Progress In Electromagnetics Research B, Vol. 63, 203–216, 2015

Electromagnetic Fields in Quasi-Fractal Waveguides Coated with Chiral Nihility Metamaterial

Samina Gulistan, Aqeel A. Syed*, and Qaisar A. Naqvi

Abstract—Solutions of Maxwell's equations for electromagnetic fields inside a waveguide coated with chiral nihility meta-material and having one axis fractal are presented in this paper. A two-dimensional line source placed at the center of the waveguide is taken as an excitation. Power of electromagnetic fields inside the waveguide is determined, and results are plotted for various fractal dimension values ranging from $1 < D \leq 2$, and thickness of the chiral nihility coating.

1. INTRODUCTION

Fractional paradigm in electromagnetics has recently gained much attention for its ability to cast new physical problems of complex geometries and to find innovative solutions of existing ones. Fractional calculus and the concept of fractional dimensional space are now well established mathematical tools employed in this paradigm [1–13].

Fractional dimension is a very useful concept to formulate the physical description of a system with complicated geometry relatively simple by introducing a fractional parameter related to the non-integer dimension space. By employing the concept of fractional space, a real confining structure of seemingly complex geometry can be theoretically replaced with an effective space, where the measurement of its confinement is characterized by the fractional dimension parameter [8, 9]. This concept becomes more important due to the fact that even our real world dimension is found to be of fractional order, $D = 3 \pm 10^{-6}$, and not exactly of integer order as indicated by several experimental measurements [10]. Additionally, it is known that even simple physical geometries of common observance have fractional dimensions on microscopic level.

Stillinger developed a formalism for constructing a generalization of integer dimensional Laplacian operator into a non-integer dimensional space [8]. Several applications of this concept were soon proposed by various researchers in physics [11–13]. The formulation of Schrodinger wave mechanics and its various applications in an arbitrary D-dimensional space is provided in [14–23]. The concepts of fractional operators and fractional space have been successfully used by various researchers in electromagnetics to pose the problems and determine the solutions related to novel physical geometries [24–36]. Some applications of the concept of fractional space in electromagnetic research include the description of fractional multipoles in fractional space [24] and the study of electromagnetic fields in fractional space by solving Poisson's equation in D-dimensional space with $2 < D \leq 3$ [32]. Also the scattering phenomenon in fractal media is discussed in [35]. Behavior of electromagnetics waves at dielectric fractal-fractal interface has been discussed in [38, 39]. The transmission and reflection of electromagnetic waves due to a quasi-fractional slab are discussed in [40]. Electromagnetic characteristics of a stratified meta-material structure placed in fractional dimension space is discussed in [41].

Received 30 May 2015, Accepted 16 September 2015, Scheduled 23 September 2015
* Corresponding author: Aqeel Abbas Syed (aqeel@qau.edu.pk).
The authors are with the Department of Electronics, Quaid-i-Azam University, Islamabad 45320, Pakistan.

In this paper, we investigate the power propagation inside a quasi-fractal waveguide coated with chiral nihility meta-material. A quasi-fractal waveguide may be realized by a confining structure of fractal order along one or more dimensions. A chiral medium is a macroscopically continuous medium composed of chiral objects, uniformly distributed and randomly aligned. Chiral medium is characterized by either a left-handedness or a right-handedness in its microstructure. When excited by an electromagnetic plane wave, such a medium is characterized by two intrinsic eigenwaves; a left-handed and a right-handed circularly polarized waves, each having a different phase velocity and refractive index [42–60]. Two wavenumbers $k^{\pm}$ associated with two eigenwaves are

$$k^{\pm} = \omega(\sqrt{\mu\epsilon} \pm \kappa)$$

where κ is the chirality parameter, and + and − correspond to the right circularly polarized (RCP) and left circularly polarized (LCP) waves, respectively. The chiral medium is described by constitutive parameters (ϵ, μ, κ) using the following relations [50]:

$$\mathbf{D} = \epsilon\mathbf{E} + i\kappa\mathbf{H}$$
$$\mathbf{B} = \mu\mathbf{H} - i\kappa\mathbf{E}$$

Chiral nihility medium is a special kind of chiral medium for which the real parts of permittivity and permeability are simultaneously zero for certain frequencies [54–60], i.e.,

$$\mathbf{D} = i\kappa\mathbf{H}$$
$$\mathbf{B} = -i\kappa\mathbf{E}$$

Therefore, the wave number of the two eigenwaves at nihility frequency become

$$k^{\pm} = \pm\omega\kappa$$

It may be noted that LCP in the chiral nihility is a backward wave, that is, direction of the phase velocity will be anti-parallel to that of the Poynting vector.

In the following section, we briefly review general plane wave solutions in fractional dimension space. Then formulation of fractional solutions for the waveguide will be presented in Section 3. By using these results, power propagation inside the waveguide is determined and the plots depicting the effect of fractionality of the dimension and nihility of the coating on the power are presented in Section 4.

2. GENERAL PLANE WAVE SOLUTIONS IN A FRACTIONAL SPACE

For source-free and lossless media, the vector wave equations for complex electric and magnetic fields are given by the Helmholtz's equation as follows [36, 37]:

$$\nabla_D^2\mathbf{E} + \beta^2\mathbf{E} = 0 \tag{1}$$

$$\nabla_D^2\mathbf{H} + \beta^2\mathbf{H} = 0 \tag{2}$$

where $\beta^2 = \omega^2\mu\epsilon$. Time dependency is taken as $\exp(i\omega t)$ and is omitted through out the paper for brevity. Here ∇_D^2 is the scalar Laplacian operator in D dimensional fractional space and is defined as follows [32]:

$$\nabla_D^2 = \frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial}{\partial z} \tag{3}$$

Here $0 < \alpha_1 \leq 1,\ 0 < \alpha_2 \leq 1,\ 0 < \alpha_3 \leq 1$, one for each axis, are fractional space parameters. The total dimension of the space can be written in terms of these parameters as simply $D = \alpha_1 + \alpha_2 + \alpha_3$. If solution of any of Equation (1) or (2) is found, other can be determined by the duality. In the following, we rehash the solution of Equation (1). In a rectangular coordinate system, a general solution for $\mathbf{E}$ can be written as

$$\mathbf{E}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \hat{\mathbf{x}}E_x(x, y, z) + \hat{\mathbf{y}}E_y(x, y, z) + \hat{\mathbf{z}}E_z(x, y, z) \tag{4}$$

Substituting (4) into (1), we arrive at three scalar wave equations:

$$\nabla_D^2 E_x(x, y, z) + \beta^2 E_x(x, y, z) = 0 \tag{5a}$$
$$\nabla_D^2 E_y(x, y, z) + \beta^2 E_y(x, y, z) = 0 \tag{5b}$$
$$\nabla_D^2 E_z(x, y, z) + \beta^2 E_z(x, y, z) = 0 \tag{5c}$$

By inserting the relation for ∇_D^2 in (5a), we get

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{\alpha_1-1}{x}\frac{\partial E_x}{\partial x}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\alpha_2-1}{y}\frac{\partial E_x}{\partial y}+\frac{\partial^2 E_x}{\partial z^2}+\frac{\alpha_3-1}{z}\frac{\partial E_x}{\partial z}+\beta^2 E_x=0 \tag{6}$$

Now using separation of variables, three ordinary differential equations are obtained. From which the x dependent is

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\alpha_1-1}{x}\frac{\partial}{\partial x}+\beta_x^2\right]f=0 \tag{7}$$

with

$$\beta_x^2+\beta_y^2+\beta_z^2=\beta^2. \tag{8}$$

Rewriting (7) as

$$\left[x\frac{\partial^2}{\partial x^2}+a\frac{\partial}{\partial x}+\beta_x^2 x\right]f=0 \tag{9}$$

with $a=\alpha_1-1$, if we insert $f=\sqrt{\frac{\pi}{2}}(\beta_x x)^{n_1}\zeta$, the equation is reduced to Bessel's equation as follows

$$\left[x^2\frac{\partial^2}{\partial x^2}+x\frac{\partial}{\partial x}+(\beta_x^2x^2-{n_1}^2)\right]\zeta=0,\quad n_1=\frac{|1-a|}{2} \tag{10}$$

The solution of Bessel's equation is given as

$$\zeta=C_1\ J_{n_1}(\beta_x x)+C_2\ Y_{n_1}(\beta_x x) \tag{11}$$

where $J_{n_1}(\beta_x x)$ is Bessel function of the first kind of order n_1, and $Y_{n_1}(\beta_x x)$ is a Bessel function of second kind of order n_1. Hence, the solution for Equation (7) becomes

$$f(x)=\sqrt{\frac{\pi}{2}}(\beta_x x)^{n_1}\left[C_1\ J_{n_1}(\beta_x x)+C_2\ Y_{n_1}(\beta_x x)\right],\quad n_1=1-\frac{\alpha_1}{2}. \tag{12}$$

Using above, the fields inside a fractional waveguide are written as

$$\begin{aligned}E_x(x,y,z)\ &=\ \left(\sqrt{\frac{\pi}{2}}\right)^3(\beta_x x)^{n_1}(\beta_y y)^{n_2}(\beta_z z)^{n_3}\left[C_1\ J_{n_1}(\beta_x x)+C_2\ Y_{n_1}(\beta_x x)\right]\\&\quad\times\left[C_3\ J_{n_2}(\beta_y y)+C_4\ Y_{n_2}(\beta_y y)\right]\times\left[C_5\ J_{n_3}(\beta_z z)+C_6\ Y_{n_3}(\beta_z z)\right]\end{aligned} \tag{13}$$

$$\begin{aligned}E_y(x,y,z)\ &=\ \left(\sqrt{\frac{\pi}{2}}\right)^3(\beta_x x)^{n_1}(\beta_y y)^{n_2}(\beta_z z)^{n_3}\left[D_1\ J_{n_1}(\beta_x x)+D_2\ Y_{n_1}(\beta_x x)\right]\\&\quad\times\left[D_3\ J_{n_2}(\beta_y y)+D_4\ Y_{n_2}(\beta_y y)\right]\times\left[D_5\ J_{n_3}(\beta_z z)+D_6\ Y_{n_3}(\beta_z z)\right]\end{aligned} \tag{14}$$

$$\begin{aligned}E_z(x,y,z)\ &=\ \left(\sqrt{\frac{\pi}{2}}\right)^3(\beta_x x)^{n_1}(\beta_y y)^{n_2}(\beta_z z)^{n_3}\left[G_1\ J_{n_1}(\beta_x x)+G_2\ Y_{n_1}(\beta_x x)\right]\\&\quad\times\left[G_3\ J_{n_2}(\beta_y y)+G_4\ Y_{n_2}(\beta_y y)\right]\times\left[G_5\ J_{n_3}(\beta_z z)+G_6\ Y_{n_3}(\beta_z z)\right]\end{aligned} \tag{15}$$

where C's, D's and G's are constants to be determined using the boundary conditions. These solution can be used to study the phenomenon of electromagnetic wave propagation in any non-integer dimensional space. Equation (12) is the generalization of the concept of wave propagation in integer dimensional space to the wave propagation in non-integer dimensional space. As a special case, for three-dimensional space, this problem reduces to classical wave propagation. That is, if we take $\alpha_1=1$ in Equation (12) then $n_1=\frac{1}{2}$ and it gives

$$f(x)=\sqrt{\frac{\pi}{2}}(\beta_x x)^{\frac{1}{2}}\left[C_1\ J_{\frac{1}{2}}(\beta_x x)+C_2\ Y_{\frac{1}{2}}(\beta_x x)\right] \tag{16}$$

where

$$J_{\frac{1}{2}}(\beta_x x)\ =\ \sqrt{\frac{2}{\pi\beta_x x}}\cos\left(\beta_x x-\frac{\pi}{2}\right) \tag{17}$$

$$Y_{\frac{1}{2}}(\beta_x x)\ =\ \sqrt{\frac{2}{\pi\beta_x x}}\sin\left(\beta_x x-\frac{\pi}{2}\right) \tag{18}$$

Equation (16) can be written as

$$f(x) = \left[\acute{C}_1 \sin(\beta_x x) + \acute{C}_2 \cos(\beta_x x)\right] \tag{19}$$

where $\acute{C}_1 = C_1$ and $\acute{C}_2 = -C_2$.

Note that for traveling wave propagation, Bessel functions are replaced with the Hankel functions of first and second kind in the above expressions, respectively.

3. FIELDS IN FRACTIONAL WAVEGUIDES

The geometry of the guiding problem under consideration is shown in Figure 1. Two perfect electric conductor (PEC) planes of infinite extent forming a parallel plate waveguide are located at $z = d_2$ and $z = -d_2$. The whole space inside the parallel plate waveguide is divided into three regions. The regions are parallel to the walls of the guide. Two regions labeled as Region 1 ($-d_2 < z < -d_1$) and Region 2 ($d_1 < z < d_2$) consist of chiral nihility material while Region 0 ($-d_1 < z < d_1$) is free space with permittivity ϵ_0 and permeability μ_0. In all three regions media is fractal in dimensions and fractionality is assumed along z-axis only, hence termed quasi-fractal. A two dimensional, time harmonic, $\exp(i\omega t)$, electric current line source is placed at the origin of the Cartesian coordinate system. Total electric and magnetic fields in the core of waveguide, region 0, are taken as combination of LCP and RCP propagating towards $\pm z$. We consider the general case of incident wave decomposed in terms of LCP and RCP instead of plane wave. As the z axis is taken fractional, therefore, Hankel functions instead of exponentials will be used for field components in this direction [43–45]. The electric and magnetic fields for Region 0, in terms of unknown coefficients, can be written as

$$\begin{aligned}\mathbf{E}_0 \; = \; & \int_{-\infty}^{\infty} dk_y \zeta(k_{0z}z)^n \exp(-ik_y y)\sqrt{\frac{\pi}{2}}[\hat{\mathbf{x}}\; H_n^{(2)}(k_{0z}z) \\ & + A^+(\mathbf{N}_R^+)\; H_n^{(2)}(k_{0z}z) + \; B^+(\mathbf{N}_L^+)\; H_n^{(2)}(k_{0z}z) \\ & + A^-(\mathbf{N}_R^-)\; H_n^{(1)}(k_{0z}z) + \; B^-(\mathbf{N}_L^-)\; H_n^{(1)}(k_{0z}z)], \quad -d_1 < y < d_1\end{aligned} \tag{20a}$$

$$\begin{aligned}\mathbf{H}_0 \; = \; & \int_{-\infty}^{\infty} dk_y \zeta(k_{0z}z)^n \exp(-ik_y y)\sqrt{\frac{\pi}{2}}\left[\left(\frac{1}{k_0\eta_0}\right)[k_{0z}\hat{\mathbf{y}}\; H_n^{(2)}(k_{0z}z) - k_y\hat{\mathbf{z}}\; H_n^{(2)}(k_{0z}z)]\right. \\ & -\frac{\imath}{\eta_0}[\; A^+(\mathbf{N}_R^+)\; H_n^{(2)}(k_{0z}z) - \; B^+(\mathbf{N}_L^+)\; H_n^{(2)}(k_{0z}z) \\ & \left. + A^-(\mathbf{N}_R^-)\; H_n^{(1)}(k_{0z}z) - \; B^-(\mathbf{N}_L^-)\; H_n^{(1)}(k_{0z}z)]\right], \quad -d_1 < y < d_1\end{aligned} \tag{20b}$$

Total electric and magnetic fields within each chiral layer, by virtue of multiple reflections at the slab boundaries, are written as combination of four type of contributions: the spectrum of LCP and RCP waves propagating towards $\pm z$ directions.

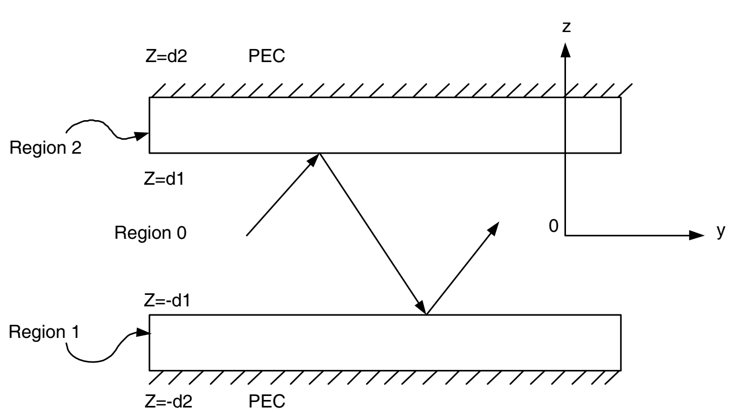


Figure 1. Three layered fractal waveguide.

The electric and magnetic fields in chiral regions of waveguide may be written in terms of unknown coefficients as [43–45]

$$\begin{aligned}\mathbf{E}_1 &= \int_{-\infty}^{\infty} dk_y \zeta \exp(-ik_y y)\sqrt{\frac{\pi}{2}}[\, C^+\mathbf{M}_R^+(k_z^+ z)^n\, H_n^{(2)}(k_z^+ z) \\ &+ D^+\mathbf{M}_L^+(k_z^- z)^n\, H_n^{(2)}(k_z^- z) + \, C^-\mathbf{M}_R^-(k_z^+ z)^n\, H_n^{(1)}(k_z^+ z)] \\ &+ D^-\mathbf{M}_L^-(k_z^- z)^n\, H_n^{(1)}(k_z^- z)], \quad -d_2 < y < -d_1 \end{aligned} \tag{20c}$$

$$\begin{aligned}\mathbf{H}_1 &= \int_{-\infty}^{\infty} dk_y \zeta \exp(-ik_y y)\sqrt{\frac{\pi}{2}}\frac{-\imath}{\eta}[\, C^+\mathbf{M}_R^+(k_z^+ z)^n\, H_n^{(2)}(k_z^+ z) \\ &- D^+\mathbf{M}_L^+(k_z^- z)^n\, H_n^{(2)}(k_z^- z) + \, C^-\mathbf{M}_R^-(k_z^+ z)^n\, H_n^{(1)}(k_z^+ z)] \\ &- D^-\mathbf{M}_L^-(k_z^- z)^n\, H_n^{(1)}(k_z^- z)], \quad -d_2 < y < -d_1 \end{aligned} \tag{20d}$$

$$\begin{aligned}\mathbf{E}_2 &= \int_{-\infty}^{\infty} dk_y \zeta \exp(-ik_y y)\sqrt{\frac{\pi}{2}}[\, E^+\mathbf{M}_R^+(k_z^+ z)^n\, H_n^{(2)}(k_z^+ z) \\ &+ F^+\mathbf{M}_L^+(k_z^- z)^n\, H_n^{(2)}(k_z^- z) + \, E^-\mathbf{M}_R^-(k_z^+ z)^n\, H_n^{(1)}(k_z^+ z)] \\ &+ F^-\mathbf{M}_L^-(k_z^- z)^n\, H_n^{(1)}(k_z^- z)], \quad d_1 < y < d_2 \end{aligned} \tag{20e}$$

$$\begin{aligned}\mathbf{H}_2 &= \int_{-\infty}^{\infty} dk_y \zeta \exp(-ik_y y)\sqrt{\frac{\pi}{2}}\frac{-\imath}{\eta}[\, E^+\mathbf{M}_R^+(k_z^+ z)^n\, H_n^{(2)}(k_z^+ z) \\ &- F^+\mathbf{M}_L^+(k_z^- z)^n\, H_n^{(2)}(k_z^- z) + \, E^-\mathbf{M}_R^-(k_z^+ z)^n\, H_n^{(1)}(k_z^+ z)] \\ &- F^-\mathbf{M}_L^-(k_z^- z)^n\, H_n^{(1)}(k_z^- z)], \quad d_1 < y < d_2 \end{aligned} \tag{20f}$$

where

$$\begin{aligned}\mathbf{N}_R^{\pm} &= \hat{\mathbf{x}} \pm \, i\frac{k_{0z}}{k_0}\hat{\mathbf{y}} - \, i\frac{k_y}{k_0}\hat{\mathbf{z}} \\ \mathbf{N}_L^{\pm} &= \hat{\mathbf{x}} \mp \, i\frac{k_{0z}}{k_0}\hat{\mathbf{y}} + \, i\frac{k_y}{k_0}\hat{\mathbf{z}} \\ \mathbf{M}_R^{\pm} &= \hat{\mathbf{x}} \pm \, i\frac{k_z^+}{k^+}\hat{\mathbf{y}} - \, i\frac{k_y}{k^+}\hat{\mathbf{z}} \\ \mathbf{M}_L^{\pm} &= \hat{\mathbf{x}} \mp \, i\frac{k_z^+}{k^+}\hat{\mathbf{y}} - \, i\frac{k_y}{k^+}\hat{\mathbf{z}}\end{aligned}$$

The exponential function is used to describe wave propagation in y direction, and Hankel function of order n is used to represent wave propagation in z direction. Hankel function of 2nd kind represent waves traveling in 'z' direction and Hankel's functions of 1st kind represents waves traveling in '$-z$' direction. The subscript R and L refer to the RCP and LCP eigen waves satisfying the dispersion relations

$$k_y^2 + (k_z^{\pm})^2 = (k^{\pm})^2 \tag{21}$$

where $k_0 = \omega\sqrt{\epsilon_0\mu_0}$, $\eta_0 = \sqrt{\mu_0/\epsilon_0}$, $\eta = \sqrt{\mu/\epsilon}$ and $\zeta = \omega\mu_0 I/4\pi k_{0z}$. k_{0z} and k_y satisfy the following dispersion relation

$$k_y^2 + (k_{0z})^2 = (k_0)^2 \tag{22}$$

The unknown coefficients in the above expressions can be determined using the appropriate boundary conditions. Application of the boundary conditions leads to following relationships for unknown coefficients

$$A^+ = A^- = B^+ = B^- = -\frac{H_n^{(2)}(k_{0z}d_1)}{4\, J_n(k_{0z}d_1)}$$

$$C^- = E^+ = \frac{1}{4T}\frac{k_z^+}{k_{0z}}\frac{H_n^{(1)}(k_z^+ d_1)}{J_n(k_{0z}d_1)}, \quad D^- = F^+ = \frac{1}{4T}\frac{k_z^+}{k_{0z}}\frac{H_n^{(2)}(k_z^+ d_1)}{J_n(k_{0z}d_1)}$$

$$D^{\pm} = -C^{\mp}, \quad F^{\pm} = -E^{\mp}, \quad T = \frac{k_0 \eta_0 k_z^+}{k_{0z} \eta k^+} \tag{23}$$

There are poles in the integration path when $J_n(k_{0z} d_1) = 0$. These poles are written as

$$k_{0z} = (m + 1/2)\frac{\pi}{d_1} + \frac{\pi}{4d_1} + \frac{n\pi}{2d_1}, \quad m = 0, 1, 2, \ldots \tag{24}$$

Using residue method of integration, the expressions in Equation (20) may be evaluated. Substitution of unknowns coefficients in these expressions, yields $\mathbf{E_1} = \mathbf{E_2} = \mathbf{0}$.

Note that for $\alpha_1 = 1$ or $n = \frac{1}{2}$ the constants takes the form

$$A^+ = A^- = B^+ = B^- = -\frac{\exp\left[-i\left(k_{0z} d_1 - \frac{\pi}{2}\right)\right]}{4\cos\left(k_{0z} d_1 - \frac{\pi}{2}\right)}$$

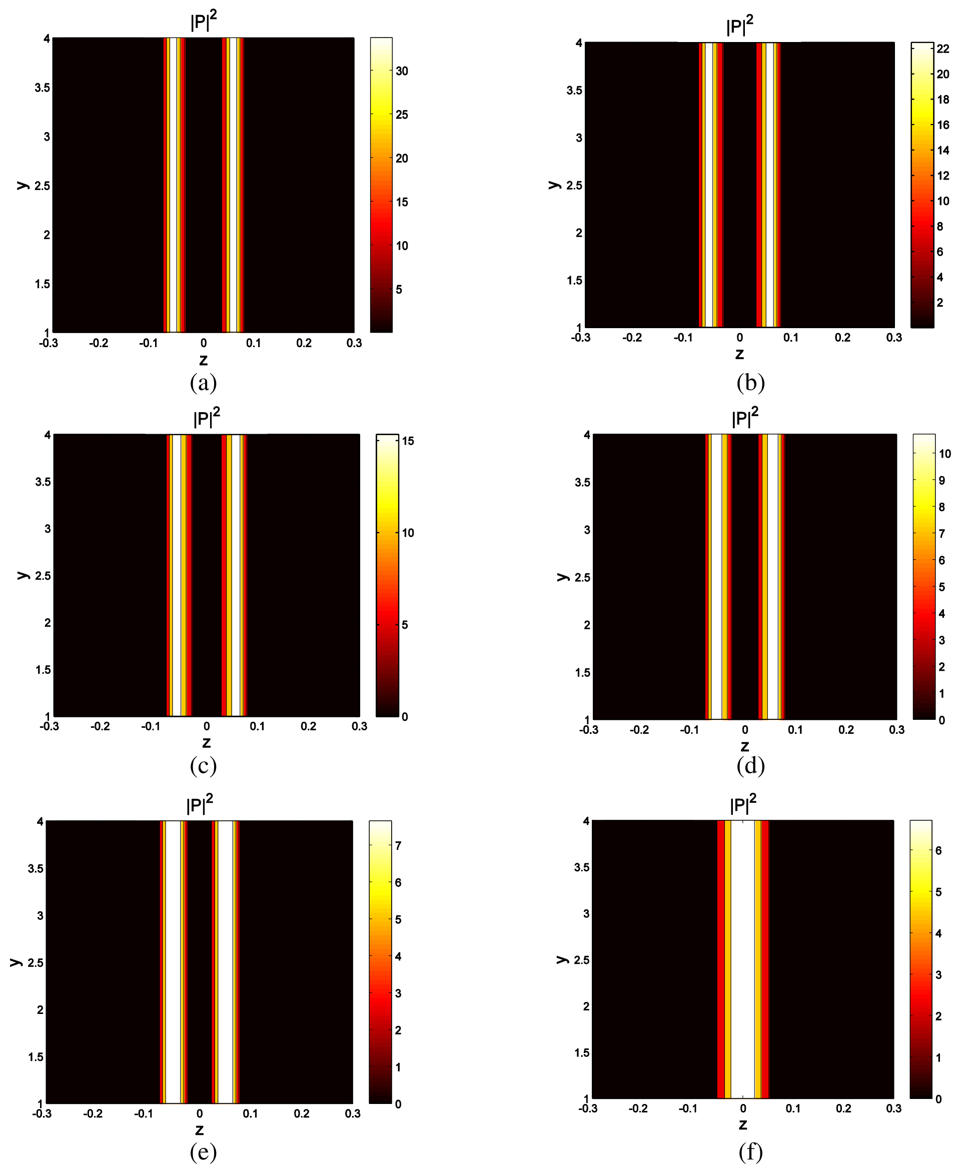


Figure 2. Power along the guide for $d_1 = 0.1$ and at (a) $D = 1.1$, (b) $D = 1.3$, (c) $D = 1.5$, (d) $D = 1.7$, (e) $D = 1.9$, (f) $D = 2$ and $m = 0$.

$$C^{-} = E^{+} = \frac{1}{4T} \frac{\exp\left[\, i\left(k_{0z}d_1 - \dfrac{\pi}{2}\right)\right]}{\cos\left(k_{0z}d_1 - \dfrac{\pi}{2}\right)} \tag{25}$$

with $k_{0z} = (m + 1/2)\frac{\pi}{d_1} + \frac{\pi}{2d_1}$. These are same results presented in [46] for a non-fractional waveguide. Also for $\alpha_1 = 1$ in Equation (7), we get back ordinary differential equation. Therefore, the results of [45] can be thought of as a special case of our present study.

4. RESULTS AND DISCUSSION

In this section, numerical results for field power inside the waveguide are presented for various values of fractional dimension D and thickness d of the nihility coating. In all figures frequency of 1 GHz is taken while chirality parameter is $\kappa = 5\mu_o\epsilon_o$. Two modes, corresponding to $m = 0$ and $m = 1$, are considered for this purpose. The results are depicted through Figures 2–7. Each figure depicts the

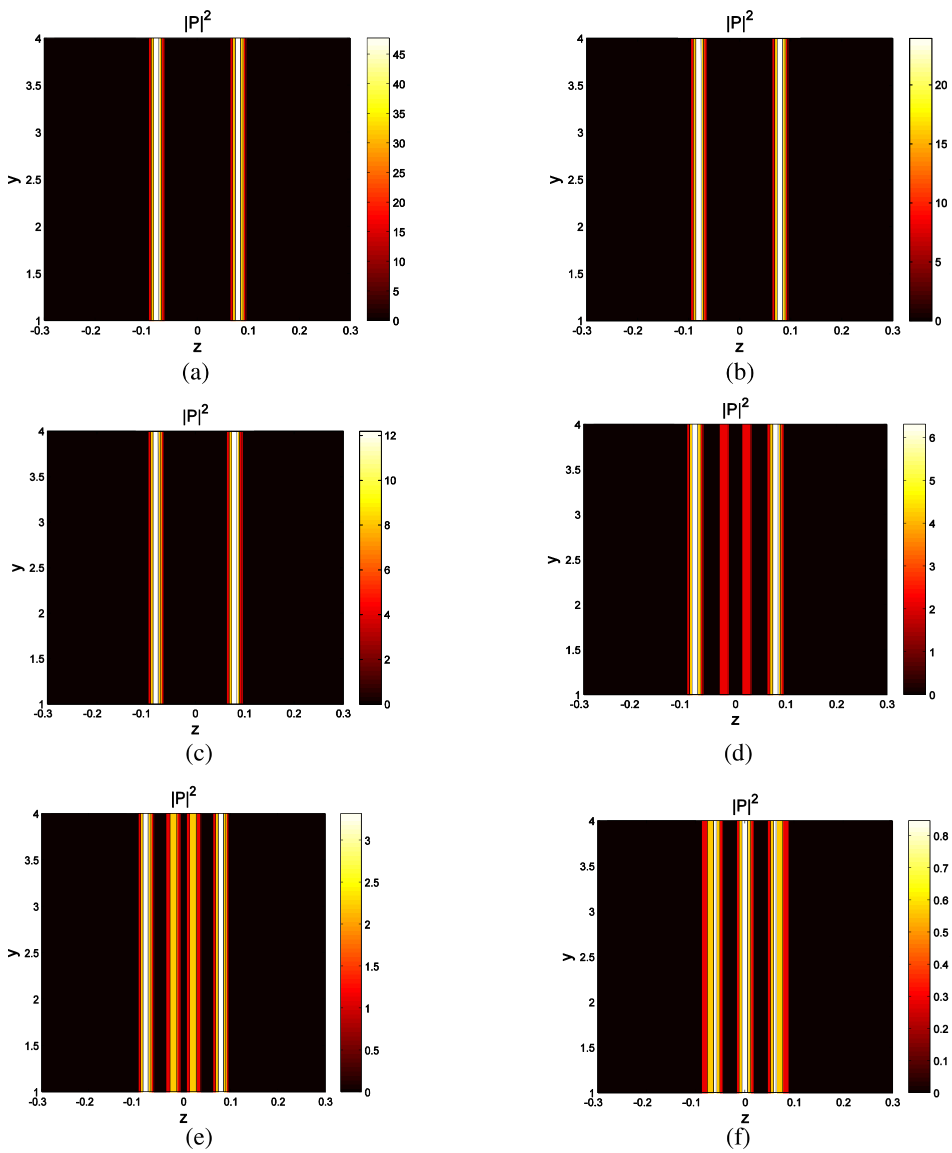


Figure 3. Power along the guide for $d_1 = 0.1$ and at (a) $D = 1.1$, (b) $D = 1.3$, (c) $D = 1.5$, (d) $D = 1.7$, (e) $D = 1.9$, (f) $D = 2$ and $m = 1$.

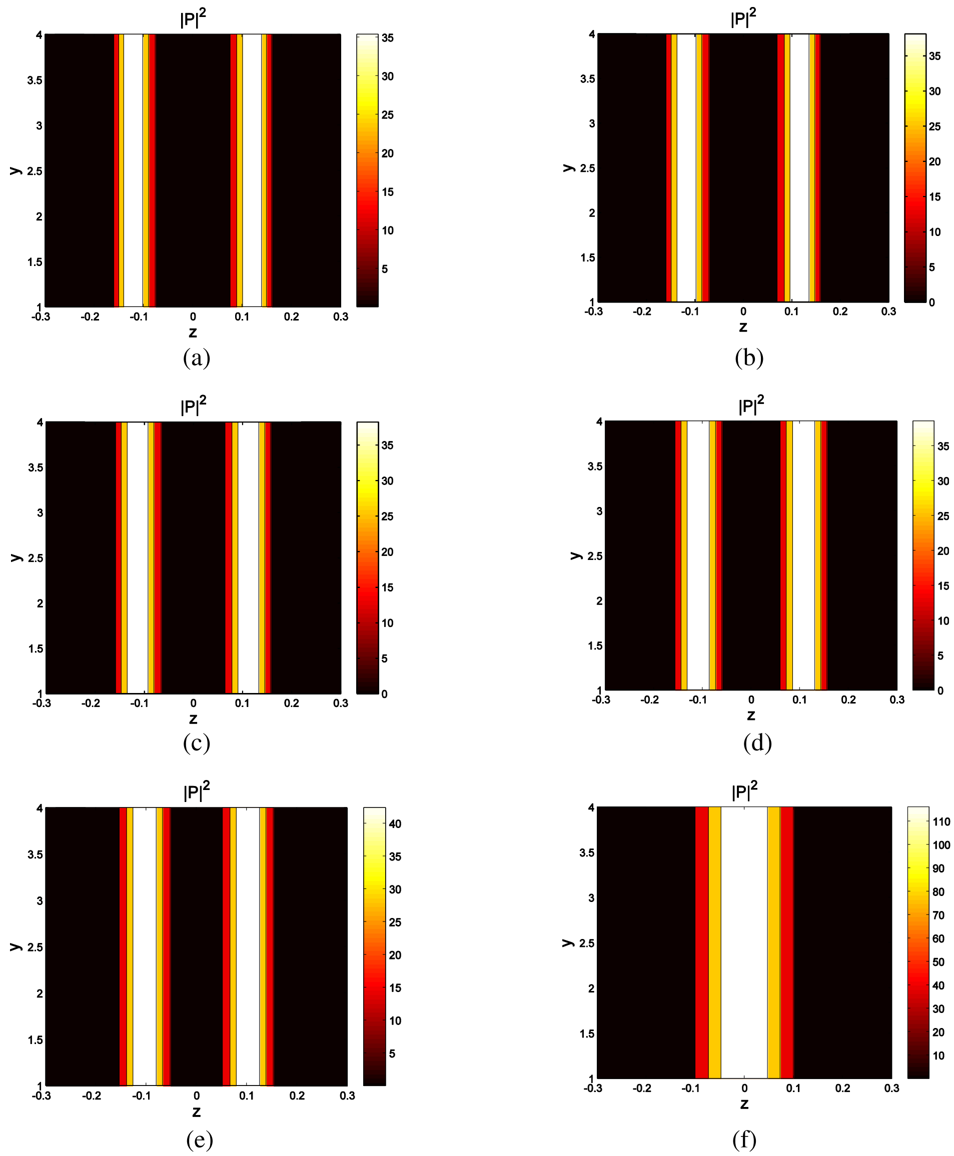
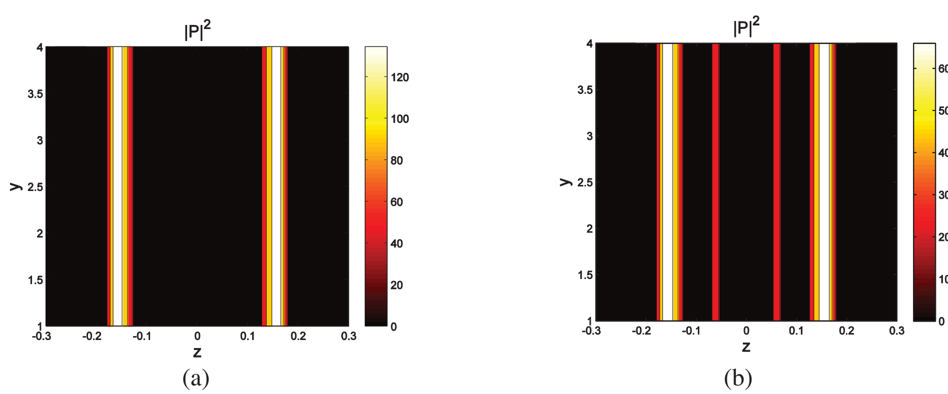


Figure 4. Power along the guide for $d_1 = 0.2$ and at (a) $D = 1.1$, (b) $D = 1.3$, (c) $D = 1.5$, (d) $D = 1.7$, (e) $D = 1.9$, (f) $D = 2$ and $m = 0$.



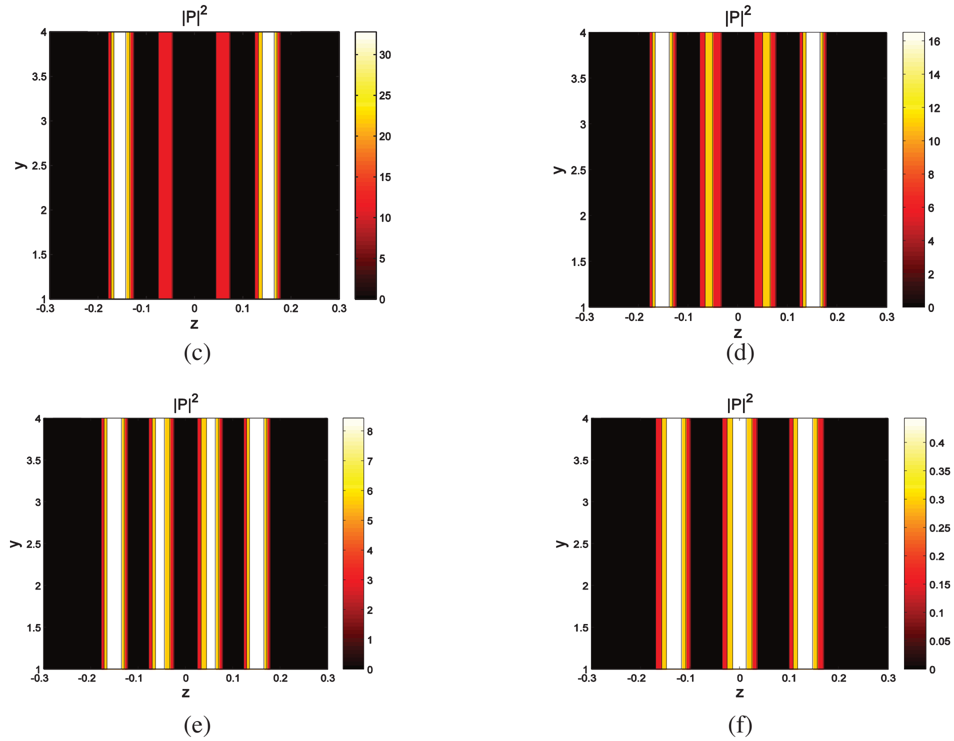
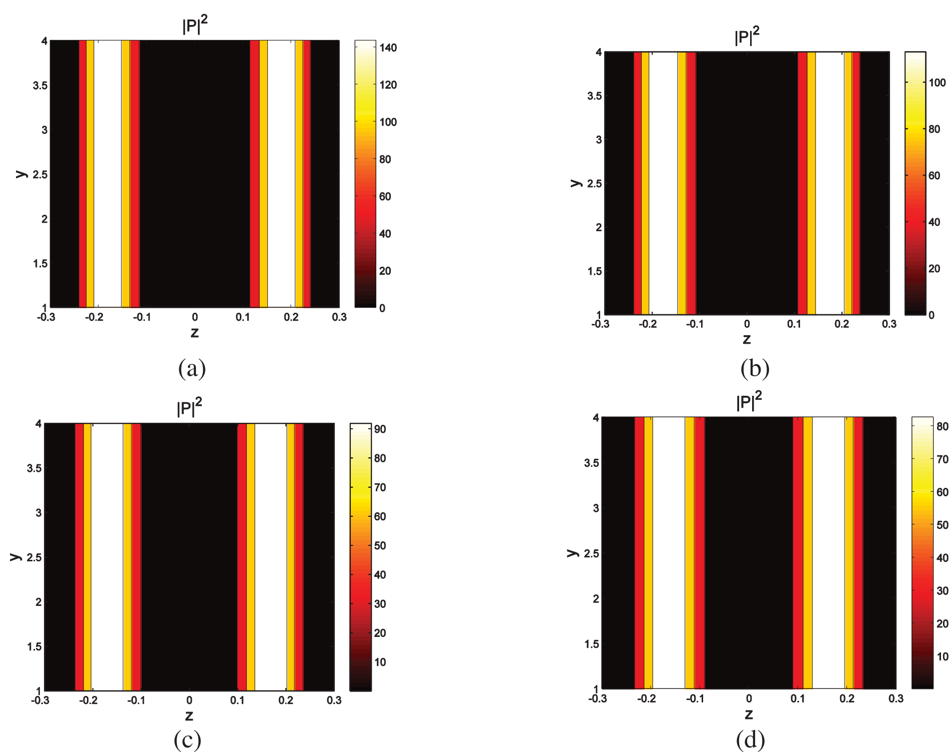


Figure 5. Power along the guide for $d_1 = 0.2$ and at (a) $D = 1.1$, (b) $D = 1.3$, (c) $D = 1.5$, (d) $D = 1.7$, (e) $D = 1.9$, (f) $D = 2$ and $m = 1$.



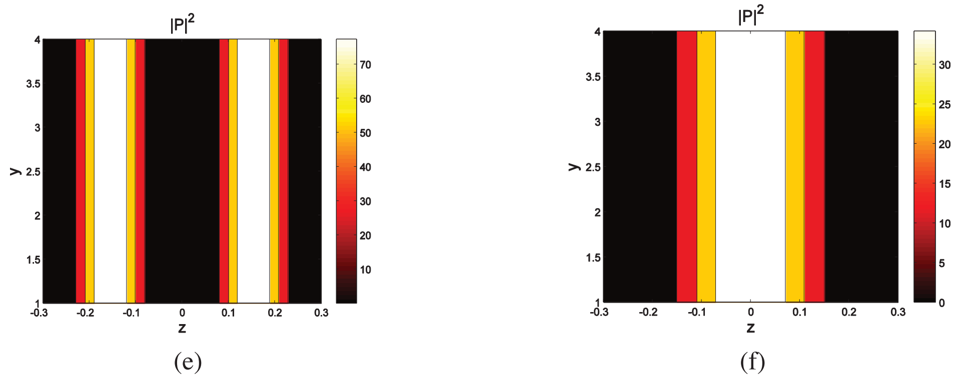


Figure 6. Power along the guide for $d_1 = 0.3$ and at (a) $D = 1.1$, (b) $D = 1.3$, (c) $D = 1.5$, (d) $D = 1.7$, (e) $D = 1.9$, (f) $D = 2$ and $m = 0$.

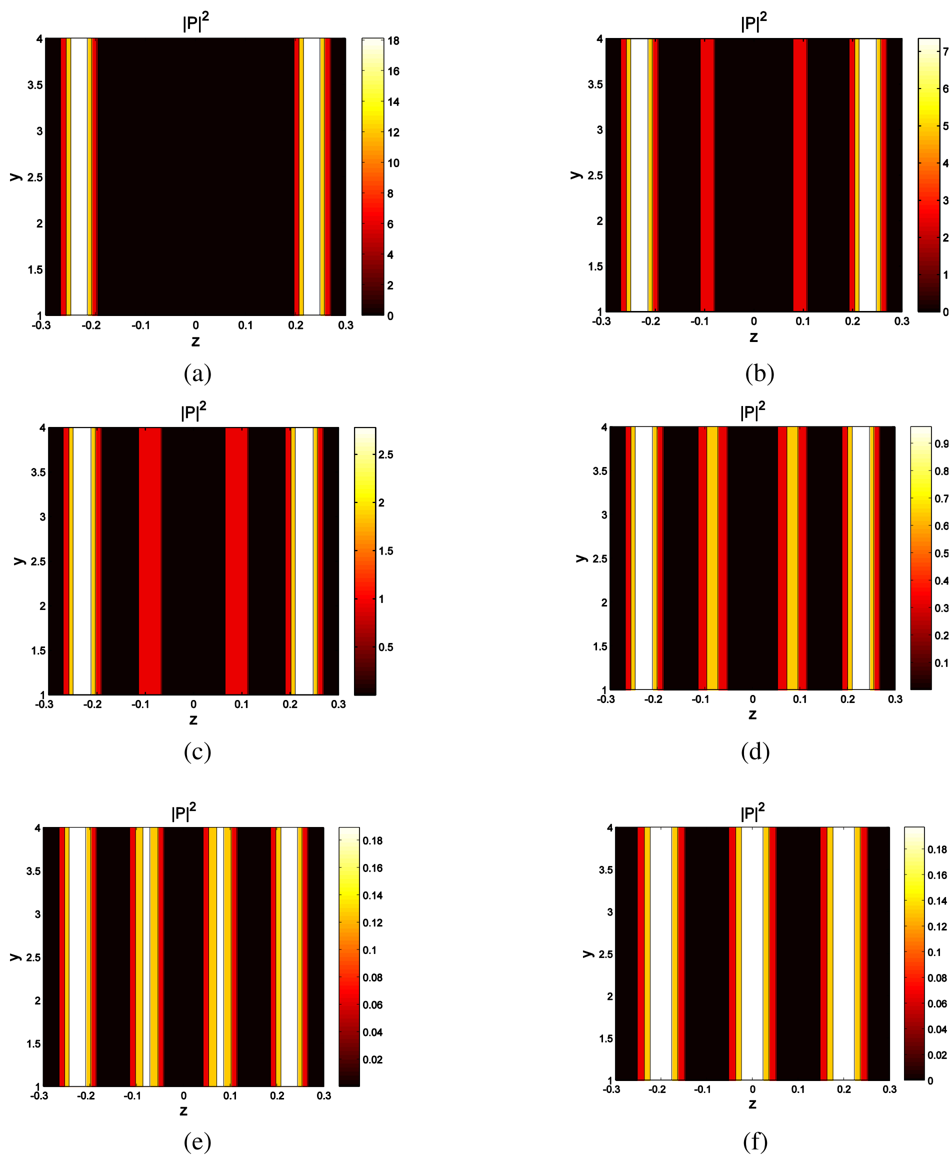


Figure 7. Power along the guide for $d_1 = 0.3$ and at (a) $D = 1.1$, (b) $D = 1.3$, (c) $D = 1.5$, (d) $D = 1.7$, (e) $D = 1.9$, (f) $D = 2$ and $m = 1$.

power distribution inside the waveguide for a fixed value of thickness of coating and various values of fractional dimensions, namely, $D = 1.1, 1.3, 1.5, 1.7, 1.9, 2$. Moreover, even numbered figures correspond to mode $m = 0$, where as odd numbered figures to mode $m = 1$. Although, the fields inside the guide show no variation along the y-axis, the results are depicted in 2-D plots for comparison with the results presented in [45].

First we consider the results for $m = 0$, i.e., Figures 2, 4, and 6. These figures correspond to thickness $d = 0.1m, 0.2m, 0.3m$, respectively. As shown in earlier results [45], chiral nihility in each case confines the power to non-nihility region. Therefore, it can be argued that the power distribution inside the waveguide bears no effect due to fractionality of the dimension. Inside the core of the waveguide, for $D = 2$, there exists only one zone, in which power is distributed around central axis of the guide. Fractionality of the dimension, on the other hand, causes the power distributed inside the core to split into two zones each moving away from the central axis with decreasing fractional dimension from $D = 2$ to $D = 1.1$. It is also interesting to note that the fractional dimensional waveguide supports additional modes despite the excitation field having only the zeroth order mode. Therefore, fractionality of the waveguide dimension causes redistribution of power inside the waveguide with power concentrating to the original mode of the waveguide with integer dimension $D = 2$. Similar trend in power distribution is observed for the $m = 1$ excitation mode as depicted in Figures 3, 5, and 7. As before, except confining the power in the non-nihility region of the waveguide, the nihility coating shows no effect due to the fractionality of the dimension. Additionally, in this case, the central zone carrying non zero power with integer dimension splits into additional zones each of them moving away from one another with fractional dimension decreasing from $D = 2$ to $D = 1.1$.

5. CONCLUSION

The development of fractional paradigm in electromagnetics, on the one hand, and advent of meta-materials on the other, motivates one to investigate the geometries composed of meta-materials possessing the features of fractionality. In this article, we present a study of electromagnetic power distribution inside a quasi-fractal waveguide coated with chiral nihility meta-material. It has been noted that although chiral nihility coating plays the role of confining power to the non-nihility region inside the waveguide, it shows no interaction with the fractional dimension of the waveguide. On the other hand, it is found that fractionality of the waveguide dimension causes the presence of additional waveguide modes, which move away from the center while increasing in power with a decrease in fractal dimension. Or conversely stating, the fractal waveguide modes come close and collapse, i.e., power concentrates, as the fractal dimension increases from $D = 1.1$ to $D = 2$. Therefore, chiral nihility coating and fractionality of the waveguide dimension can be used to control the power distribution inside a fractal waveguide. Moreover, fractionality of the dimension can be used to support additional modes inside a fractal waveguide.

REFERENCES

1. Oldham, K. B. and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.
2. Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific, 2000.
3. Debnath, L., "Recent applications of fractional calculus to science and engineering," *International Journal of Mathematics and Mathematical Sciences*, Vol. 54, 3413–3442, 2003.
4. Engheta, N., "A note on fractional calculus and the image method for dielectric spheres," *Journal of Electromagnetic Waves and Applications*, Vol. 9, No. 9, 1179–1188, 1995.
5. Engheta, N., "Use of fractional integration to propose some "fractional" solutions for the scalar helmholtz equation," *Progress In Electromagnetics Research*, Vol. 12, 107–132, 1996.
6. Engheta, N., "Electrostatic fractional image methods for perfectly conducting wedges and cones," *IEEE Transactions on Antennas and Propagation*, Vol. 44, 1565–1574, 1996.
7. Engheta, N., "On the role of fractional calculus in electromagnetic theory," *IEEE Antennas and Propagation Magazine*, Vol. 39, 35–46, 1997.

8. Stillinger, F. H., "Axiomatic basis for spaces with non-integer dimension," *Journal of Mathematical Physics*, Vol. 18, No. 6, 1224–1234, 1977.
9. Leibbrandt, G., "Introduction to the technique of dimensional regularization," *Rev. Mod. Phys.*, Vol. 47, No. 4, 849–876, 1975.
10. Wilson, K. G. and M. E. Fisher, "Critical exponents in 3.99 dimensions," *Phys. Rev. Lett.*, Vol. 28, No. 4, 240–243, 1972.
11. He, X. F., "Anisotropy and isotropy: A model of fraction-dimensional space," *Solid State Communications*, Vol. 75, No. 2, 111–114, 1990.
12. Tarasov, V. E., "Vector calculus in non-integer dimensional space and its applications to fractal media," *Commun. Nonlinear Sci. Numer. Simul.*, Vol. 20, No. 2, 360–374, 2015.
13. Palmer, C. and P. N. Stavrinou, "Equations of motion in a non-integer dimension space," *J. Phys. A*, Vol. 37, 6987–7003, 2004.
14. He, X.-F., "Fractional dimensionality and fractional derivative spectra of interband optical transitions," *Phys. Rev. B*, Vol. 42, No. 18, 11751–1756, 1990.
15. He, X.-F., "Excitons in anisotropic solids: The model of fractional-dimensional space," *Phys. Rev. B*, Vol. 43, No. 3, 2063–2069, 1991.
16. Lohe, M. A. and A. Thilagam, "Quantum mechanical models in fractional dimensions," *J. Phys. A*, Vol. 37, No. 23, 61–81, 2004.
17. De Dios-Leyva, M., A. Bruno-Alfonso, A. Matos-Abiague, and L. E. Oliveira, "Fractional-dimensional space and applications in quantum-confined semiconducting heterostructures," *J. Appl. Phys.*, Vol. 82, No. 6, 3155–3157, 1997.
18. Matos-Abiague, A., "Free particle in fractional-dimensional space," *Bulg. J. Phys.*, Vol. 27, No. 3, 54–57, 2000.
19. Matos-Abiague, A., "Deformation of quantum mechanics in fractional-dimensional space," *J. Phys. A*, Vol. 34, No. 49, 11059–1068, 2001.
20. Matos-Abiague, A., "Bose-like oscillator in fractional-dimensional space," *J. Phys. A*, Vol. 34, No. 14, 3125–3138, 2001.
21. Eid, R., S. I. Muslih, D. Baleanu, and E. Rabei, "On fractional Schrodinger equation in-dimensional fractional space," *Nonlinear Anal.: Real World Appl.*, Vol. 10, No. 3, 1299–1304, 2009.
22. Muslih, S. I. and O. P. Agrawal, *Schrodinger Equation in Fractional Space, in Fractional Dynamics and Control*, D. Baleanu, J. A. Tenreiro Machado, and A. C. J. Luo (eds.), Chap. 17, 209–215, Springer, New York, 2012.
23. Sandev, T., I. Petreska, and E. K. Lenzi, "Harmonic and anharmonic quantum-mechanical oscillators in noninteger dimensions," *Phys. Lett. A*, Vol. 378, No. 3, 109–116, 2013.
24. Muslih, S. and D. Baleanu, "Fractional multi-poles in fractional space," *Nonlinear Anal.: Real World Appl.*, Vol. 8, 198–203, 2007.
25. Baleanu, D., A. K. Golmankhaneh, and A. K. Golmankhaneh, "On electromagnetic field in fractional space," *Nonlinear Anal.: Real World Appl.*, Vol. 11, No. 1, 288–292, 2010.
26. Tarasov, V. E., "Electromagnetic fields on fractals," *Modern Physics Letters A*, Vol. 21, No. 20, 1587–1600, 2006.
27. Baleanu, D., A. K. Golmankhaneh, and A. K. Golmankhaneh, "On electromagnetic field in fractional space," *Nonlinear Anal.: Real World Appl.*, Vol. 11, No. 1, 288–292, 2010.
28. Naqvi, Q. A. and A. A. Rizvi, "Fractional dual solutions and corresponding sources," *Progress In Electromagnetics Research*, Vol. 25, 223–238, 2000.
29. Naqvi, Q. A., "Fractional dual interface in chiral nihility medium," *Progress In Electromagnetics Research Letters*, Vol. 8, 135–142, 2009.
30. Hussain, A., S. Ishfaq, and Q. A. Naqvi, "Fractional curl operator and fractional waveguides," *Progress In Electromagnetics Research*, Vol. 63, 319–335, 2006.
31. Zubair, M., M. J. Mughal, Q. A. Naqvi, and A. A. Rizvi, "Differential electromagnetic equations in fractional space," *Progress In Electromagnetics Research*, Vol. 114, 255–269, 2011.

32. Zubair, M., M. J. Mughal, and Q. A. Naqvi, "The wave equation and general plane wave solutions in fractional space," *Progress In Electromagnetics Research Letters*, Vol. 19, 137–146, 2010.
33. Zubair, M., M. J. Mughal, and Q. A. Naqvi, "On electromagnetic wave propagation in fractional space," *Nonlinear Anal.: Real World Appl.*, Vol. 12, No. 5, 2844–2850, 2011.
34. Zubair, M., M. J. Mughal, and Q. A. Naqvi, "An exact solution of the spherical wave equation in D-dimensional fractional space," *Journal of Electromagnetic Waves and Applications*, Vol. 25, No. 10, 1481–1491, 2011.
35. Song, W. Z. and L. B. Wei, "The scattering of electromagnetic waves in fractal media," *Waves in Random and Complex Media*, Vol. 4, No. 1, 97–103, 1994.
36. Balanis, C. A., *Advanced Engineering Electro-magnetics*, Wiley, New York, 1989.
37. Harrington, R. F., *Time-harmonic Electromagnetic Fields*, McGraw-Hill Inc., New York, 1961.
38. Omer, M. and M. J. Mughal, "Behavior of electromagnetic waves at dielectric fractal-fractal interface in fractional spaces," *Progress In Electromagnetics Research M*, Vol. 28, 229–244, 2013.
39. Asad, H., M. Zubair, and M. J. Mughal, "Reflection and transmission at dielectric-fractal interface," *Progress In Electromagnetics Research*, Vol. 125, 543–558, 2012.
40. Attiya, A. M., "Reflection and transmission of electromagnetic wave due to a quasi-fractional-space slab," *Progress In Electromagnetics Research Letters*, Vol. 24, 119–128, 2011.
41. Marwat, S. K. and M. J. Mughal, "Characteristics of multilayered metamaterial structures embedded in fractional space for tera-hertz applications," *Progress In Electromagnetics Research*, Vol. 144, 229–239, 2014.
42. Lindell, I. V., A. H. Sihvola, S. A. Tretyakov, and A. J. Viitanen, *Electromagnetic Waves in Chiral and Bi-isotropic Media*, Artech House, Boston, 1994.
43. Cheng, Q. T., J. Cui, and C. Zhang, "Waves in planar waveguide containing chiral nihility metamaterial," *Optics Communications*, Vol. 276, 317–321, 2007.
44. Chew, W. C., *Waves and Fields in In-homogenous Media*, Van Nostrand Reinhold, New York, 1990.
45. Naqvi, A., A. Hussain, and Q. A. Naqvi, "Waves in fractional dual planar waveguides containing chiral nihility metamaterial," *Journal of Electromagnetic Waves and Applications*, Vol. 24, Nos. 11–12, 1575–1586, 2010.
46. Pelet, P. and N. Engheta, "The theory of chirowaveguides," *IEEE Transactions on Antennas and Propagation*, Vol. 38, No. 1, 90–98, 1990.
47. Lakhtakia, A., *Beltrami Fields in Chiral Media*, World Scientific, Singapore, 1994.
48. Jaggard, D. L., A. R. Mickelson, and C. H. Papas, "On electromagnetic waves in chiral media," *Applied Physics*, Vol. 18, 16–21, 1979.
49. Bassiri, S., C. H. Papas, and N. Engheta, "Electromagnetic wave propagation through a dielectric-chiral interface and through a chiral slab," *Journal of Optical Society of America A*, Vol. 5, 145–209, 1988.
50. Lindell, I. V., A. H. Sihvola, S. A. Tretyakov, and A. J. Viitanen, *Electromagnetic Waves in Chiral and Bi-isotropic Media*, Artech House, Boston, 1994.
51. Lakhtakia, A., V. K. Varadan, and V. V. Varadan, *Time Harmonic Electromagnetic Fields in Chiral Media*, Springer, Berlin, 1989.
52. Zarifi, D., A. Abdolali, M. Soleimani, and M. V. Nayyeri, "Inhomogeneous planar layered chiral media: Analysis of wave propagation and scattering using Taylor's series expansion," *Progress In Electromagnetics Research*, Vol. 125, 119–135, 2012.
53. Zarifi, D., M. Soleimani, and V. A. Nayyeri, "Novel dual-band chiral metamaterial structure with giant optical activity and negative refractive index," *Journal of Electromagnetic Waves and Applications*, Vol. 26, Nos. 2–3, 251–263, 2012.
54. Tretyakov, S., I. Nefedov, A. Sihvola, S. Maslovski, and C. Simovski, "Waves and energy in chiral nihility," *Journal of Electromagnetic Waves and Applications*, Vol. 17, No. 5, 695–706, 2003.
55. Lakhtakia, A., "An electromagnetic trinity from negative permittivity and negative permeability," *International Journal of Infrared and Millimeter Waves*, Vol. 22, 173–214, 2001.

56. Dong, J.-F., J. Li, and F.-Q. Yang, "Guided modes in the four-layer slab waveguide containing chiral nihility core," *Progress In Electromagnetics Research*, Vol. 112, 241–255, 2011.
57. Rahim, A. A., M. J. Mughal, and Q. A. Naqvi, "Fractional rectangular waveguide internally coated with chiral nihility metamaterial," *Progress In Electromagnetics Research M*, Vol. 17, 197–211, 2011.
58. Naqvi, Q. A., "Planar slab of chiral nihility metamaterial backed by fractional DUAL/PEMC interface," *Progress In Electromagnetics Research*, Vol. 85, 381–391, 2009.
59. Tuz, V. R. and C.-W. Qiu, "Semi-infinite chiral nihility photonics: Parametric dependence, wave tunneling and rejection," *Progress In Electromagnetics Research*, Vol. 103, 139–352, 2010.
60. Cheng, X., H. Chen, X.-M. Zhang, B. Zhang, and B.-I. Wu, "Cloaking a perfectly conducting sphere with rotationally uniaxial nihility media in monostatic radar system," *Progress In Electromagnetics Research*, Vol. 100, 285–298, 2010.