

# Reciprocity Relations for Nonlinear Galvanomagnetic Transducer

Vyacheslav Ignatjev, Andrey Orlov\*, and Sergey Perchenko

**Abstract**—This paper deals with reciprocity relations derivation for a nonlinear, stationary, homogeneous and isotropic plasma-like medium in an external homogeneous magnetic field. A special case of such a medium is the charge carriers collective in semiconductors. It is shown that the classical reciprocity relations will be valid even in the presence of nonlinearity, and they can be used for Hall magnetometer bias compensation.

## 1. INTRODUCTION

Phenomena of interaction between the cross-transfer processes have been studied for a long time. Kelvin studied the thermoelectric effect, which occurs with the simultaneous flow of electric current and heat in 1854. Also he obtained the first reciprocity relations on the basis of thermodynamic arguments. Helmholtz obtained such relations in 1876. He researched transfer processes in electrolytes. Eastman developed these relations for the diffusion and heat flow in 1926 [1]. The general theory was built in 1931 by Lars Onsager, for which he was awarded the Nobel Prize in 1968.

Onsager's reciprocity relations are fundamental, and they are valid not only in thermodynamics. For example, in the mechanics they are expressed in the form of reciprocity displacements (Maxwell-Betti law) [2]. Reciprocity is widely used in acoustics [3], in the calculation and measurement of the antennas characteristics [4], and to enhance the accuracy of magnetic measurements [5].

Onsager's symmetry relations of the kinetic coefficients [6] are obtained in the linear approximation. They are based on two assumptions: the invariance of the macroscopic motion under time reversal and the fact that the average relaxation of spontaneous fluctuations in the system takes place in accordance with the macroscopic laws. The theory of nonlinear permeability, based on the solution of kinetic equations by expansion in powers of the electromagnetic field, is developed in modern nonlinear electrodynamics [7]. This method can be used in the particular case of plasma-like media, which include group of carriers in semiconductors, but the reciprocity relations for susceptibilities are not considered.

A number of studies present reciprocity relations for special cases [8]. The example of these studies is the conductivity of weakly nonlinear finite size two-dimensional media [9], but as long as they have not been confirmed by any precise and general method (or experiment) [10]. The influence of magnetic field on the reciprocity relations has never been investigated for nonlinear media, in contrast to the linear theory of Onsager.

There is a problem of optimizing the galvanomagnetic transducers design for a specific task [11]. Semiconductor materials with high sensitivity to temperature changes and high nonlinearity are often used in sensor production. Thus, there are many problems of nonlinearity and temperature dependence compensation. Studying the conditions under which nonlinear galvanomagnetic converters are reciprocal allows using standard methods of compensation for their errors well [5].

The question of reciprocity magnetically nonlinear functional electronics devices has not been considered previously. Generally, this problem does not seem to be solved, because general statistical

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\* Corresponding author: Andrey Orlov (orlwork@inbox.ru).

The authors are with the Department of Radiophysics, Volgograd State University, Russia.

methods reciprocal relations are not applicable to the case of the nonlinear relationship between the forces and flows [10]. Reciprocity relations are postulated [12] for linear systems in thermodynamics of nonequilibrium processes, and they are sometimes considered as the fourth law of thermodynamics. Magnetically active plasma-environment in which the dynamics of charge carriers is described by the kinetic equation in the relaxation approximation is of great interest. Reciprocity relations for this special but important case can be obtained for a nonlinear medium in a magnetic field, described by the Vlasov equation.

## 2. NONLINEAR CONDUCTIVITY OF PLASMA-LIKE MEDIA

Transport processes in the relaxation approximation are described by the kinetic Boltzmann's equation. It takes Vlasov's equation form for the medium in electric and magnetic fields:

$$\frac{\partial f}{\partial t} + \bar{v} \cdot \frac{\partial f}{\partial \bar{r}} + q(\bar{E} + \bar{v} \times \bar{B}) \cdot \frac{\partial f}{\partial \bar{p}} = -\frac{f - f_0}{\tau}, \quad (1)$$

where  $f(t, \bar{r}, \bar{p})$  — distribution function of charge carriers (electrons or holes) on the coordinates and momenta,  $f_0(p^2)$  — the equilibrium distribution function,  $q$  — the charge of the carrier,  $\tau$  — the ensemble average relaxation time. Equation (1) is not invariant to simultaneous inversion of the electric and magnetic fields. Rotation  $\bar{E} \rightarrow -\bar{E}$  and  $\bar{B} \rightarrow -\bar{B}$  translates it to:

$$\frac{\partial f}{\partial t} + \bar{v} \cdot \frac{\partial f}{\partial \bar{r}} - q(\bar{E} + \bar{v} \times \bar{B}) \cdot \frac{\partial f}{\partial \bar{p}} = -\frac{f - f_0}{\tau} \quad (2)$$

If function  $f^+(\bar{p}) = f(\bar{p}, \bar{E}, \bar{B})$  is the solution of Equations (1) and the function  $f^-(\bar{p}) = f(\bar{p}, -\bar{E}, -\bar{B})$  is a solution to Equation (2), then generally  $f^+(\bar{p}) \neq f^-(\bar{p})$ . Accordingly, the current density is not invariant to the field inversion:

$$\bar{j}(\bar{E}, \bar{B}) = \int \frac{qc\bar{p}}{m} f(\bar{p}, \bar{E}, \bar{B}) d^3p, \quad (3)$$

where  $c$  is the concentration of charge carriers  $q$ . We cannot directly get the connection between functions  $f^+(\bar{p})$  and  $f^-(\bar{p})$  in the momentum representation, to obtain relationship material equations  $\bar{j}(\bar{E}, \bar{B})$  and  $\bar{j}(-\bar{E}, -\bar{B})$ . However, integral (3) calculation is usually done by changing the variables [9, 13]. We can perform transformation of variables, which is not invariant to field inversion. Then, Equation (1) takes the invariant form. Let us turn to the new variables. We consider the case, where magnetic and electric fields are constant, uniform, not in the same direction and not equal to zero. Let's

$$\xi_1 = \bar{p} \cdot \bar{E}, \quad \xi_2 = \bar{p} \cdot \bar{B}, \quad \xi_3 = p^2. \quad (4)$$

Let us obtain expression of momentum as the function of fields  $\bar{E}$ ,  $\bar{B}$  and also variables (4). Then, after expanding by vectors  $\bar{E}$ ,  $\bar{B}$  and  $\bar{E} \times \bar{B}$ ,

$$\bar{p} = A_1 \bar{E} + A_2 \bar{B} + A_3 \bar{B} \times \bar{E}, \quad (5)$$

where  $A_i$  — scalar coefficients. After scalar multiplying (5) on vectors  $\bar{E}$ ,  $\bar{B}$  and  $\bar{p}$  we obtain:

$$\begin{aligned} \xi_1 &= A_1 E^2 + A_2 \bar{E} \cdot \bar{B}; & \xi_2 &= A_1 \bar{E} \cdot \bar{B} + A_2 B^2; \\ \xi_3 &= p^2 = A_1^2 E^2 + A_2^2 B^2 + A_3^2 |\bar{B} \times \bar{E}|^2 + 2A_1 A_2 \bar{B} \cdot \bar{E}. \end{aligned}$$

Solving this system allows us to find  $A_i$ :

$$\bar{p} = \frac{B^2 \xi_1 - \bar{E} \cdot \bar{B} \xi_2}{|\bar{B} \times \bar{E}|^2} \bar{E} + \frac{E^2 \xi_2 - \bar{E} \cdot \bar{B} \xi_1}{|\bar{B} \times \bar{E}|^2} \bar{B} + \frac{\sqrt{D}}{|\bar{B} \times \bar{E}|^2} \bar{B} \times \bar{E}. \quad (6)$$

Conversions (4)–(5) are mutually unambiguous in solution region  $V$

$$D = D(\bar{\xi}, \bar{E}, \bar{E}) = \xi_3 |\bar{B} \times \bar{E}|^2 - (\xi_2 \bar{E} + \xi_1 \bar{B})^2 \geq 0 \quad (7)$$

where  $\bar{\xi}$  is the vector with components  $\xi_1$ ,  $\xi_2$ ,  $\xi_3$  and

$$\frac{\partial f}{\partial \bar{p}} = \bar{E} \frac{\partial F}{\partial \xi_1} + \bar{B} \frac{\partial F}{\partial \xi_2} + 2\bar{p} \frac{\partial F}{\partial \xi_3}. \quad (8)$$

Equations (4), (6) and (8) are transformed to

$$\xi_1 = -\bar{p} \cdot \bar{E}, \quad \xi_2 = -\bar{p} \cdot \bar{B}, \quad \xi_3 = p^2, \quad (9)$$

$$\bar{p} = -\frac{B^2 \xi_1 - \bar{E} \cdot \bar{B} \xi_2}{|\bar{B} \times \bar{E}|^2} \bar{E} - \frac{E^2 \xi_2 - \bar{E} \cdot \bar{B} \xi_1}{|\bar{B} \times \bar{E}|^2} \bar{B} + \frac{\sqrt{D}}{|\bar{B} \times \bar{E}|^2} \bar{B} \times \bar{E}, \quad (10)$$

$$\frac{\partial f}{\partial \bar{p}} = -\bar{E} \frac{\partial F}{\partial \xi_1} - \bar{B} \frac{\partial F}{\partial \xi_2} + 2\bar{p} \frac{\partial F}{\partial \xi_3}. \quad (11)$$

Expression (7) is invariant to the field inversion.

Equation (1) by means of (4), (6) and (8), and Equation (2) by means of (9), (10) and (11) are converted in the same equation, which is invariant to field inversion

$$\left( E^2 + \sqrt{D} \right) \frac{\partial F(\bar{\xi})}{\partial \xi_1} + \bar{E} \cdot \bar{B} \frac{\partial F(\bar{\xi})}{\partial \xi_2} + 2\xi_1 \frac{\partial F(\bar{\xi})}{\partial \xi_3} = \frac{F(\bar{\xi}) - F_0(\xi_3)}{q\tau}. \quad (12)$$

New distribution function is

$$\begin{aligned} F(\bar{\xi}, \bar{E}, \bar{B}) &= f^+ \left( \bar{p} = \frac{B^2 \xi_1 - \bar{E} \cdot \bar{B} \xi_2}{|\bar{B} \times \bar{E}|^2} \bar{E} + \frac{E^2 \xi_2 - \bar{E} \cdot \bar{B} \xi_1}{|\bar{B} \times \bar{E}|^2} \bar{B} + \frac{\sqrt{D}}{|\bar{B} \times \bar{E}|^2} \bar{B} \times \bar{E} \right) \\ &= f^- \left( \bar{p} = -\frac{B^2 \xi_1 - \bar{E} \cdot \bar{B} \xi_2}{|\bar{B} \times \bar{E}|^2} \bar{E} - \frac{E^2 \xi_2 - \bar{E} \cdot \bar{B} \xi_1}{|\bar{B} \times \bar{E}|^2} \bar{B} + \frac{\sqrt{D}}{|\bar{B} \times \bar{E}|^2} \bar{B} \times \bar{E} \right). \end{aligned}$$

An additional condition to Equation (12) can be obtained from physical considerations. In the zero electric field  $\xi_1 = 0$ , the function of the electron momentum distribution is determined only by the magnetic field. This function should not depend on the magnetic field for an isotropic medium:  $f(\bar{p}, \bar{E} = 0, -\bar{B}) = f(\bar{p}, \bar{E} = 0, \bar{B})$ . The distribution function in the zero electric field depends on  $p^2$  and  $\bar{p} \cdot \bar{B}$ . Therefore, in view of the relations (4)

$$F(0, \xi_2, \xi_3 \geq 0) = \Phi(\xi_2, \xi_3), \quad (13)$$

where  $\Phi(\xi_2, \xi_3)$  — defined (known) function, and  $\Phi(\xi_2, \xi_3, \bar{B}) = \Phi(\xi_2, \xi_3, -\bar{B})$ .

The partial differential equation of the first order (12) with an additional condition (13) has a unique solution in  $V$  (7) [13, 14], and

$$F(\bar{\xi}, -\bar{E}, -\bar{B}) = F(\bar{\xi}, \bar{E}, \bar{B}). \quad (14)$$

We can note that in Equation (4) variables  $\xi_1, \xi_2$  are linearly dependent on the  $\bar{E}$  and  $\bar{B}$ , but in Equation (12), they are considered as independent variables, which are expressed by the momentum components (6). Therefore, signs of the fields  $\bar{E}$  and  $\bar{B}$  and variables  $\xi_1, \xi_2, \xi_3$  in Equations (12) and (13) are independent.

The sign change of fields  $\bar{E}$  and  $\bar{B}$  can change momentum components  $\bar{p}$ , but the Jacobian  $J(\bar{\xi})$  of transformation (4) does not change. Indeed, using formula (4), we obtain:

$$J(\bar{\xi}, \bar{E}, \bar{B}) = \frac{\partial(p_1, p_2, p_3)}{\partial(\xi_1, \xi_2, \xi_3)} = \left( \frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(p_1, p_2, p_3)} \right)^{-1} = \begin{vmatrix} E_1 & E_2 & E_3 \\ B_1 & B_2 & B_3 \\ 2p_1 & 2p_2 & 2p_3 \end{vmatrix}^{-1} = \frac{1}{2\bar{p} [\bar{B} \times \bar{E}]} = \frac{1}{2\sqrt{D}}.$$

Thereby from expression (7) we have

$$D(\bar{\xi}, -\bar{E}, -\bar{B}) = D(\bar{\xi}, \bar{E}, \bar{B}). \quad (15)$$

Hence,  $J(\bar{\xi}, -\bar{E}, -\bar{B}) = J(\bar{\xi}, \bar{E}, \bar{B})$ .

Then we can obtain for the constitutive Equation (3), which is based on formulas (6) and designation  $\bar{\xi} = \{\xi_1, \xi_2, \xi_3\}$

$$\begin{aligned} j_i(\bar{E}, \bar{B}) &= \int \frac{qc p_i}{m} f^+(\bar{p}, \bar{E}, \bar{B}) d^3 p = \frac{qc E_i/m}{|\bar{B} \times \bar{E}|^2} \iiint_V \{B^2 \xi_1 - \bar{E} \cdot \bar{B} \xi_2\} F(\bar{\xi}, \bar{E}, \bar{B}) |J(\bar{\xi}, \bar{E}, \bar{B})| d^3 \xi \\ &+ \frac{qc B_i/m}{|\bar{B} \times \bar{E}|^2} \iiint_V \{E^2 \xi_2 - \bar{E} \cdot \bar{B} \xi_1\} F(\bar{\xi}, \bar{E}, \bar{B}) |J(\bar{\xi}, \bar{E}, \bar{B})| d^3 \xi \\ &+ \frac{qc \varepsilon_{ijk} E_j B_k/m}{|\bar{B} \times \bar{E}|^2} \iiint_V \sqrt{D(\bar{\xi}, \bar{E}, \bar{B})} F(\bar{\xi}, \bar{E}, \bar{B}) |J(\bar{\xi}, \bar{E}, \bar{B})| d^3 \xi = \sigma_{ij}(\bar{E}, \bar{B}) E_j, \end{aligned} \quad (16)$$

where

$$\begin{aligned} \sigma_{ij}(\bar{E}, \bar{B}) &= \delta_{ij} \{K_2(\bar{E}, \bar{B}) B^2 - K_1(\bar{E}, \bar{B}) \bar{E} \cdot \bar{B}\} \\ &+ K_1(\bar{E}, \bar{B}) B_i E_j - K_2(\bar{E}, \bar{B}) B_i B_j + \varepsilon_{ijk} K_3(\bar{E}, \bar{B}) B_k, \end{aligned} \quad (17)$$

— quasi-linear conductivity tensor,  $\varepsilon_{ijk}$  — the unit antisymmetric tensor (Levi-Civita's symbol)

$$\begin{aligned} K_1(\bar{E}, \bar{B}) &= \frac{qc/m}{|\bar{B} \times \bar{E}|^2} \iiint_V \xi_1 F(\bar{\xi}, \bar{E}, \bar{B}) |J(\bar{\xi}, \bar{E}, \bar{B})| d^3 \xi; \\ K_2(\bar{E}, \bar{B}) &= \frac{qc/m}{|\bar{B} \times \bar{E}|^2} \iiint_V \xi_2 F(\bar{\xi}, \bar{E}, \bar{B}) |J(\bar{\xi}, \bar{E}, \bar{B})| d^3 \xi; \\ K_3(\bar{E}, \bar{B}) &= \frac{qc/m}{|\bar{B} \times \bar{E}|^2} \iiint_V \sqrt{D(\bar{\xi}, \bar{E}, \bar{B})} F(\bar{\xi}, \bar{E}, \bar{B}) |J(\bar{\xi}, \bar{E}, \bar{B})| d^3 \xi. \end{aligned} \quad (18)$$

and repeated indices imply summation.

From (6), (14) and (15) it follows that

$$K_n(-\bar{E}, -\bar{B}) = K_n(\bar{E}, \bar{B}), \quad n = 1, 2, 3. \quad (19)$$

In the case of the fields inversion, similar to derivation (16) expression with the help of integral (3) and (10), we can obtain:

$$\begin{aligned} j_i(-\bar{E}, -\bar{B}) &= \int \frac{qc p_i}{m} f^-(\bar{p}, -\bar{E}, -\bar{B}) d^3 p = \sigma_{ij}(-\bar{E}, -\bar{B}) E_j, \\ \sigma_{ij}(-\bar{E}, -\bar{B}) &= \delta_{ij} \{K_2(-\bar{E}, -\bar{B}) B^2 - K_1(-\bar{E}, -\bar{B}) \bar{E} \cdot \bar{B}\} \\ &+ K_1(-\bar{E}, -\bar{B}) B_i E_j - K_2(-\bar{E}, -\bar{B}) B_i B_j - \varepsilon_{ijk} K_3(-\bar{E}, -\bar{B}) B_k. \end{aligned} \quad (20)$$

Let's introduce the vector

$$\tilde{j}_i(\bar{E}, \bar{B}) = (j_i(\bar{E}, \bar{B}) + j_i(-\bar{E}, -\bar{B})) / 2 = \{\sigma_{ij}(\bar{E}, \bar{B}) - \sigma_{ij}(-\bar{E}, -\bar{B})\} E_j / 2. \quad (21)$$

If we propose

$$\tilde{\sigma}_{ij}(\bar{E}, \bar{B}) = \{\sigma_{ij}(\bar{E}, \bar{B}) - \sigma_{ij}(-\bar{E}, -\bar{B})\} / 2, \quad (22)$$

than

$$\tilde{\sigma}_{ij}(\bar{E}, \bar{B}) = \tilde{\sigma}_{ji}(\bar{E}, -\bar{B}). \quad (23)$$

### 3. DISCUSSION

Equation (21) represents the reciprocity relationship for stationary, homogeneous and isotropic medium in a uniform magnetic field. This equation was obtained in the relaxation approximation of the Vlasov kinetic Equation (1), and it is valid for plasma-like media described by this equation. Note that the

following from the Vlasov Equation (1) and Equation (12) based on the Equations (4) and (7) in zero electric field has trivial solution  $F(\bar{\xi}) = F_0(\xi_3)$ , i.e.,  $f(\bar{p}, \bar{E} = 0, \bar{B}) = f_0(p^2)$ .

Thus, the magnetic field does not perturb the distribution function in zero electric field. It follows from the stationary Vlasov Equation (1). This condition does not work for all systems. Ultrahigh magnetic fields create primarily carrier motion in the direction of  $\bar{B}$ . It takes place when the cyclotron frequency  $\omega_C = qB/m$  is higher than the frequency of collisions  $\nu = 1/\tau$ . Therefore, the relaxation approximation (1) does not apply to a magnetic field greater than  $B_C = m/(q\tau) = 1/\beta$ . For semiconductors, the characteristic value of  $B_C$  is about 200 mT. Reciprocal relations for a nonlinear medium in a strong magnetic fields require special consideration.

Furthermore, reciprocity relations (21) are not applicable to the case of superstrong electric fields when a plasma medium has fluctuations and other forms of instability. The uniqueness of Equation (12) solution with the additional condition (13) physically corresponds to the stability of a plasma environment. Equations (1) and (12) are valid at a constant carrier density. Therefore, reciprocity relation (21) will not be executed in the modes of carriers avalanche multiplication and intervalley transitions. The unstable state of a plasma environment and change of the charge carriers concentration in a solid is possible when the energy acquired by a charge carrier mean free path is comparable to the material bandgap. In semiconductors, this corresponds to the electric field of the  $10^6$  V/m order.

Finally, we note that the reciprocity relations (21) were obtained for the solid-state plasma, homogeneous and isotropic crystal lattice which is a thermostat for the carriers group. It provides stationary and the homogeneity of the distribution function in the fields, which vary little during the Maxwell relaxation time and at a free path distance. In a weakly ionized plasma the role of the thermostat may perform not the ionized gas molecules; however, the applicability of (21) to such cases requires special consideration.

Thus, for quasi conductivity (23) classical reciprocity relations are held. They can be used to improve the accuracy of the magnetosphere diagnosis [5] and optimization of working elements of functional electronics [15]. The current distribution (21) can be obtained by combining the results in various modes of the galvanomagnetic element. It follows from (23) that a method for reducing errors of the Hall magnetometer, based on the use of linear reciprocal relations in crossed fields [5], will be valid for a highly sensitive transducer Hall with nonlinear current-voltage characteristic. This will improve the accuracy of magnetic measurements in problems of magnetic microstructural analysis and nondestructive testing [16].

#### 4. CONCLUSION

We prove the reciprocity relation for quasi-linear conductivity tensor for a plasma environment for collective carrier of solid. We consider the relaxation approximation. Temporal and spatial scales of field changes significantly exceed the Maxwell relaxation time and the free path of the charge carriers. Proven reciprocity relations help to investigate a method using a magnetic device of electronics, including the Hall effect transducer with significantly more metrological characteristics improving.

Further research will extend the relations of reciprocity from the quasi-linear conductivity tensor to the integral relations to the defined geometry of magnetoactive devices.

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