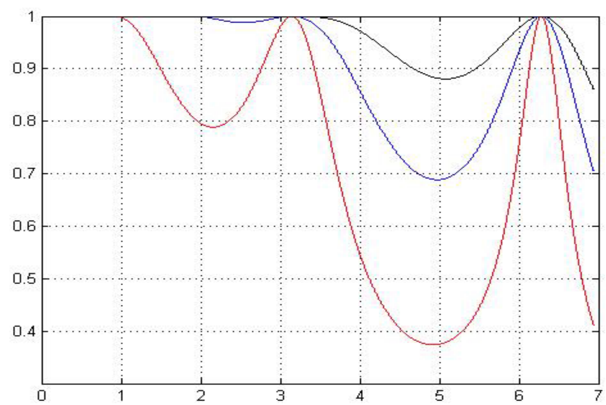


Figure A1. Modulus of the transmission coefficient $F_1 = F_1(z)$, $C_0 - 1 < z < C_0 + 1$; $C = C_0 - 1$ (red), C_0 (blue), $C_0 + 1$ (black). C_0 is determined according to Lemma 2.

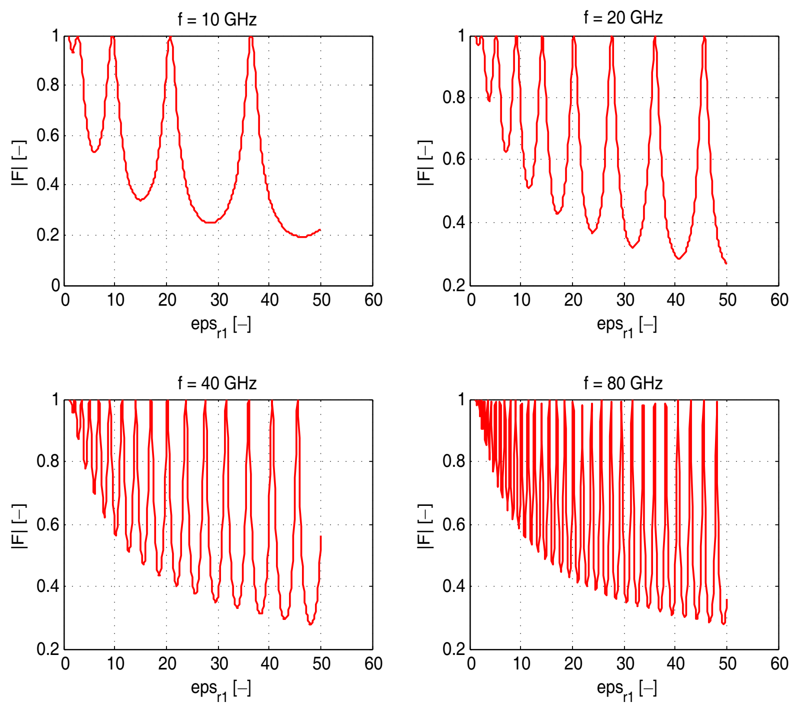


Figure A2. Modulus of the transmission coefficient $F_1 = F_1(\varepsilon_1)$ vs. relative permittivity for increasing values of the frequency parameter $f = 10$, 20, 40, 80 GHz..

The dependence of the modulus of transmission coefficient $F_1 = F_1(z)$ on real parameter z shown in Figs. 5 and A1 is similar to the data in Fig. A2 where this quantity is calculated vs. relative permittivity. In fact, C and z may be considered as independent variables if all geometric quantities a, b, and l_1 and frequency are taken as nonspectral parameters (fixed) and ϵ_1 is varied; therefore, in view of the definition (A1), the dependence on ϵ_1 is similar to that on z. We see that zeros of function $g(z)$ exist for certain values of C when both real and imaginary parts of z are positive; therefore zeros of $g(z)$ (poles of the transmission coefficient F) lie in a domain defined by $\text{sign}\,(x)\,\text{sign}\,(y) = 1$.

Using (A11), (A12), and (A8) it is easy to prove the following

Lemma 3.

$$\lim_{n\to\infty} u_n = 2\alpha - 1 = 1,$$
$$\lim_{n\to\infty} v_n = 2\sqrt{\alpha-\alpha^2} = 0, \tag{A15}$$
$$\alpha = \lim_{n\to\infty} \alpha_n = 1.$$

From Lemma 3 it follows that zeros $\epsilon_{1,n}$ determined according to (A12) of $g(z(\zeta))$ considered as a function of the complex variable $\zeta = \epsilon_1$ form a sequence of complex numbers $\{\epsilon_{1,n}\}$ such that

$$\lim_{n\to\infty} \epsilon_{1,n} = 1. \tag{A16}$$

For real z it is easy to obtain from (A1)

$$|g\,(z)\,| = \sqrt{1+\frac{1}{4}g_d(z)}, \quad g_d(z) = \sin^2 z\left(\frac{z}{C}-\frac{C}{z}\right)^2 \tag{A17}$$

so that

$$\min_{\Im z=0} |g\,(z)\,| = |g\,(\pi n)\,| = |g\,(C)\,| = 1. \tag{A18}$$

Calculating the derivative of $|g\,(z)\,|$ (or of $g_d(z)$) it is easy to check that the local maxima of $|g\,(z)\,|$ are at the points $z = x_n^* \in (\pi/2(2n-1),\, \pi n)$ $(n = 1, 2, \ldots)$, where x_n^* are the roots of the equation

$$\tan z = z\frac{C^2 - z^2}{C^2 + z^2}. \tag{A19}$$

Figure A3 illustrates the location of the first two roots $x_1^* \in (\pi/2,\, \pi)$ and $x_2^* \in (3\pi/2,\, 2\pi)$.

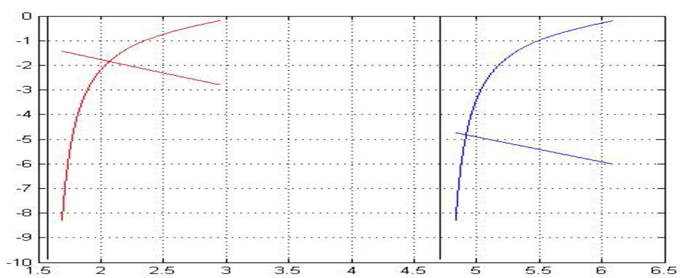


Figure A3. Location of roots of Equation (A19). $C = 0.5$.

Consequently, for the transmission coefficient F

$$T = \left|\frac{F}{A}\right| = \frac{1}{|g\,(z)|}, \tag{A20}$$
$$\max_{\Im z=0} T = |g\,(\pi n)\,| = |g\,(C)\,| = 1,$$

and the local minima of T are at the points $z = x_n^*$ $(n = 0, 1, \ldots)$. We see that the transmission coefficient F considered as a function of complex variable z has poles z_n at the points (A14) referred to as RSs associated with the inhomogeneity in the waveguide in the form of a dielectric insert (a parallel-plane diaphragm). In addition, the transmission coefficient considered as a function of real variable z has local minima and maxima at the 'resonance' points $z = x_n^*$ each associated with a particular $x_n = \Re z_n$ $(n = 0, 1, \ldots)$ and, respectively, $z_r^{ex}(1) = \pi r$ $(r = 1, 2, \ldots)$. The latter fact can be directly applied to the diffraction problem at real frequencies and has a clear physical meaning. Also

$$\lim_{n\to\infty} |x_n^* - x_n| = 0. \tag{A21}$$

The corresponding resonance frequency values are

$$f_n = \frac{c}{2}\sqrt{\frac{1}{\epsilon_1}\left[\left(\frac{2n-1}{2l_1}\right)^2 + \frac{1}{a^2}\right]}. \tag{A22}$$

A.3. Some Properties of the Extrema of the Transmission Coefficient

Consider now the functions (57)

$$g_{\mp}(z) = \cos tz \mp \frac{i}{2}\left(\sqrt{\varepsilon_1}\frac{z}{\sqrt{z^2-q}} + \frac{1}{\sqrt{\varepsilon_1}}\frac{\sqrt{z^2-q}}{z}\right)\sin tz, \quad t = l_1, \tag{A23}$$

(A23) as well as function $g(z)$ in (A1), can be written in the form

$$\begin{aligned} g_{\mp}(z) &= \cos tz \mp ih(X)\sin tz,\ z > q, \quad g_{\mp}(z) = \cos tz \mp h_1(X)\sin tz,\ z < q, \\ g\,(z) &= \cos z + iZ(z,C)\sin z, \\ X &= \frac{z\sqrt{\varepsilon_1}}{\sqrt{|z^2-q|}}, \quad h(X) = Z(X,1) = \frac{1}{2}\left(X + \frac{1}{X}\right), \quad h_1(X) = Z_1(X,1); \end{aligned} \tag{A24}$$

here

$$Z_1(s,c) = \frac{1}{2}\left(\frac{s}{c} - \frac{c}{s}\right), \tag{A25}$$

and $h(X)$ and $Z(s,c)$ (the Zhukovsky function and the modified Zhukovsky function (19)) have the following properties for real $s, c > 0$:

$$\min_{s>0} h(s) = h(1) = 1 \quad (h(s) \geq 1,\ s > 0); \tag{A26}$$

$$Z(s,c) = Z(c,s), \quad \min_{s>0} Z(s,c) = Z(c,c) = 1 \quad (Z(s,c) \geq 1,\ s > 0). \tag{A27}$$

Function $Z_1(s,c)$ is monotonically increasing for $s, c > 0$, take all real values for $s \in (0,\infty)$, and

$$\begin{aligned} Z_1(s,c) &= -Z_1(c,s),\ Z(c,c) = 0;\ \min_{s>0}|Z_1(s,c)| = \min_{s>0} Z_1^2(s,c) = Z_1(c,c) = 0, \\ Z(s,c) &< 0,\ s < c; \quad Z(s,c) > 0,\ s > c. \end{aligned} \tag{A28}$$

For real $z, c, q, t > 0$ we have, in view of (A26)–(A28), the result obtained above using a different proof

$$|g\,(z)\,|^2 = \cos^2 z + Z^2(z,C)\sin^2 z \geq \cos^2 z + \sin^2 z = 1 = |g\,(\pi n)\,|, \tag{A29}$$

and

$$|g_{\mp}(z)|^2 = \cos^2 tz + h^2(X)\sin^2 tz \geq \cos^2 z + \sin^2 z = 1 = \left|g_{\mp}\left(\frac{\pi r}{t}\right)\right| \quad (z > q). \tag{A30}$$

Thus $z_r^{ex}(t) = \dfrac{\pi r}{t}$ and $\lambda_{1\,mr}$ $(r, m = 1, 2, \ldots)$ given by (67) are points of extremum of the transmission coefficient $F = F_1(z)$ given by (18) and (23) with $s = 1$ (considered as a function of real variable z) where $F = F_1(z)$ attains unity (Fig. 5).

REFERENCES

1. Shestopalov, Y., Y. Smirnov, and E. Derevyanchuk, "Permittivity reconstruction of layered dielectrics in a rectangular waveguide from the transmission coefficient at different frequencies," *Springer Proceedings in Mathematics and Statistics*, Vol. 52, No. XII, 169–181, 2013.
2. Shestopalov, Y., Y. Smirnov, and E. Derevyanchuk, "Inverse problem method for complex permittivity reconstruction of layered media in a rectangular waveguide," *Phys. Status Solidi C*, 1–6, 2014.
3. Gamow, G., "Zur quantentheorie des atomkernes," *Z. Phys.*, Vol. 51, 204–212, 1928.
4. Armitage, L. J., et al, "Resonant-state expansion applied to planar waveguides," *Phys. Rev. A*, Vol. 89, 053831, 2014.
5. Shestopalov, Y. and V. Shestopalov, *Spectral Theory and Excitation of Open Structures*, The IEE Peter Peregrinus, London, 1996.
6. Brovenko, A., P. Melezhik, and A. Poedinchuk, "Spectral problems in the theory of diffraction of waves on layered media," *Telecommunications and Radio Engineering*, Vol. 72, 1821–1838, 2013.
7. Colton, A. and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, Berlin, 1998.
8. Shestopalov, V. and Y. Sirenko, *Dynamical Theory of Gratings*, Naukova Dumka, Kiev, 1989.