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An Efficient Algorithm for the Novel Weakly Conditionally Stable FDTD Method

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Abstract—In this paper, we present an efficient formulation of the novel weakly conditionally stable finite-difference time-domain (NWCS-FDTD) method for the electromagnetic problems with very fine structures in one or two directions. The formulation is obtained by using only algebraic manipulation of the original method, and therefore the numerical stability and dispersion properties can be preserved. Moreover, due to its simpler right-hand sides of the updating equations, the proposed algorithm is more efficient than the existing WCS-FDTD methods, allowing a significant reduction in the cost of CPU time. Numerical experiments are finally given to verify the accuracy and efficiency of the proposed method.

1. INTRODUCTION

The finite difference time domain (FDTD) method has attracted much interest for numerical analysis of electromagnetic problems due to its robustness and simplicity [1]. However, since the conventional FDTD method is based on the explicit central difference scheme, its time step size is constrained by the Courant-Friedrich-Levy (CFL) condition, making it less suitable for the problems with fine structures. To overcome this shortcoming, various unconditionally stable FDTD methods, such as the alternating direction implicit (ADI) FDTD [2–4], locally 1-D (LOD) FDTD [5–7] and split step (SS) FDTD [8, 9], have been proposed. In these methods, there is no limitation for the time step size. Nevertheless, their numerical dispersion errors dramatically increase when the time step size increases beyond a certain level. Thus, the applicability of these methods is limited by the accuracy requirements.

For solving a particular class of problems that only require fine spatial discretization in one or two directions, a weakly conditionally stable (WCS) FDTD method has been developed [10, 11]. Although the CFL condition is not completely removed, the WCS-FDTD method can maintain stability when using a time step size much larger than the CFL limit of the conventional FDTD method. Besides, by combining the explicit and the implicit time integration schemes, this method has higher computation efficiency and lower numerical dispersion error in comparison with the unconditionally stable ADI-FDTD method.

More recently, a novel WCS-FDTD method, hereafter designated as NWCS-FDTD, has been proposed [12]. The stability condition of the new method is more relax than the WCS-FDTD method, which may lead to a further improvement of the efficiency by using larger time steps. However, the WCS-FDTD and the NWCS-FDTD algorithms are still rather complicated because the right-hand sides of their updating equations contain numerous terms that require considerable arithmetic operations. In order to further improve the computation efficiency, their updating equations need to be simplified.

In this letter, a new efficient algorithm for implementing the NWCS-FDTD method is presented by using the technique introduced in [13, 14]. The updating equations of the novel algorithm have more

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efficient and much simpler right-hand sides than the conventional methods, leading to a significant reduction in the number of arithmetic operations required in the implementation. A comparative study on numerical properties and computation costs is conducted between the efficient algorithm and the existing WCS-FDTD methods. It is found that the proposed algorithm not only achieves an appreciable improvement on the computation efficiency, but also retains the weakly conditionally stability feature of the NWCS-FDTD method.

2. GENERALIZED FORMULATIONS

In this section, based on the NWCS-FDTD method introduced in [12], the updating equations are developed for the new efficient algorithm. Without loss of generality, we assume that the largest space discretization is along the x-direction. In such a case, the numerical formulations of the original NWCS-FDTD method for each sub-procedure are given in (1)–(4), respectively, as follows:

$$\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{A}_1\right)\mathbf{u}_1^{n+\frac{1}{2}}=\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{B}_1\right)\mathbf{u}_1^{n}+\frac{\Delta t}{2}\mathbf{D}_1\mathbf{u}_2^{n} \tag{1}$$

$$\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{A}_2\right)\mathbf{u}_2^{n+\frac{1}{2}}=\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{B}_2\right)\mathbf{u}_2^{n}+\frac{\Delta t}{2}\mathbf{D}_2\mathbf{u}_1^{n+\frac{1}{2}} \tag{2}$$

$$\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{B}_1\right)\mathbf{u}_1^{n+1}=\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}_1\right)\mathbf{u}_1^{n+\frac{1}{2}}+\frac{\Delta t}{2}\mathbf{D}_1\mathbf{u}_2^{n+\frac{1}{2}} \tag{3}$$

$$\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{B}_2\right)\mathbf{u}_2^{n+1}=\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}_2\right)\mathbf{u}_2^{n+\frac{1}{2}}+\frac{\Delta t}{2}\mathbf{D}_2\mathbf{u}_1^{n+1} \tag{4}$$

where $\mathbf{u}_1=[H_y,H_z,E_x]^T$, $\mathbf{u}_2=[E_y,E_z,H_x]^T$ are the field component vectors, and

$$\mathbf{A}_1=-\begin{bmatrix}0&0&\partial_z/\mu\\0&0&0\\\partial_z/\varepsilon&0&0\end{bmatrix},\quad \mathbf{B}_1=\begin{bmatrix}0&0&0\\0&0&\partial_y/\mu\\0&\partial_y/\varepsilon&0\end{bmatrix}$$

$$\mathbf{D}_1=\begin{bmatrix}0&\partial_x/\mu&0\\-\partial_x/\mu&0&0\\0&0&0\end{bmatrix}$$

$$\mathbf{A}_2=-\mathbf{A}_1^T,\quad \mathbf{B}_2=-\mathbf{B}_1^T,\quad \mathbf{D}_2=\frac{\mu}{\varepsilon}\mathbf{D}_1^T$$

are the matrices that contain the spatial differential operators originated from Maxwell's curl equations.

Similar to the original method, the updating procedure of the efficient algorithm is performed in four sub-procedures. With the same algebraic manipulations described in [13], the updating equation for the first sub-procedure (1) can be reformulated by introducing the intermediate variable$\mathbf{v}_1$, expressed as the following matrix forms

$$\mathbf{v}_1^n=\tilde{\mathbf{u}}_1^n-\mathbf{v}_1^{n-\frac{1}{2}}+\frac{\Delta t}{4}\mathbf{D}_1\tilde{\mathbf{u}}_2^n \tag{5}$$

$$\left(\frac{1}{2}\mathbf{I}-\frac{\Delta t}{4}\mathbf{A}_1\right)\tilde{\mathbf{u}}_1^n=\mathbf{v}_1^n \tag{6}$$

Here, the other two intermediate variables $\tilde{\mathbf{u}}_1=2\mathbf{u}_1$, $\tilde{\mathbf{u}}_2=2\mathbf{u}_2$ are also introduced for the purpose of reducing the arithmetic operations. The updating equations for the other three sub-procedures can be derived in the same manner.

Then let us take the first sub-procedure as an example to illustrate the implementation of the algorithm. For convenience, we shall adopt the following notations:

$$b=\frac{\Delta t}{2\varepsilon},\quad d=\frac{\Delta t}{2\mu} \tag{7}$$

Therefore, the steps in carrying out the first updating sub-procedure of the proposed algorithm are as follows:

(i) Auxiliary updating for $\mathbf{v}_1^n$ is given as

$$h_y^n = \tilde{H}_y^n - h_y^{n-\frac{1}{2}} + \frac{d}{2}\partial_x \tilde{E}_z^n \tag{8a}$$

$$h_z^n = \tilde{H}_z^n - h_z^{n-\frac{1}{2}} - \frac{d}{2}\partial_x \tilde{E}_y^{n-\frac{1}{2}} \tag{8b}$$

$$e_x^n = \tilde{E}_x^n - e_x^{n-\frac{1}{2}} \tag{8c}$$

(ii) Implicit updating for $\tilde{E}_x^{n+\frac{1}{2}}$ is given as

$$\frac{1}{2}\tilde{E}_x^{n+\frac{1}{2}} - \frac{bd}{2}\partial_z^2 \tilde{E}_x^{n+\frac{1}{2}} = e_x^n - b\partial_z h_y^n \tag{9}$$

(iii) Explicit updating for $\tilde{H}_y^{n+\frac{1}{2}}$ and $\tilde{H}_z^{n+\frac{1}{2}}$ is given as

$$\tilde{H}_y^{n+\frac{1}{2}} = 2h_y^n - d\partial_z \tilde{E}_x^{n+\frac{1}{2}} \tag{10a}$$

$$\tilde{H}_z^{n+\frac{1}{2}} = 2h_z^n \tag{10b}$$

In fact, additional savings of operations can be achieved by simplifying the updating equations. Taking the previous time step of updating procedure, the calculation of (10b) can be removed. Accordingly, the updating equations, (8a) and (8b), need to be reformulated as

$$h_y^n = h_y^{n-\frac{1}{2}} + \frac{d}{2}\partial_x \tilde{E}_z^n \tag{11a}$$

$$h_z^n = \tilde{H}_z^n - h_y^{n-\frac{1}{2}} - d\partial_x e_y^{n-\frac{1}{2}} \tag{11b}$$

Equations in the other three sub-procedures can be similarly treated as those done in the first sub-procedure. Therefore, it can be seen that the updating equations of this method have much simpler right-hand sides in comparison with the WCS-FDTD and the NWCS-FDTD methods.

3. COMPARISONS AND DISCUSSIONS

In this section, the proposed algorithm is compared with the WCS-FDTD and the NWCS-FDTD methods.

Let us first compare the numerical properties of these three WCS-FDTD methods based on the analysis results for the stability and dispersion properties provided in [10, 12]. Since our formulation is obtained by using only algebraic manipulation of the original formulation, the numerical properties of the proposed algorithm should be identical with those of the NWCS-FDTD method. Thus, the maximum time step of the proposed method is $2\Delta x/c$, not $\Delta x/c$ in the WCS-FDTD method.

Figure 1 shows the normalized numerical phase velocity versus the propagation angle θ for the ADI-FDTD and the WCS-FDTD methods with different time step sizes. For clarify, an nonuniform grid cell ($\Delta x = 10\Delta y = 10\Delta z = \lambda/15$) are employed, and we define CFLN as the ratio of the time step size to the CFL limit of the FDTD method. From Figure 1, it can be seen that all the WCS-FDTD methods have the same dispersion error as that of the ADI-FDTD method with a small time step size (CFLN = 1) or with the propagation angle on the $k_y - k_z$ plane ($\varphi = 90°$). When the propagation angle $\varphi = 0°$ or 45° and the time step size increases (CFLN = 14.17), the normalized numerical phase velocity errors of the NWCS-FDTD method and the proposed efficient algorithm are a little larger than that of the WCS-FDTD method, but still are much smaller than that of the ADI-FDTD method.

After the comparative analysis of numerical stability and dispersion properties, Table 1 presents the comparisons of the floating-point operations (flops) counts and the efficiency gains of the WCS-FDTD methods. We adopt the same analysis method introduced in [14]. By counting the numbers of the arithmetic operations on the right hand side (RHS) of their updating equations in one complete time step, the numbers of multiplications/divisions (M/D) or additions/subtractions (A/S) operations are determined. In addition, the RHS efficiency gains and the overall efficiency gains are given by taking the WCS-FDTD method as the benchmark. The RHS efficiency gains are calculated by dividing

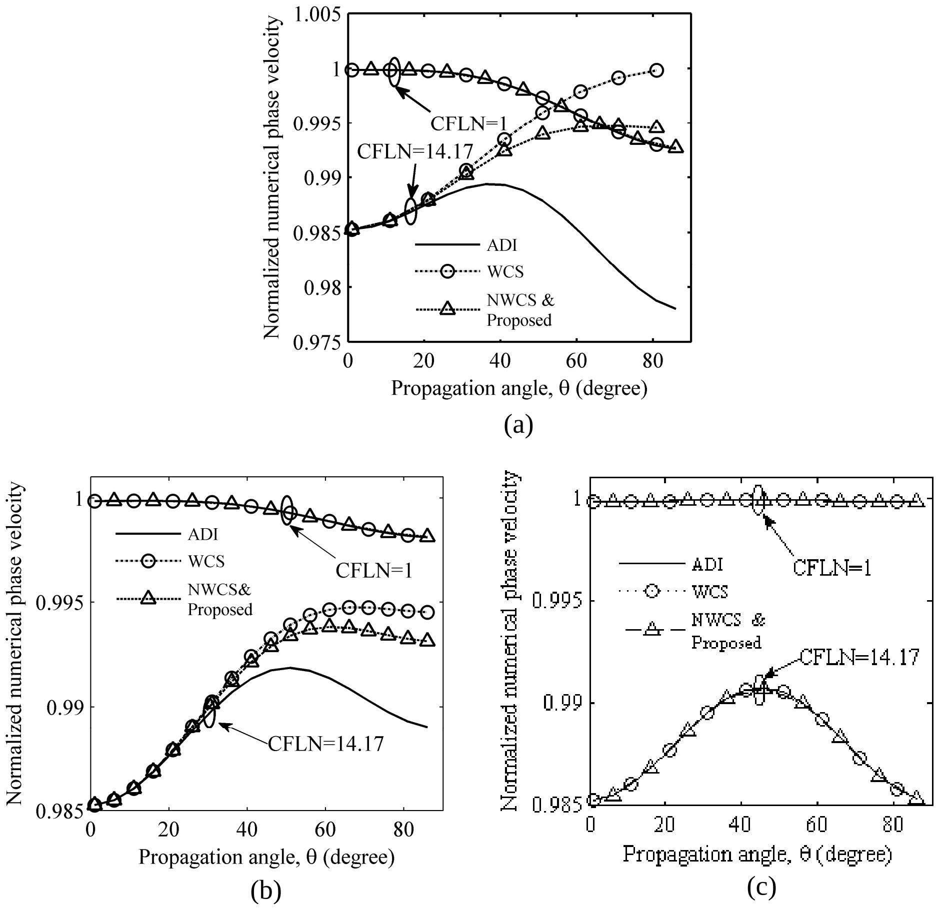

Figure 1. Normalized phase velocities versus propagation angle $\theta$for different methods: (a) $\varphi = 0°$, (b) $\varphi = 45°$, and (c) $\varphi = 90°$.

Table 1. Comparison of algorithms.

Scheme		WCS	NWCS	Proposed
Implicit	M/D	20	12	4
	A/S	32	32	8
Explicit	M/D	6	16	16
	A/D	20	32	32
Total	M/D + A/S	78	92	60
Efficiency Gain	RHS	1	0.85	1.30
	Overall	1	0.88	1.23

the total flops count for the WCS-FDTD method by the counts for the other two methods, and the overall efficiency gains are calculated by taking into account of the additional cost for solving (implicit) tridiagonal systems, which is typically $5N$ flops for a system of order N. It can be easily seen from Table 1 that the RHS and the overall efficient gains of the NWCS-FDTD method are 0.85 and 0.88, which are less than 1. It means that this method takes more flops count than the WCS-FDTD in one complete time step. Therefore, it is difficult to ensure that the NWCS-FDTD can obtain relatively higher computation efficiency in comparison with the WCS-FDTD method, despite its more relaxed

CFL stability condition. However, the RHS and the overall efficient gains of the proposed algorithm are 1.30 and 1.23, respectively, which indicates that the proposed algorithm can get a large flops count reduction, while it retains the much weaker stability condition of the NWCS-FDTD method.

4. NUMERICAL EXAMPLE

To demonstrate the accuracy and the efficiency of the proposed algorithm, a simple three-dimensional air-filled cavity with the size of 5 cm × 5 cm × 5 cm is calculated. A nonuniform mesh $\Delta x = 5\Delta y = 5\Delta z = 5$ mm is chosen as the space increment, and therefore the total mesh dimensions are $10 \times 50 \times 50$ cells in the x-, y- and z-directions, respectively. To initialize the fields inside the cavity, the field values are set to be those of TE_{111} mode (5.19 GHz). The total simulation time is selected to be 100 ns. The observation point is set at 3 cells away from the center point along the x-direction.

Figure 2 shows the waveforms of E_y in time domain simulated by the conventional FDTD, the NWCS-FDTD and the proposed efficient methods, respectively, with CFLN = 1 for the FDTD and CFLN = 7.14 for the other two methods. It can be seen that the results are in quite good agreement each other.

Table 2 shows the relative errors of the resonance frequency and the CPU time for the four methods with different time step sizes. The WCS-FDTD method becomes unstable when CFLN > 7.14. It can be seen that the proposed method consumes the least CPU time for the simulation, and has the same accuracy as the conventional NWCS-FDTD method. The numerical errors of the proposed method are a litter larger than those of the WCS-FDTD method, but are smaller than those of the ADI-FDTD method, which is in accordance with the previous theoretical analysis.

Figure 3 shows a common model in electromagnetic compatibility (EMC) issues. The upper portion is a three-dimensional diagram of the enclosure, and the lower portion is a front view of the thin metal

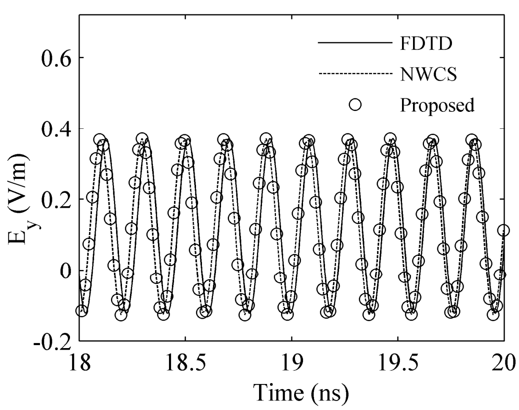

Figure 2. Time domain waveforms of E_y.

Table 2. Computation results.

Schemes	CFLN = 1		CFLN = 7.14		CFLN = 14.28	
	Error	Time	Error	Time	Error	Time
FDTD	0.05%	720 s				
ADI	0.24%	4231 s	1.39%	596 s	4.87%	307 s
WCS	0.24%	2858 s	0.97%	405 s		
NWCS	0.24%	3884 s	1.01%	546 s	3.71%	275 s
Proposed	0.24%	2117 s	1.01%	298 s	3.71%	151 s

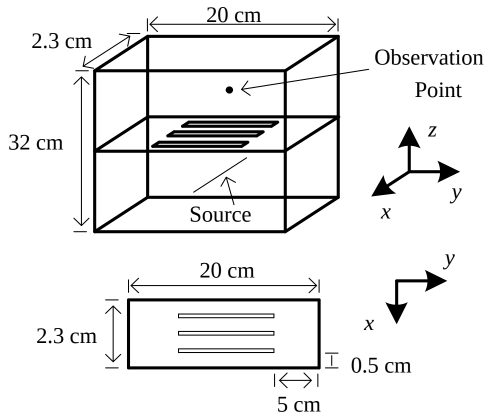

Figure 3. Enclosure with a thin metal plate.

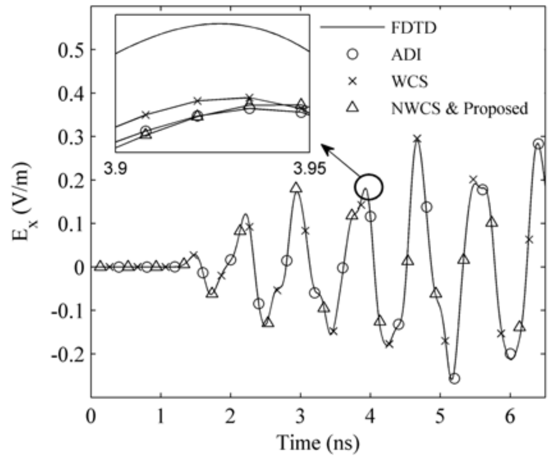

Figure 4. Time domain waveforms of E_x excited by Gaussian pulse.

Table 3. CPU time and memory requirement for different methods.

Method	Memory (MB)	CPU Time (s)
FDTD	3.5	710.3
ADI	5.3	268.2
WCS	4.7	136.7
NWCS	4.1	215.9
Proposed	7.1	87.5

plate that is set in the middle of the enclosure to divide it into two equal parts. The enclosure is assumed to be a cuboid box with the size of $2.3\,\text{cm} \times 20\,\text{cm} \times 32\,\text{cm}$. Three narrow slots of 10 cm length and 0.1 cm width are cut on the thin metal plate. A current source along x-direction with a Gaussian-pulse temporal variation is placed at the center of the lower part of the enclosure. The observation point is set at the central point of the upper part of the enclosure. To resolve the fine-scale geometric detail of the narrow slots in the x-direction, we choose the spatial grid sizes $\Delta x = 0.02\,\text{cm}$ and $\Delta y = \Delta z = 1\,\text{cm}$. Thus, the total mesh dimensions are $115 \times 20 \times 32$ cells in the x-,y- and z-directions, respectively.

Figure 4 shows the electric field component E_x at the observation point calculated by using the five methods mentioned above. The CFLN is 1 for the FDTD method and 20 for the other methods. It can be seen that the numerical results of these methods are close to each other. Moreover, due to the large dispersion error, the results of the ADI-FDTD method and the three WCS-FDTD methods slightly deviated from that of the FDTD method.

Table 3 shows the comparisons of the computation time and the memory requirement of the simulation for the five methods. It can see that the memory requirement of the proposed method is larger than that of the NWCS-FDTD method, but the CPU time of the proposed method is much less than those of the other methods.

5. CONCLUSION

In this paper, a new efficiency-improved algorithm for implementing the NWCS-FDTD method is presented. The proposed algorithm has the same numerical stability and dispersion properties as the recently proposed NWCS-FDTD method, but requires the least amount of floating-point operations. Therefore, the proposed method is particularly suitable for solving electromagnetic problems with fine features in one or two spatial directions.

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