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Robust Superdirective Beamforming under Uncertainty Set Constraint

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Abstract—Based on gain optimization methods, superdirective beamformers can achieve high beam directivity with small aperture array. However, the extreme sensitivity to array uncertainty is a main obstacle to engineering application. In this work, a robust gain optimization algorithm under uncertainty set constraint is proposed. Considering steering vector mismatch is the result of combined effect of various array errors, and it is a measurable indicator especially in receiving system. We apply its uncertainty as a constraint on gain optimization method, which is more intuitive in physical sense. Different from existing solutions, it makes a better tradeoff among directive gain, robustness and radiation efficiency. Experimental analysis verifies its good performance in engineering.

1. INTRODUCTION

For arbitrary sensor arrays, many indexes, such as directive gain, sensitivity factor, beam efficiency, signal-to-noise ratio (SNR), can be used for evaluating their performance [1]. Among these, directive gain often receives the most attention. It is well known that conventional array with half-wavelength aperture can easily achieve optimal directive gain by adding equal amplitude excitations and proper phase shifts. However, this does not result in the same effect on mini-arrays which have much smaller element space than half wavelength. Seeking gain maximization with small aperture array forms a meaningful exploration, and all relative solutions can be classified as superdirective beamforming methods [2–10]. These methods can make sufficiently small array, which acquires the same directivity as conventional array. Actually, practice is not as ideal as theory assumption. Superdirective beamforming is very sensitive to various errors. It requires array precisely calibrated. Besides, high directive gain is obtained by large excitation currents of opposite signs in neighboring elements, which makes radiation efficiency rather low [5]. In other words, if we scale down the excitation currents to affordable range, the summed signal may be substantially attenuated and even drowned into internal noise, which means the loss of SNR. Consequently, blind pursuit of high gain is unrealistic.

Aiming to enhance robustness and improve efficiency, many works have been done. Newman et al. [8] and Barrik and Lilleboe [8, 9] find, in a frequency band with background noise dominant, such as high frequency band (HF), that if attenuated external noise is greater than internal receiver noise with diminished excitation currents, the SNR of summed signal is always proportional to directive gain. Under this condition, low efficiency becomes an acceptable index. Further, some constrained gain optimization methods are developed [11–17]. In summary, the common attribute that they share is constraining other indexes such as SNR, sensitivity factor on the basis of maximizing directive gain. However, real array system is complex. The accuracy of excitation weights, channel errors, mutual coupling effect, azimuth error of incident signal, etc. should not be ignored. Therefore, constraint on single index does not work effectively. Considering that steering vector mismatch is the result of

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combined effect of various errors and is a measurable indicator especially in receiving system, we apply its uncertainty as a constraint on gain optimization method, which is more practical and adaptable in real magnetic environment compared with existing solutions. Generally speaking, many reasons may cause array mismatch [18–20]. For superdirective array, there are some dominant reasons. For transmitting antenna array, exciting current in each element is large, and phase difference between channels is also significant, which means that transmitting system must adjust weights of predetermined orientation very accurately. This is too hard to guarantee because of obstruction of non-linearities in amplifiers and inconsistent channel impedance. Besides, large exciting currents in compact array will produce obvious mutual coupling effect. These two reasons are sources of steering vector mismatch in transmitting beamforming. For receiving antenna array, in addition to mutual coupling errors, channel errors existing in analog receivers also cannot be ignored. Their internal electronic components should be carefully selected to guarantee consistent electrical characteristics. As for array distribution errors, unlike large-size traditional array, superdirective array distribution can be precisely measured. Position deviation can be controlled in centimeter or less. Compared to HF wavelength of dozens of meters, the impact is negligible, which belongs to secondary reason. To sum up, the steering vector mismatch mainly comes from transceiver channel errors and mutual coupling effect. In order to overcome this challenge, Vorobyov et al., Gershman et al., Li et al., etc. have proposed a series of robust adaptive beamformers (RABs) with uncertainty set constraint [19–29]. This RAB design principle is based on modeling the actual desired signal steering vector as a sum of the presumed steering vector and a deterministic norm bounded mismatch vector. As a class of data-dependent RAB methods, array efficiency and directive gain are not urgent issues worth considering [20]. However, for superdirective beamforming, both cannot be too low, or it will lose application value. Moreover, the two indexes essentially have close links with array robustness. Establishing these links and making a suitable compromise is helpful in designing practical superdirective array.

The rest of this paper is arranged as follows. In Section 2, problems and relative solutions are reviewed. Section 3 gives out specific derivation of proposed method. In order to analyze performance of proposed method, several numeric examples and comparative tests are shown in Sections 4 and 5. The final conclusion is drawn in Section 6.

2. PROBLEMS AND EXISTING SOLUTIONS

In order to facilitate analysis, we assume that an antenna array consists of M isotropic elements which uniformly distribute at known locations. Applying a set of complex excitation weights $\mathbf{w}$, the radiation pattern of array can be steered towards a predetermined direction (θ_0, ϕ_0):

$$F(\theta,\phi) = \mathbf{w}^H(\theta_0,\phi_0)\mathbf{a}(\theta,\phi) \tag{1}$$

where $\mathbf{a}(\theta,\phi)$ is steering vector of array, and $(\cdot)^H$ denotes Hermitian transpose. The corresponding directive gain is represented as:

$$G(\theta_0,\phi_0) = \frac{4\pi|F(\theta_0,\phi_0)|^2}{\int_0^{2\pi}\int_0^{\pi}\sin\theta|F(\theta,\phi)|^2 d\theta d\phi} \tag{2}$$

which denotes radiation intensity in desired direction under given total radiated power. It can be further simplified as:

$$G(\theta_0,\phi_0) = \frac{\mathbf{w}^H\mathbf{N}\mathbf{w}}{\mathbf{w}^H\mathbf{R}\mathbf{w}} \tag{3}$$

where $\mathbf{N} = \mathbf{a}(\theta_0,\phi_0)\mathbf{a}^H(\theta_0,\phi_0)$ and $\mathbf{R} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta\mathbf{a}(\theta,\phi)\mathbf{a}^H(\theta,\phi)d\theta d\phi$. $\mathbf{R}$ is positive definite matrix, which can be easily proved.

According to matrix theorem and description in paper [4], the optimal directive gain G_{opt} equals to the maximum eigenvalue of regular pencil of matrix $\mathbf{N}-\lambda\mathbf{R}$, i.e., G_{opt} is the largest root of determinantal equation $det(\mathbf{N}-\lambda\mathbf{R}) = 0$, where $det(\cdot)$ represents determinantal operator. Omitting derivation, the optimal directive gain expression is given as:

$$\mathbf{w}_{opt} = \mathbf{R}^{-1}\mathbf{a}(\theta_0,\phi_0), \quad G_{opt}(\theta_0,\phi_0) = \mathbf{a}^H(\theta_0,\phi_0)\mathbf{R}^{-1}\mathbf{a}(\theta_0,\phi_0) \tag{4}$$

Although, in this manner, a theoretical maximum directive gain can be obtained, the high sensitivity to array errors makes it unsuitable in practical engineering. In order to improve robustness, sensitivity factor or tolerance factor must be considered [10]. It is defined as:

$$K = \frac{\mathbf{w}^H \mathbf{w}}{\mathbf{w}^H \mathbf{N} \mathbf{w}} \tag{5}$$

which is used for evaluating susceptibility of pattern to random errors. The smaller the K is, the more robust the array will be. In [6], a constrained nearly optimal gain method is given. By introducing a new indicator called Q factor, it establishes a close relationship between sensitivity and directive gain. The Q factor is described as:

$$Q = \frac{\mathbf{w}^H \mathbf{w}}{\mathbf{w}^H \mathbf{R} \mathbf{w}} = KG(\theta_0, \phi_0) \tag{6}$$

As array aperture decreases, Q changes in nearly the same proportion as K. For a given Q, the constrained optimal gain method can be written in Lagrange function:

$$L = G(\theta_0, \phi_0) + vQ \tag{7}$$

where v denotes Lagrange multiplier. The relative solution is:

$$\hat{\mathbf{w}}_{opt} = (\mathbf{R} + \delta \mathbf{I})^{-1} \mathbf{a}(\theta_0, \phi_0), \quad \hat{G}_{opt} = \mathbf{a}^H(\theta_0, \phi_0)(\mathbf{R} + \delta \mathbf{I})^{-1} \mathbf{a}(\theta_0, \phi_0) \tag{8}$$

where δ is the scalar constant and $\mathbf{I}$ an identity matrix. As a classic diagonal loading method, its defect is the difficulty in choosing a proper loading value, which is not practical in engineering.

Revolving around sensitivity index, another effective optimization method based on double constraints appears [12, 15]. Making the desired radiation direction distortionless, we get $\mathbf{w}^H \mathbf{a}(\theta_0, \phi_0) = 1$. The expressions of $G(\theta_0, \phi_0)$ and K can be simplified as:

$$G(\theta_0, \phi_0) = \frac{1}{\mathbf{w}^H \mathbf{R} \mathbf{w}}, \quad K = \mathbf{w}^H \mathbf{w} \tag{9}$$

For maximum directive gain G, we just need to minimize denominator. Thus, if sensitivity factor K is given, the constrained optimal gain method is represented as:

$$\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R} \mathbf{w}, \quad \textit{subject to } \mathbf{w}^H \mathbf{a}(\theta_0, \phi_0) = 1, \quad \|\mathbf{w}\|^2 = K. \tag{10}$$

The corresponding excitation weight is:

$$\hat{\mathbf{w}}_{opt} = \frac{(\mathbf{R} + \lambda \mathbf{I})^{-1} \mathbf{a}(\theta_0, \phi_0)}{\mathbf{a}^H(\theta_0, \phi_0)(\mathbf{R} + \lambda \mathbf{I})^{-1} \mathbf{a}(\theta_0, \phi_0)} \tag{11}$$

where λ is a scaler multiplier associated with K. The smaller K value results in better robustness but worse directive gain. Therefore, making a proper tradeoff between these two indexes is the key. Nevertheless, in some situation, people cannot always acquire its solution of Equation (11). According to [12], K subjects to the following interval:

$$\frac{1}{M} \leq K \leq \min\left(\frac{1}{M\eta_0}, \frac{\mathbf{a}^H(\theta_0, \phi_0)\mathbf{R}^{-2}\mathbf{a}(\theta_0, \phi_0)}{[\mathbf{a}^H(\theta_0, \phi_0)\mathbf{R}^{-1}\mathbf{a}(\theta_0, \phi_0)]^2}\right) \tag{12}$$

where η_0 denotes array efficiency and is defined as $\eta_0 = \frac{\mathbf{w}^H \mathbf{N} \mathbf{w}}{M \mathbf{w}^H \mathbf{w}}$ [9, 10, 12]. For a given array distribution, the right bound may be smaller than the left bound, which makes the constraints in Equation (10) meaningless. In other words, this method has a limitation on array aperture. Therefore, a more general optimization approach needs to be developed.

3. PROPOSED METHOD

In expression (10), excluding the constraint on sensitivity index K, the gain optimization method based on distortionless response can be rewritten as:

$$\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R} \mathbf{w}, \quad \textit{subject to } \mathbf{w}^H \mathbf{a}(\theta_0, \phi_0) = 1. \tag{13}$$

Lagrange method can be applied to solve it for $\mathbf{w}$. Related derivation can be found in [30]. The final computation formula is:

$$\hat{\mathbf{w}}_{opt} = \frac{\mathbf{R}^{-1}\mathbf{a}(\theta_0, \phi_0)}{\mathbf{a}^H(\theta_0, \phi_0)\mathbf{R}^{-1}\mathbf{a}(\theta_0, \phi_0)} \tag{14}$$

Inserting $\hat{\mathbf{w}}_{opt}$ into Equation (3), we can get the corresponding directive gain $\hat{G}_{opt}(\theta_0, \phi_0)$:

$$\hat{G}_{opt}(\theta_0, \phi_0) = \mathbf{a}^H(\theta_0, \phi_0)\mathbf{R}^{-1}\mathbf{a}(\theta_0, \phi_0) \tag{15}$$

For formulation (15), we can only get its estimate, because steering vector mismatch always exists in real antenna system. Assume $\mathbf{a}(\theta_0, \phi_0)$ as desired steering vector and $\hat{\mathbf{a}}$ as mismatched steering vector. The relationship between $\mathbf{a}$ and $\hat{\mathbf{a}}$ meets following uncertainty constraint:

$$[\hat{\mathbf{a}} - \mathbf{a}(\theta_0, \phi_0)]^H \mathbf{C}^{-1} [\hat{\mathbf{a}} - \mathbf{a}(\theta_0, \phi_0)] \leq 1 \tag{16}$$

Apparently, it is a Euclid ellipsoidal constraint problem. Without losing generality, we make $\mathbf{C} = \mu\mathbf{I}$, where $\mathbf{I}$ is identity matrix, and μ takes the maximum axle length of ellipsoidal. Thus, the inequation above can degenerate into sphere constraint problem:

$$\|\hat{\mathbf{a}} - \mathbf{a}(\theta_0, \phi_0)\|^2 \leq \mu \tag{17}$$

On the other hand, in formula (15), replacing $\mathbf{a}(\theta_0, \phi_0)$ with $\hat{\mathbf{a}}$ yields estimation value $\hat{G}$. Considering these two aspects, the proposed method can be represented as:

$$\max_{\hat{\mathbf{a}}} \hat{G} \quad \textit{subject to } \|\hat{\mathbf{a}} - \mathbf{a}(\theta_0, \phi_0)\|^2 \leq \mu \tag{18}$$

The expression (18) has a very clear physical meaning. μ denotes the mismatch amount by summed array error. Because $\mathbf{R}^{-1}$ is positive definite, $\hat{G} = \hat{a}^H \mathbf{R}^{-1} \hat{a}$ is a concave function. Its maximum value will appear at boundaries of the constraint interval. Therefore, we can reformulate the proposed method with quadratic equation constraint:

$$\max_{\hat{\mathbf{a}}} \hat{\mathbf{a}}^H \mathbf{R}^{-1} \hat{\mathbf{a}} \quad \textit{subject to } \|\hat{\mathbf{a}} - \mathbf{a}(\theta_0, \phi_0)\|^2 = \mu \tag{19}$$

In order to acquire its solution, we construct target function as:

$$L = \hat{\mathbf{a}}^H \mathbf{R}^{-1} \hat{\mathbf{a}} + \lambda(\|\hat{\mathbf{a}} - \mathbf{a}(\theta_0, \phi_0)\|^2 - \mu) \tag{20}$$

where λ is the Lagrange multiplier. Taking its gradient with respect to $\hat{\mathbf{a}}$ and making $\nabla_{\hat{\mathbf{a}}}(L) = 0$, we get:

$$\hat{\mathbf{a}} = (\mathbf{R}^{-1}/\lambda + \mathbf{I})^{-1} \mathbf{a}(\theta_0, \phi_0) \tag{21}$$

Replacing $\mathbf{a}(\theta_0, \phi_0)$ with $\hat{\mathbf{a}}$ in formulation (14) yields estimation value $\hat{\mathbf{w}}$:

$$\hat{\mathbf{w}} = \frac{\mathbf{R}^{-1}\hat{\mathbf{a}}}{\hat{\mathbf{a}}^H \mathbf{R}^{-1} \hat{\mathbf{a}}} = \frac{(\mathbf{R} + \frac{1}{\lambda}\mathbf{I})^{-1}\mathbf{a}(\theta_0, \phi_0)}{\mathbf{a}^H(\theta_0, \phi_0)(\mathbf{R} + \frac{1}{\lambda}\mathbf{I})^{-1}\mathbf{R}(\mathbf{R} + \frac{1}{\lambda}\mathbf{I})^{-1}\mathbf{a}(\theta_0, \phi_0)} \tag{22}$$

As seen, the solution is very similar to diagonal loading method. The key is to find a proper loading value λ. Consider that λ is linked with given mismatch μ by the following equation:

$$\|\hat{\mathbf{a}} - \mathbf{a}(\theta_0, \phi_0)\|^2 = \mu \tag{23}$$

Using matrix inversion lemma [31], $\hat{\mathbf{a}}$ in (21) can be reformulated as:

$$\hat{\mathbf{a}} = \mathbf{a}(\theta_0, \phi_0) - (\mathbf{I} + \lambda\mathbf{R})^{-1}\mathbf{a}(\theta_0, \phi_0) \tag{24}$$

Inserting (24) into (23) leads to:

$$\|(\mathbf{I} + \lambda\mathbf{R})^{-1}\mathbf{a}(\theta_0, \phi_0)\|^2 = \mu \tag{25}$$

The matrix $\mathbf{R}$ can be decomposed as $\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$, in which $\mathbf{U}$ represents the eigenvector matrix, and $\mathbf{\Lambda}$ is a diagonal matrix composed by the eigenvalues of $\mathbf{R}$. Therefore, applying $(\mathbf{I} + \lambda\mathbf{R})^{-1} = [\mathbf{U}(\mathbf{I} + \lambda\mathbf{\Lambda})^{-1}\mathbf{U}^H]^{-1}$, Equation (25) is further factorized as:

$$[\mathbf{U}^H\mathbf{a}(\theta_0, \phi_0)]^H (\mathbf{I} + \lambda\mathbf{\Lambda})^{-2} [\mathbf{U}^H\mathbf{a}(\theta_0, \phi_0)] = \mu \tag{26}$$

Making $\mathbf{z} = \mathbf{U}^H \mathbf{a}(\theta_0, \phi_0)$, we reformulate Equation (26) as:

$$g(\lambda) = \sum_{i=1}^{M} \frac{|z_i|^2}{(1+\lambda\gamma_i)^2} = \mu \tag{27}$$

where z_i is the element of vector $\mathbf{z}$, and $\gamma_i (i = 1, 2, \ldots, M)$ denotes the diagonal element of $\mathbf{\Lambda}$, satisfying $\gamma_1 \geq \gamma_2 \geq, \ldots, \geq \gamma_M$. It is intricate to yield all the solutions to λ in entire real number field, especially when the array distribution is unknown. Applying the descending relationship between γ_i and the monotonically decreasing nature of $g(\lambda)$ when λ is positive, Ref. [22] gives its incomplete solution. Here, we will give out a more comprehensive and clear discussion about the solutions to λ.

Construct a subfunction $g_i(\lambda) = 1/(1+\lambda\gamma_i)^2$. Geometrically speaking, it is an even function whose symmetric axis is $x = -1/\gamma_i$. When $\lambda = 0$, $g_i(0) = 1$. In Fig. 1, we plot the right half of subfunctions with respect to their symmetric axles.

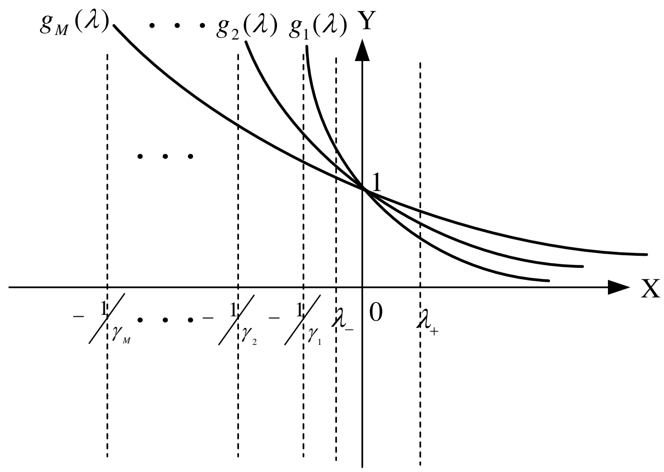


Figure 1. Subfunction curve.

As the figure shows, each subfunction monotonically decreases. Consequently, for linear summation function $g(\lambda) = \sum_{i=1}^{M} |z_i|^2 g_i(\lambda)$, it also monotonically decreases in interval $[-1/\gamma_1, +\infty]$. Hence, λ can be obtained by numeric methodology, such as bisection method or Newton's method.

Further, when $-1/\gamma_1 \leq \lambda_- \leq 0$, we yield:

$$\sum_{i=1}^{M} |z_i|^2 g_M(\lambda_-) \leq g(\lambda_-) \leq \sum_{i=1}^{M} |z_i|^2 g_1(\lambda_-) \tag{28}$$

i.e.,

$$\frac{\|\mathbf{z}\|^2}{(1+\lambda_-\gamma_M)^2} \leq \mu \leq \frac{\|\mathbf{z}\|^2}{(1+\lambda_-\gamma_1)^2} \tag{29}$$

By simplifying, the solution boundary in (29) is:

$$\frac{\sqrt{\frac{\|\mathbf{z}\|^2}{\mu}} - 1}{\gamma_M} \leq \lambda_- \leq \frac{\sqrt{\frac{\|\mathbf{z}\|^2}{\mu}} - 1}{\gamma_1} \tag{30}$$

Consider that the λ_- is negative, hence, $\mu > \|\mathbf{z}\|^2$ must be guaranteed. Combining $-1/\gamma_1 \leq \lambda_- \leq 0$, we derive a tighter solution boundary on λ_-:

$$\max\left\{\frac{-1}{\gamma_1}, \frac{\sqrt{\frac{\|\mathbf{z}\|^2}{\mu}} - 1}{\gamma_M}\right\} \leq \lambda_- \leq \frac{\sqrt{\frac{\|\mathbf{z}\|^2}{\mu}} - 1}{\gamma_1} \tag{31}$$

Likewise, for $\lambda_+ > 0$, we get:

$$\sum_{i=1}^{M} |z_i|^2 g_1(\lambda_+) \leq g(\lambda_+) \leq \sum_{i=1}^{M} |z_i|^2 g_M(\lambda_+) \tag{32}$$

Omitting derivation, the corresponding solution boundary on λ_+ is:

$$\frac{\sqrt{\frac{\|\mathbf{z}\|^2}{\mu}} - 1}{\gamma_1} \leq \lambda_+ \leq \frac{\sqrt{\frac{\|\mathbf{z}\|^2}{\mu}} - 1}{\gamma_M} \tag{33}$$

In order to ensure $\lambda_+ > 0$, let $\mu < \|\mathbf{z}\|^2$.

For $\mu = \|\mathbf{z}\|^2$, special attention is needed. According to inequality (31) or (33), we yield $\lambda = 0$. Inserting $\lambda = 0$ into formula (24) will lead to $\hat{\mathbf{a}} = 0$, which is a trivial solution. Therefore, $\mu \neq \|\mathbf{z}\|^2$ must be guaranteed. For $\mu = 0$, according to constraint (19), $\hat{\mathbf{a}} = \mathbf{a}(\theta_0, \phi_0)$. It means that the steer vector mismatch does not exist, and only idealized array model complies with this situation.

In summary, when $0 \leq \mu < \|\mathbf{z}\|^2$, the solution boundary of λ belongs to (33). When $\mu > \|\mathbf{z}\|^2$, the solution boundary belongs to (31). Here, we do not discuss the solution to $g(\lambda)$, when $\lambda < -1/\gamma_1$. Because, its monotonous nature is complicated and strongly dependent on the actual value of γ_i. Therefore, the discussion does not have universality.

As a conclusion, we implement the proposed method as following steps:

Step 1) Conduct eigenvalue decomposition of R and compute corresponding $\|\mathbf{z}\|^2$.

Step 2) Compare the practical mismatch amount μ with $\|\mathbf{z}\|^2$ and choose the corresponding solution boundary on λ.

Step 3) Solve (27) for λ via, for example, bisection method.

Step 4) Use the λ obtained in Step 3 to calculate out the optimal excitation weight $\hat{\mathbf{w}}$ according to formula (22).

4. PERFORMANCE ANALYSIS

In this section, we will discuss how the changes of mismatch value affect array's radiation performance. Assume a circular array model which consists of 5 idealized short vertical dipole elements. The array aperture is 4 meters, and working frequency is 10 MHz. Classic optimal gain method and conventional beamforming method are drawn together to form contrast. As shown in Fig. 2(a), when $\mu = 0$, radiation pattern by proposed method is exactly consistent with optimal gain method, which means that the array has no error. As μ value increases, in order to keep robustness, its beam width also correspondingly increases but still demonstrates a higher directive gain than conventional beam width does. Further, in Fig. 2(b), when μ gradually approaches $\|\mathbf{z}\|^2$, the pattern will be increasingly similar to conventional beam pattern. Under condition of $\mu > \|\mathbf{z}\|^2$, the array even acquires worse directive gain than that by conventional method, just as shown in Fig. 2(c). Therefore, in practical engineering, design of superdirective array has a strict theoretical constraint on total system mismatch amount. Beyond this constraint value, the so-called supergain will not be achieved.

To evaluate impact of proposed method on array performance more comprehensively, some index curves such as directive gain, beam efficiency, sensitivity factor and Q factor are plotted in Fig. 3. These curves are helpful to the design of a robust superdirective array by choosing a proper μ value. According to Fig. 3(a) and Fig. 3(d), we find that directive gain and radiation efficiency have contradictory relationship. For $\mu = 0$, although the array gain is up to 19.43 dB, the corresponding radiation efficiency is only 6.35%, which makes it have no practical value in engineering. Therefore, when system mismatch value $\mu0$ is given, if we choose the given $\mu0$ to calculate excitation coefficients, the array efficiency may be too low to meet power demand. In this situation, a larger μ value should be selected, which sacrifices some gains to guarantee efficiency. In Fig. 3(b) and Fig. 3(c), Q factor and sensitivity factor have almost the same trajectory change. Both of them can be used to assess the robustness of array. The smaller they are, the more robust the array will be. Further, we can find that in the vicinity of axis

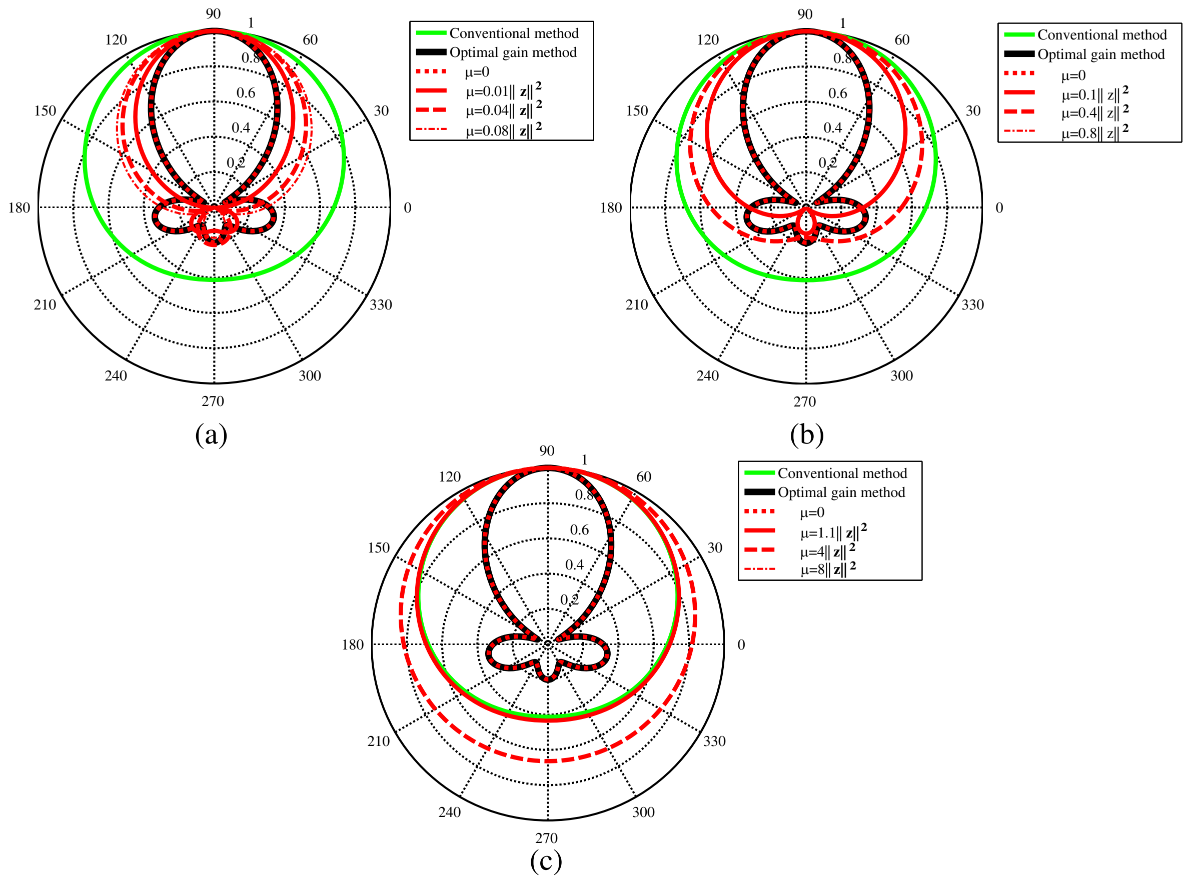


Figure 2. Radiation pattern under different μ values.

$\mu = 0.05\|\mathbf{z}\|^2$, both curves have an inflection point. This inflection point is meaningful in engineering. In the range on its left, the curves decline deeply. It means that if mismatch amount is controlled in this interval, any slight fluctuation such as excitation current, antenna deformation, etc. may make radiation pattern change drastically, which should be avoided in engineering. In the range on its right, the curves are relatively flat, which means that array has a higher tolerance for mismatch errors. Besides, in this interval, radiation efficiency is more than 50% (Fig. 3(d)), and directive gain is still higher than that of conventional method (Fig. 3(a)), which is practical in implementation.

5. ROBUSTNESS COMPARISON

As we discussed in section one, transceiver channel errors and mutual coupling effect form two types of incentives causing steering vector mismatch. According to the research of [32–34], mutual coupling between array elements can be equivalent to additional amplitude and phase errors associated with direction of arrival (DOA). This equivalence is especially suitable for uniform circular array. In other words, all summed mismatches in practical measurement are presented as amplitude and phase errors. The higher error tolerance means stronger robustness against mismatch. Therefore, it is meaningful to explore relationship between the two. Further, the maximum affordable array error can be regarded as a measure of array robustness. According to the numerical method provided by [12], we obtain the maximum affordable array error curves on amplitude and phase, just as shown in Fig. 4. It gives out an accurate tradeoff between robustness and array uncertainty as mismatch value μ changes. Combining the analysis of Section 4, a meaningful value μ should be between $[0.05, 1) \cdot \|\mathbf{z}\|^2$. Under this condition, the maximum affordable amplitude error and phase error can be regarded as $-5.86\,\text{dB}$ and 21°. In implementation, if measured array error is beyond this maximum value, the robustness will be hard to

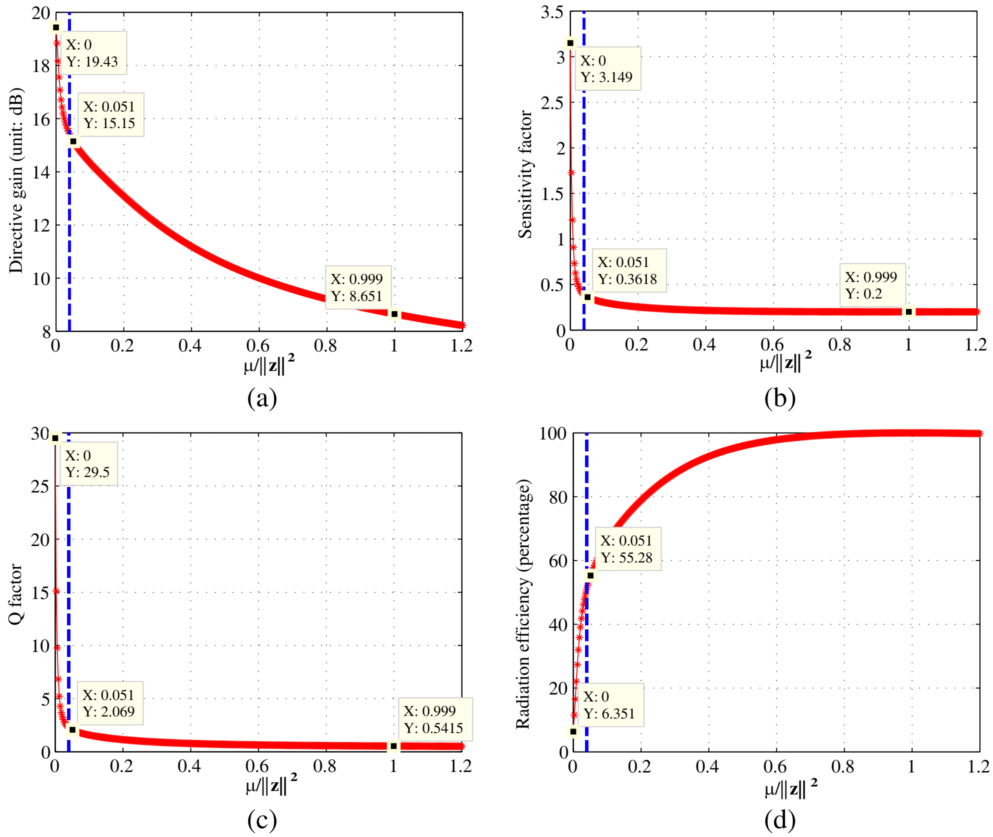


Figure 3. Performance analysis curves. (a) Directive gain corresponding to a given μ. (b) Sensitivity factor corresponding to a given μ. (c) Q factor corresponding to a given μ. (d) Radiation efficiency corresponding to a given μ.

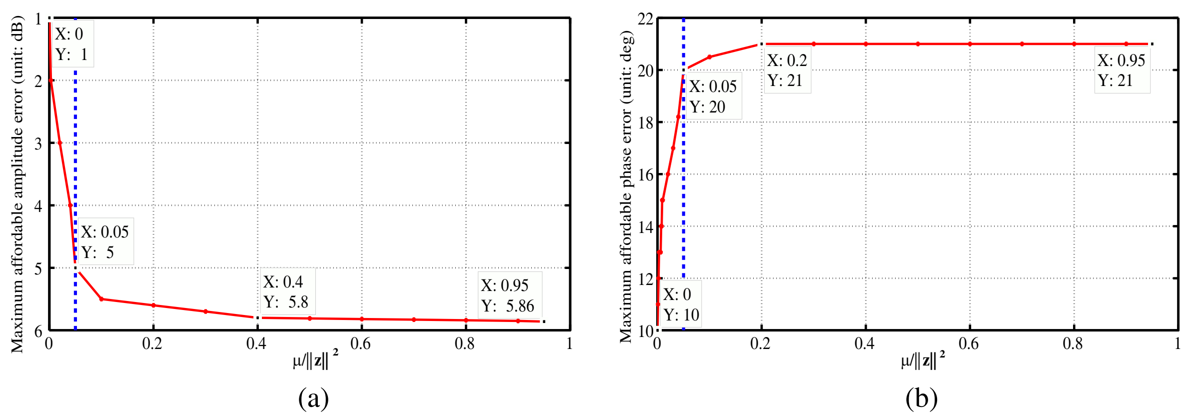
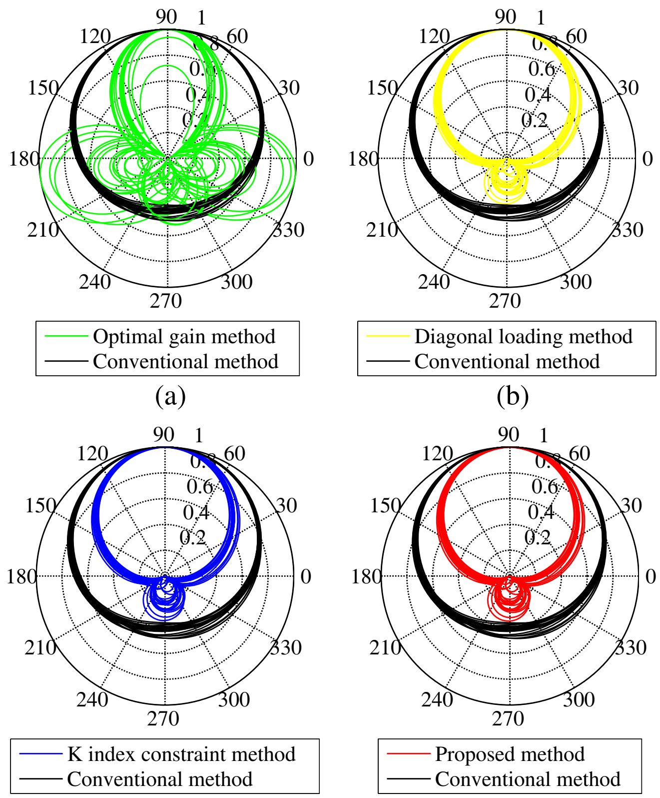


Figure 4. The maximum affordable array error corresponding to a given μ. (a) The maximum affordable amplitude error. (b) The maximum affordable phase error.

guarantee.

In order to compare robustness of the proposed method and existing methods, $-5\,\text{dB}$ random amplitude error and 5° random phase error are added into the array model. These two errors are independent, and both meet Gaussian distribution. In Fig. 5(a), the radiation pattern by classic optimal gain method is very unstable, which cannot be applied in engineering. Although the radiation patterns in Figs. 5(b), 5(c) and 5(d) are all favorable and almost identical, they are acquired in different means. In Fig. 5(b), the selection of loading value $\delta = 0.2$ relies on experience, which needs multiple tests to achieve desired effect. Figs. 5(c) and 5(d) are relatively more convenient to obtain. According to the analysis in Sections 4 and 5, $\mu = 0.05\|\mathbf{z}\|^2$ is proper. As we can see in Figs. 3(a) and 3(d), its directive gain is 6.5 dB higher than conventional method, and radiation efficiency is up to 55%. Meanwhile, by referring to the coordinate display in Fig. 3(b), we get corresponding sensitivity factor $K = 0.36$. Consequently, the radiation pattern in Fig. 5(c) can achieve the same robustness as that in Fig. 5(d).



(a) (b) (c) (d)

Figure 5. Robustness comparison. (a) Optimal gain method. (b) Diagonal loading method. (c) K index constraint method. (d) Proposed method.

6. CONCLUSION

In this work, a novel superdirective beamforming method is proposed. By adding steering vector mismatch as a constraint, array can make a better tradeoff among robustness, directive gain and radiation efficiency. Although the proposed method cannot replace array calibration, it highly reduces requirements on accuracy of calibration. Considering that the mismatch amount directly reflects the extend of array error, this method is also more intuitive in physical sense.

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