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One-Step Leapfrog HIE-FDTD Method for Lossy Media

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Abstract—The one-step leapfrog hybrid implicit-explicit finite-difference time-domain (HIE-FDTD) method for lossy media is presented. By adopting the Crank-Nicolson and Peaceman-Rachford schemes, the derived method involves calculations of the lossy terms at two different time steps. Different from the original HIE-FDTD method, the proposed method can also be considered as a second order perturbation of the conventional FDTD method. To verify the effectiveness of the proposed method, numerical experiments are performed by using different FDTD methods. It is shown that the proposed method can be more efficient than the conventional HIE-FDTD method with almost the same accuracy.

1. INTRODUCTION

The conventional finite-difference time-domain (FDTD) is an explicit difference method which is restricted by the Courant-Friedrich-Levy (CFL) stability condition [1]. To remove this limitation, the unconditional stable alternating-direction-implicit FDTD (ADI-FDTD) method was proposed in [2] where one full time step calculations are divided into two sub-step computations. To improve its efficiency, one-step leapfrog ADI-FDTD method was developed based on the conventional ADI-FDTD method where the mid time step calculations are removed [3]. Further the leapfrog ADI-FDTD method was extended to model lossy and other complex media [4–6]. However, the leapfrog ADI-FDTD method needs to solve six implicit equations in one time step which is very time consuming. To overcome this drawback, a large CFL number should be chosen. Therefore the leapfrog ADI-FDTD method is optimal to simulate 3D electromagnetic problems with fine structures at low frequency.

On the other hand, some electromagnetic structures are only fine in one or two directions in which the update equations should be calculated implicitly in order to remove the relations of stability condition with such directions. Specifically, one hybrid implicit-explicit FDTD (HIE-FDTD) method [7] and weakly conditionally stable FDTD (WCS-FDTD)method [8] were developed for modelling structures with fine grid in one and two directions, respectively. Recently, the one-step leapfrog HIE-FDTD method for lossless media was proposed in [9] where the stability condition is only determined by one grid size and a larger CFL number can be selected than that in [7].

In this paper, we further extend the one-step leapfrog HIE-FDTD method for modelling lossy media. The update equations are derived by using both the Crank-Nicolson (CN) and Peaceman-Rachford (PR) schemes. Numerical results show that the computational time of one single step is slightly more than that of conventional HIE-FDTD method but the CFL condition is more relaxed. Therefore the proposed method can be more efficient by using a larger CFL number.

2. LEAPFROG HIE-FDTD FORMULATION FOR LOSSY MEDIA

In a lossy, non-dispersive medium with permittivity ε, permeability μ, and electric conductivity $[\sigma]$, the time-dependent Maxwell's curl equations can be written in a concise form as

$$\frac{\partial u}{\partial t} = [R]u + J \tag{1a}$$

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where $u = [E_x, E_y, E_z, H_x, H_y, H_z]^T$, $J = [J_e, 0]^T$, $J_e = [J_x, J_y, J_z]^T = [\sigma]E$ and

$$[R] = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{\varepsilon}\frac{\partial}{\partial z} & \frac{1}{\varepsilon}\frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{1}{\varepsilon}\frac{\partial}{\partial z} & 0 & -\frac{1}{\varepsilon}\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -\frac{1}{\varepsilon}\frac{\partial}{\partial y} & \frac{1}{\varepsilon}\frac{\partial}{\partial x} & 0 \\ 0 & \frac{1}{\mu}\frac{\partial}{\partial z} & -\frac{1}{\mu}\frac{\partial}{\partial y} & 0 & 0 & 0 \\ -\frac{1}{\mu}\frac{\partial}{\partial z} & 0 & \frac{1}{\mu}\frac{\partial}{\partial x} & 0 & 0 & 0 \\ \frac{1}{\mu}\frac{\partial}{\partial y} & -\frac{1}{\mu}\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1b}$$

$$[\sigma] = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \tag{1c}$$

By splitting the space operator $[R]$ into two operators as $[R] = [A] + [B]$ and applying the CN scheme to (1a) at time step $n + 1/2$, we obtain

$$\left(I - \frac{\Delta t}{2}[A] - \frac{\Delta t}{2}[B]\right) u^{n+1} = \left(I + \frac{\Delta t}{2}[A] + \frac{\Delta t}{2}[B]\right) u^n + \Delta t J^{n+1/2}. \tag{2}$$

Further, by introducing a second order perturbation $\Delta t^2[A][B](u^{n+1} - u^n)/4$, (2) can be approximated by

$$\left(I - \frac{\Delta t}{2}[A]\right)\left(I - \frac{\Delta t}{2}[B]\right) u^{n+1} = \left(I + \frac{\Delta t}{2}[A]\right)\left(I + \frac{\Delta t}{2}[B]\right) u^n + \Delta t J^{n+1/2}. \tag{3}$$

By introducing an intermediate term $u^{n+1/2}$, (3) can be divided into two sub time step computations as

$$\left(I - \frac{\Delta t}{2}[A]\right) u^{n+1/2} = \left(I + \frac{\Delta t}{2}[B]\right) u^n + \frac{\Delta t}{2} J^{n+1/2} \tag{4a}$$

$$\left(I - \frac{\Delta t}{2}[B]\right) u^{n+1} = \left(I + \frac{\Delta t}{2}[A]\right) u^{n+1/2} + \frac{\Delta t}{2} J^{n+1/2}. \tag{4b}$$

Note that different $[A]$ and $[B]$ selections can result in different FDTD methods. Assumed that the narrow grid is along the y direction, to obtain the HIE-FDTD method, $[A]$ and $[B]$ can be set as [9]

$$[A] = \begin{bmatrix} 0 & \frac{C}{\varepsilon} \\ \frac{D}{\mu} & 0 \end{bmatrix}, \ [B] = \begin{bmatrix} 0 & -\frac{D}{\varepsilon} \\ -\frac{C}{\mu} & 0 \end{bmatrix}, \ [C] = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ [D] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 \end{bmatrix}. \tag{5}$$

Specifically, from (4a), the update equations from n to $n + 1/2$ can be obtained as

$$E^{n+1/2} - \frac{\Delta t}{2\varepsilon}[C]H^{n+1/2} = E^n - \frac{\Delta t}{2\varepsilon}[D]H^n + \frac{\Delta t}{2} J_e^{n+1/2} \tag{6a}$$

$$H^{n+1/2} - \frac{\Delta t}{2\mu}[D]E^{n+1/2} = H^n - \frac{\Delta t}{2\mu}[C]E^n. \tag{6b}$$

Similarly, from (4b), the update equations from $n + 1/2$ to $n + 1$ can be obtained as

$$E^{n+1} + \frac{\Delta t}{2\varepsilon}[D]H^{n+1} = E^{n+1/2} + \frac{\Delta t}{2\varepsilon}[C]H^{n+1/2} + \frac{\Delta t}{2} J_e^{n+1/2} \tag{7a}$$

$$H^{n+1}+\frac{\Delta t}{2\mu}[C]E^{n+1} = H^{n+1/2}+\frac{\Delta t}{2\mu}[D]E^{n+1/2}. \tag{7b}$$

To obtain a leapfrog scheme, the similar derivative process as proposed in [3] is performed as follows. First substituting (6b) into (6a), we have

$$\left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C][D]\right)E^{n+1/2}=\left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C]^2\right)E^{n}-\frac{\Delta t}{2\varepsilon}\left([D]-[C]\right)H^{n}+\frac{\Delta t}{2}J_e^{n+1/2}. \tag{8}$$

Then replacing n with $n-1$ in (7a) and (7b), we have

$$E^{n}+\frac{\Delta t}{2\varepsilon}[D]H^{n} = E^{n-1/2}+\frac{\Delta t}{2\varepsilon}[C]H^{n-1/2}+\frac{\Delta t}{2}J_e^{n-1/2} \tag{9a}$$

$$H^{n}+\frac{\Delta t}{2\mu}[C]E^{n} = H^{n-1/2}+\frac{\Delta t}{2\mu}[D]E^{n-1/2}. \tag{9b}$$

Substituting (9b) into (9a), we obtain

$$\left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C]^2\right)E^{n}=\left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C][D]\right)E^{n-1/2}-\frac{\Delta t}{2\varepsilon}\left([D]-[C]\right)H^{n}+\frac{\Delta t}{2}J_e^{n-1/2} \tag{10}$$

Adding (8) and (10) on their both sides, we obtain the leapfrog equation for E as

$$\begin{aligned}\left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C][D]\right)E^{n+1/2} &= \left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C][D]\right)E^{n-1/2}-\frac{\Delta t}{\varepsilon}\left([D]-[C]\right)H^{n}\\ &\quad+\frac{\Delta t}{2}J_e^{n-1/2}+\frac{\Delta t}{2}J_e^{n+1/2}\end{aligned} \tag{11}$$

Similarly, the leapfrog equation for H can be obtained as follows.

First substituting (6a) into (6b), we have

$$\left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C]^2\right)H^{n+1/2}=\left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C][D]\right)H^{n}+\frac{\Delta t}{2\mu}\left([D]-[C]\right)E^{n+1/2}+\frac{\Delta t^2}{4\mu}[C]J_e^{n+1/2}. \tag{12}$$

Then substituting (7a) into (7b), we have

$$\left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C][D]\right)H^{n+1}=\left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C]^2\right)H^{n+1/2}+\frac{\Delta t}{2\mu}\left([D]-[C]\right)E^{n+1/2}-\frac{\Delta t^2}{4\mu}[C]J_e^{n+1/2}. \tag{13}$$

Adding (12) and (13) on their both sides, the leapfrog equation for H can be obtained as

$$\left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C][D]\right)H^{n+1}=\left(I-\frac{\Delta t^2}{4\varepsilon\mu}[C][D]\right)H^{n}+\frac{\Delta t}{\mu}\left([D]-[C]\right)E^{n+1/2}. \tag{14}$$

Note that (11) and (14) are the one-step leapfrog HIE-FDTD method for lossy media which can be seen as the FDTD method with a second order perturbation term $\Delta t^2[C][D]/4\varepsilon\mu$.

For clarity, the equations of field components are given as

$$\left(1-\frac{\Delta t}{2}\sigma_y\right)E_y^{n+1/2} = \left(1+\frac{\Delta t}{2}\sigma_y\right)E_y^{n+1/2}+\frac{\Delta t}{\varepsilon}\left(\frac{\partial H_x^n}{\partial z}-\frac{\partial H_z^n}{\partial x}\right) \tag{15a}$$

$$\left(1-\frac{\Delta t}{2}\sigma_z\right)E_z^{n+1/2} = \left(1+\frac{\Delta t}{2}\sigma_z\right)E_z^{n+1/2}+\frac{\Delta t}{\varepsilon}\left(\frac{\partial H_y^n}{\partial x}-\frac{\partial H_x^n}{\partial y}\right) \tag{15b}$$

$$\begin{aligned}\left(I-\frac{\Delta t^2}{4\varepsilon\mu}\frac{\partial^2}{\partial y^2}-\frac{\Delta t}{2}\sigma_x\right)E_x^{n+1/2} &= \left(I-\frac{\Delta t^2}{4\varepsilon\mu}\frac{\partial^2}{\partial y^2}+\frac{\Delta t}{2}\sigma_x\right)E_x^{n-1/2}+\frac{\Delta t}{\varepsilon}\left(\frac{\partial H_z^n}{\partial y}-\frac{\partial H_y^n}{\partial z}\right)\\ &\quad-\frac{\Delta t^2}{4\varepsilon\mu}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(E_y^{n+1/2}-E_y^{n-1/2}\right)\\ &\quad-\frac{\Delta t^2}{4\varepsilon\mu}\frac{\partial}{\partial x}\frac{\partial}{\partial z}\left(E_z^{n+1/2}-E_z^{n-1/2}\right)\end{aligned} \tag{15c}$$

$$H_y^{n+1} = H_y^n - \frac{\Delta t}{\mu}\left(\frac{\partial E_x^{n+1/2}}{\partial z} - \frac{\partial E_z^{n+1/2}}{\partial x}\right), \tag{15d}$$

$$H_z^{n+1} = H_z^n - \frac{\Delta t}{\mu}\left(\frac{\partial E_y^{n+1/2}}{\partial x} - \frac{\partial E_x^{n+1/2}}{\partial y}\right) \tag{15e}$$

$$\left(I - \frac{\Delta t^2}{4\varepsilon\mu}\frac{\partial^2}{\partial y^2}\right) H_x^{n+1/2} = \left(I - \frac{\Delta t^2}{4\varepsilon\mu}\frac{\partial^2}{\partial y^2}\right) H_x^n - \frac{\Delta t}{\mu}\left(\frac{\partial E_z^{n+1/2}}{\partial y} - \frac{\partial E_y^{n+1/2}}{\partial z}\right)$$
$$-\frac{\Delta t^2}{4\varepsilon\mu}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(H_y^{n+1} - H_y^n\right) - \frac{\Delta t^2}{4\varepsilon\mu}\frac{\partial}{\partial x}\frac{\partial}{\partial z}\left(H_z^{n+1} - H_z^n\right) \tag{15f}$$

3. NUMERICAL RESULTS AND DISCUSSION

To check the accuracy and stability of the proposed leapfrog HIE-FDTD method, a PEC cavity filled with lossy media was studied as depicted in Figure 1.

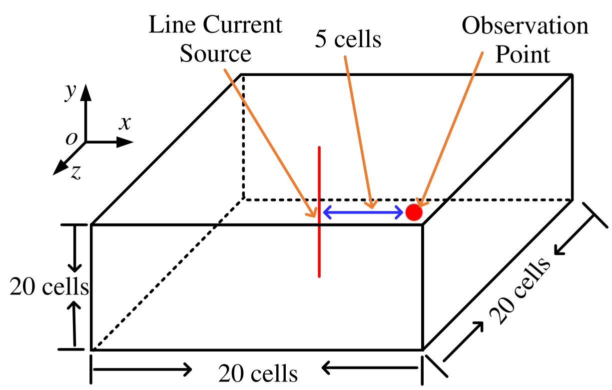

Figure 1. The schematic of the computational domain: A PEC cavity filled with lossy media.

In the simulations, a nonuniform mesh $\Delta x = \Delta z = 10\Delta y = 1\,\text{mm}$ was used and the cavity was described by $20 \times 20 \times 20$ cells. The electromagnetic field was excited by a line current source centered at the computational domain extended from bottom to top along the y-direction as

$$J_y = \exp\left[-4\pi\left(t - t_0\right)^2/\tau^2\right] \tag{16}$$

where $\tau = 150\Delta t_{\text{FDTD}}/\text{CFLN}$ and $t_0 = 2\tau$ with $\Delta t_{\text{FDTD}} = 1/(c_0\sqrt{1/\Delta x^2 + 1/\Delta y^2 + 1/\Delta z^2})$ being the maximum time step of the traditional FDTD method, c being the speed of light in vacuum, CFLN being the CFL number. The electric conductivity was set as $\sigma_x = \sigma_y = \sigma_z = 0.2\,\text{S/m}$. The observation point was placed at five cells away from the center of the cavity along the x-direction.

Figure 2 shows the recorded E_y at the observation point versus time computed by the proposed HIE-FDTD method for lossy media. Because the maximum time step of the leapfrog HIE-FDTD method is $\Delta x/c_0$ [9] which leads to the maximum CFLN $= (\Delta x/c_0)/\Delta t_{\text{FDTD}} = 10.1$, the result simulated with CFLN $= 10.1$ is provided. Moreover the results simulated with CFLN $= 1$, 5, and the traditional FDTD with CFLN $= 1$ are also given for comparison. It is seen that the result obtained with the proposed method with CFLN $= 1$ agrees well with that of FDTD; the errors increase as CFLN becomes larger; even for CFLN $= 10.1$, the numerical errors are acceptable.

Figure 3 provides the recorded E_y versus time computed by original HIE-FDTD method [10], leapfrog ADI-FDTD method [5], and proposed HIE-FDTD method. All are computed with CFLN $==$ 7.1 which is the maximum time step of the original HIE-FDTD method [10], i.e., $[1/(c_0\sqrt{1/\Delta x^2 + 1/\Delta z^2})]/\Delta t_{\text{FDTD}} = 7.1$. To comparison, the result obtained by FDTD with CFLN $= 1$

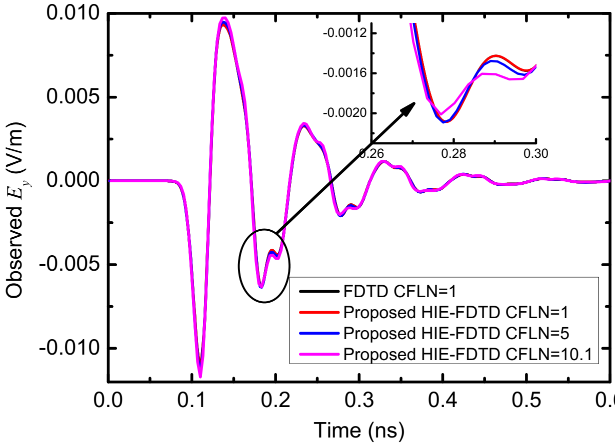

Figure 2. Recorded E_y at the observation point versus time computed by the proposed HIE-FDTD with CFLN = 1, 5, and its maximum value 10.1. The result computed by the conventional FDTD method is also given for comparison.

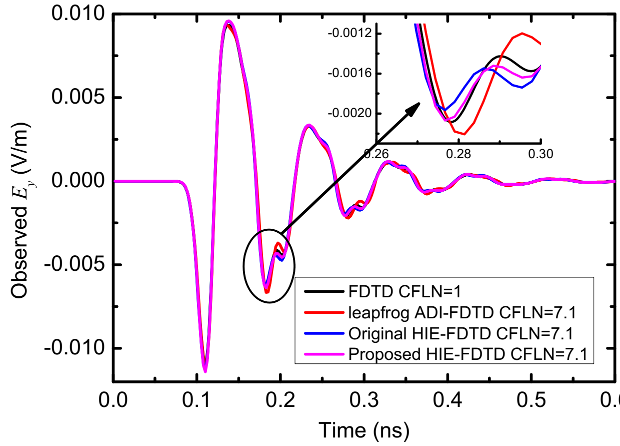

Figure 3. Recorded E_y computed by the leapfrog ADI-FDTD, original HIE-FDTD, and proposed HIE-FDTD, all with CFLN = 7.1 which is the maximum time step of the original HIE-FDTD method. The result computed by FDTD with CFLN = 1 is also given for comparison.

Table 1. Comparison of the CPU time and Memory used by FDTD, leapfrog ADI-FDTD, original HIE-FDTD, and proposed HIE-FDTD.

	FDTD	leapfrog ADI-FDTD	Original HIE-FDTD	Proposed HIE-FDTD	
CFLN	1	7.1	7.1	7.1	10.1
Number of iterations	1,000,000	140845	140845	140,845	99,010
Time (s)	140.4	79.6	50.8	55.8	39.7
Memory (Mb)	0.366	0.366	0.427	0.488	

is also given. It can be seen that both the errors computed by original HIE-FDTD and proposed HIE-FDTD are in the same level but smaller than that of leapfrog ADI-FDTD method. This is because, as a hybrid explicit-implicit method, the numerical dispersion of the original HIE-FDTD method is nearly the same as that of leapfrog HIE-FDTD method but both are smaller than that of leapfrog ADI-FDTD method [9].

To make a further comparison in terms of CPU time and Memory, the results obtained by FDTD, leapfrog ADI-FDTD, original HIE-FDTD, and proposed HIE-FDTD are given in Table 1. The simulation platform was Lenovo PC with Intel i7 4770 k of 3.5 GHz and RAM of 16 GB; the source code was written in Fortran language. It is seen that when CFLN = 7.1 the simulation time of the proposed method is 55.8 s which is larger than that of original HIE method. This is because (15c) and (15f) of the proposed method in this paper are a little bit more complex than those of the original HIE-FDTD method [10]. However, this drawback can be made up by a large CFLN selection, e.g., CFLN = 10.1 which is the largest time step of the proposed HIE method. In terms of the memory used, the leapfrog ADI-FDTD and FDTD method uses the same and the least. This is because there are only six field components need to be stored for leapfrog ADI-FDTD and FDTD methods while the original HIE-FDTD needs to store another one and the proposed HIE-FDTD method needs to store another two field components of the passed time step.

To check the stability condition, numerical experiments were carried out by using the proposed HIE-FDTD method with CFLN = 10.2 which is not satisfied the condition given in [9]. As a result, it was found that the field was unstable. To make a further investigation, more experiments were carried out with different parameters and it was found that the stability condition of the proposed HIE-FDTD method for lossy media was always the same as that proposed in [9].

4. CONCLUSION

The one-step leapfrog HIE-FDTD method for lossy media has been provided in this paper. This method is derived based on the CN and PR schemes and can be extended for developing other one-step methods for lossy media. Numerical results have shown that the proposed method can be more efficient than the original HIE-FDTD method.

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REFERENCES

1. Taflove, A. and S. C. Hagness, *Computational Electrodynamics: The Finite-difference Time-domain Method*, 3rd Edition, Artech House, Norwood, MA, 2005.
2. Zheng, F., Z. Chen, and J. Zhang, "Toward the development of a three dimensional unconditionally stable finite-different time-domain method," *IEEE Trans. Microwave Theory Tech.*, Vol. 48, No. 9, 1550–1558, 2000.
3. Cooke, S. J., M. Botton, T. M. Antonsen, and B. Levush, "A leapfrog formulation of the 3D ADI-FDTD algorithm," *Int. J. Numer. Model*, Vol. 22, No. 2, 187–200, 2009.
4. Gan, T. H. and E. L. Tan, "Unconditionally stable leapfrog ADI-FDTD method for lossy media," *Progress In Electromagnetics Research M*, Vol. 26, 173–186, 2012.
5. Gao, J. Y. and H. X. Zheng, "One-step leapfrog ADI-FDTD method for lossy media and its stability analysis," *Progress In Electromagnetics Research Letters*, Vol. 40, 49–60, 2013.
6. Wang, X. H., W. Y. Yin, and Z. Chen, "One-step Leapfrog ADI-FDTD method for simulating electromagnetic wave propagation in general dispersive media," *Optics Express*, Vol. 21, No. 18, 20565–20576, 2013.
7. Chen, J. and J. Wang, "A 3-D hybrid implicit-explicit FDTD schemewith weakly conditional stability," *Microw. Opt. Technol. Lett.*, Vol. 48, 2291–2294, 2006.
8. Chen, J. and J. Wang, "A novel WCS-FDTD method with weakly conditionalstability," *IEEE Trans. Electromagn. Compat.*, Vol. 49, No. 2, 419–426, 2007.
9. Wang, J., B. Zhou, L. Shi, C. Gao, and B. Chen, "A novel 3-D HIE-FDTD method with one-step leapfrog scheme," *IEEE Trans. Microw. Theory Tech.*, Vol. 62, No. 6, 1275–1283, 2014.
10. Unno, M. and H. Asai, "HIE-FDTD method for hybrid system with lumped elements and conductive media," *IEEE Microw. Wireless Compon. Lett.*, Vol. 21, No. 9, 453–455, 2011.