# Relativistic Bateman-Hillion Solutions for the Electromagnetic 4-Potential in Hermite-Gaussian Beams

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**Abstract**—The electromagnetic field equations are solved to give the 4-potential in Hermite-Gaussian beams as a function of both the 4-positions of the beam waist and each point in the field. These solutions are the sums of products of position-dependent complex 4-vectors and modified Bateman-Hillion functions. It is assumed that the time difference between the beam waist and each other point is equal to the distance between the points divided by the speed of light. This method is shown to generate solutions that preserve their forms under Lorentz transformations that also correspond to the well known paraxial solutions for the case of nearly parallel beams.

## 1. INTRODUCTION

Bateman [1] discovered a class of exact solutions to the linear wave equation over a hundred years ago. Hillion [2, 3] later complexified these solutions for application to wave packet and wave beam problems.

The purpose of this paper is to present exact Bateman-Hillion related solutions to the electromagnetic field equations for the 4-potential $A_\mu(x_\nu, x_\nu^r)$ in Hermite-Gaussian beams where $x_\nu$ and $x_\nu^r$ denote the 4-positions ($\mu$, $\nu = 0, 1, 2, 3$) of a point in the field and the waist of the beam respectively. It is further intended to demonstrate the usage of these solutions in calculating the field properties for continuous wave sources [4, 5].

A review of the literature on wave beams for both the continuous wave and pulsed [6] cases can be found in [7]. This review provides a lot of historical context for the current paper as it discusses Bateman type solutions in relationship to other mathematical approaches [8, 9]. In addition to the predominately linear methods treated in [7] there is also an interesting and practical body of work relating to the transmission of optical pulses in nonlinear dispersive media [10, 11].

The origin of the coordinate system for a beam is usually at the center of the waist. It follows that the spatial coordinates $x_i^r$ ($i = 1, 2.3$) of this point are zeroed out but translating the beam will make them explicit. For relativistic solutions, it must be recognized $x_i^r$ is part of a 4-vector such that the 4-potential is also dependent on the time coordinate $x_0^r$.

The fact that $A_\mu$ depends on two 4-position vectors creates a problem familiar from the treatment of two interacting relativistic particles [12, 13] that the field cannot evolve in two independent time coordinates. The known solution to be applied here is to use Dirac delta function notation to impose a relationship between the relative space-time coordinates. One notable application of this idea, in classical electrodynamics, is the derivation of the Liénard-Wiechert potentials [14] for the field experienced at one point owing to the presence of a point charge at another. In this case, it is assumed that the time difference between the points is equal to the distance between them divided by the speed of light.

The field equations for the 4-potential $A_\mu$ comprising Maxwell's equations and the Lorenz gauge condition are presented in Section 2. It is assumed that $A_\mu$ for a Gaussian beam can be expressed as

the product $a_\mu(x_\mu)\Psi(x_\mu)$ where $a_\mu(x_\mu)$ is a complex 4-vector and $\Psi$ is a Bateman-Hillion function. The exact form of $A_\mu$ is determined on inserting the trial product form into the field equations.

The solution method for Gaussian beams is generalized in section 3 to determine the 4-potentials for all modes of Hermite-Gaussian beams. The difference for the higher modes is they must be expanded in up to three products of complex vectors and Bateman-Hillion functions. The notation for expressing the solutions is made concise through the use of ladder operators for raising and lowering the beam mode.

The behavior of the exact solutions for Hermite-Gaussian beams is investigated under Lorentz transformations in Section 4. It is shown that for a relativistic observer moving parallel to the axis of the beam that wavelength and frequency of the radiation appear Doppler shifted as is to be expected. It is further confirmed that the solutions for all the beam modes are form preserving under Lorentz transformations and therefore fully relativistic.

It is conventional in treating paraxial beams to assume time-harmonic solutions and neglect the second order axial derivative term in the wave equation. In Section 5 it is shown that an equivalent method is to start from the exact 4-potentials in the beam and restrict the time separation between the waist of the beam and other points in the field to equal the axial distance between the points divided by the speed of light. The clear assumption here being the beam is parallel enough for the axial displacement of a point in the beam from the waist to be a good approximation to the total distance from the waist.

One practical motivation for investigating Bateman-Hillion solutions for beam potentials is to go beyond the paraxial approximation. It is proposed in Section 5 that progress may well be possible using a more general form of the constraint condition on the relative coordinates though a detailed investigation of this approach remains as a problem for the future.

## 2. GAUSSIAN BEAMS

Electromagnetic radiation [14] can be represented using a 4-potential $A_\mu(x_\nu)$ where $\mu$, $\nu = 0, 1, 2, 3$ and $x_\mu = (x_i, ct)$ is position in Minkowski space. The classical field equations for $A_\mu$ consist of Maxwell's equations

$$\frac{\partial^2 A_\mu}{\partial x_1^2} + \frac{\partial^2 A_\mu}{\partial x_2^2} + \frac{\partial^2 A_\mu}{\partial x_3^2} - \frac{1}{c^2}\frac{\partial^2 A_\mu}{\partial t^2} = 0 \tag{1}$$

and the Lorenz gauge condition

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{1}{c}\frac{\partial A_0}{\partial t} = 0 \tag{2}$$

where $c$ is the velocity of light. It will be assumed that $A_\mu$ also depends on the 4-position coordinates $x_\nu^r = (x_i^r, ct^r)$ of the beam waist. The superscript $r$ is used since $x_\nu^r$ will eventually be interpreted as retarded coordinates but $x_\mu$ and $x_\nu^r$ can be treated as independent for now.

The objective of this section is to find an exact circularly polarized Gaussian solution $A_\mu^{\beta 00}$ to the field Equations (1) and (2) for a beam moving in the $x_3$-direction. It will be assumed that this takes the Bateman inspired form

$$A_\mu^{\beta 00} = a_\mu^{\beta 00}(\xi_1, \xi_2, \xi_3 + c\tau)\Phi_{00}(\xi_1, \xi_2, \xi_3 + c\tau)\exp\left[\imath\left(k_3 x_3 - \omega t\right)\right] \tag{3}$$

where

$$\xi_i = x_i - x_i^r, \quad \tau = t - t^r \tag{4}$$

denote relative coordinates, $a_\mu^{\beta 00}$ is the complex position-dependent unit vector

$$a_\mu^{\beta 00}(x_\nu) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \imath\beta \\ a_3^{\beta 00}(\xi_1, \xi_2, \xi_3 + c\tau) \\ a_0^{\beta 00}(\xi_1, \xi_2, \xi_3 + c\tau) \end{pmatrix} \tag{5}$$

$\Phi_{00}$ is a scalar function, and $\beta(=\pm 1)$ is the polarization state. It is further notable that the wave number $k_3$ and the angular frequency $\omega(= ck_3)$ are both components of a wave vector $k_\mu$ such that $k_\mu x_\mu = k_3 x_3 - \omega t$.

The definition of $a_\mu^{\beta 00}$ as a unit vector implies

$$\left| a_\mu^{\beta 00} \right| = 1 + \left( a_3^{\beta 00} \right)^2 - \left( a_0^{\beta 00} \right)^2 = 1 \tag{6}$$

giving

$$a_0^{\beta 00} = -a_3^{\beta 00} \tag{7}$$

Equation (6) also admits $a_0^{\beta 00} = a_3^{\beta 00}$ though only the more analytically convenient $a_0^{\beta 00} = -a_3^{\beta 00}$ branch will be developed here.

Inserting Equation (3) into Maxwell's Equation (1) gives

$$\left[ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 2\imath k_3 \left( \frac{\partial}{\partial x_3} + \frac{1}{c}\frac{\partial}{\partial t} \right) \right] \left( a_\mu^{\beta 00} \Phi_{00} \right) = 0 \tag{8}$$

having spotted

$$\left( \frac{\partial^2}{\partial x_3^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) (a_\mu^{\beta 00} \Phi_{00}) = 0 \tag{9}$$

owing to the dependence of $a_3^{\beta 00} \Phi_{00}$ on $\xi_3$ and $\tau$ in the linear combination $\xi_3 + c\tau$. Similarly, inserting Equation (3) into the Lorenz gauge condition (2) gives

$$a_3^{\beta 00} = \frac{\imath}{2k_3} \left( \frac{a_1^{\beta 00}}{\Phi_{00}} \frac{\partial \Phi_{00}}{\partial x_1} + \frac{a_2^{\beta 00}}{\Phi_{00}} \frac{\partial \Phi_{00}}{\partial x_2} \right) \tag{10}$$

having made use of Equation (7).

The $a_1^{\beta 00}$ and $a_2^{\beta 00}$ coefficients are constants. It follows that Equation (8) reduces to the form

$$\left[ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 2\imath k_3 \left( \frac{\partial}{\partial x_3} + \frac{1}{c}\frac{\partial}{\partial t} \right) \right] \Phi_{00} = 0 \tag{11}$$

for these cases.

Subtracting Equation (11) from Equation (8) for $\mu = 3$ gives

$$\left[ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{2}{\Phi_{00}} \left( \frac{\partial \Phi_{00}}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial \Phi_{00}}{\partial x_2}\frac{\partial}{\partial x_2} \right) + 2\imath k_3 \left( \frac{\partial}{\partial x_3} + \frac{1}{c}\frac{\partial}{\partial t} \right) \right] a_3^{\beta 00} = 0 \tag{12}$$

The next step is therefore to calculate $A_\mu^{\beta 00}$ from Equations (10), (11) and (12).

Equation (11) has a Gaussian solution in the Bateman-Hillion form

$$\Phi_{00} = \frac{C_{00} L_R}{L_R + \imath\frac{1}{2}(\xi_3 + c\tau)} \exp\left[ \frac{\imath k_3(\xi_1^2 + \xi_2^2)}{\xi_3 + c\tau - \imath 2 L_R} \right] \tag{13}$$

where $C_{00}$ and $L_R$ are constants. Putting this result into Equation (10) gives

$$a_3^{\beta 00} = -\frac{a_1^{\beta 00}\xi_1 + a_2^{\beta 00}\xi_2}{\xi_3 + c\tau - \imath 2 L_R} \tag{14}$$

Equations (13) and (14) now generate the derivatives

$$\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) a_3^{\beta 00} = 0 \tag{15}$$

$$\frac{2}{\Phi_{00}} \left( \frac{\partial \Phi_{00}}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial \Phi_{00}}{\partial x_2}\frac{\partial}{\partial x_2} \right) a_3^{\beta 00} = -4\imath k_3 \frac{a_1^{\beta 00}\xi_1 + a_2^{\beta 00}\xi_2}{(\xi_3 + \tau - \imath 2 L_R)^2} \tag{16}$$

$$2\imath k_3 \left( \frac{\partial}{\partial x_3} + \frac{1}{c}\frac{\partial}{\partial t} \right) a_3^{\beta 00} = 4\imath k_3 \frac{a_1^{\beta 00}\xi_1 + a_2^{\beta 00}\xi_2}{(\xi_3 + \tau - \imath 2 L_R)^2} \tag{17}$$

confirming $\Phi_{00}$ and $a_{003}^{00}$ satisfy (12) exactly. This implies the trial solution (3) has worked giving

$$A_\mu^{\beta 00} = \frac{C_{00}L_R}{\sqrt{2}} \begin{pmatrix} 1 \\ \beta\imath \\ \frac{\xi_1+\beta\imath\xi_2}{2\imath L_R-\xi_3-c\tau} \\ \frac{-\xi_1-\beta\imath\xi_2}{2\imath L_R-\xi_3-c\tau} \end{pmatrix} \frac{\exp\left[\frac{\imath k_3(\xi_1^2+\xi_2^2)}{\xi_3+c\tau-\imath 2L_R} + \imath(k_3 x_3 - \omega t)\right]}{L_R + \imath\frac{1}{2}(\xi_3+c\tau)} \tag{18}$$

to be an exact solution of Maxwell's Equation (1) and Lorenz gauge condition (2) for the electromagnetic 4-potential in a Gaussian mode laser beam.

## 3. HERMITE-GAUSSIAN BEAMS

The calculation of the 4-potential $A_\mu^{\beta mn}$ in Hermite-Gaussian beams for values the $m$ and $n$ integers greater than zero is similar to the calculation of $A_\mu^{\beta 00}$ in Section 2 except that it will be necessary to start from a more complicated trial solution of the form

$$A_\mu^{mn} = \sum_p \sum_q a_{\mu pq}^{\beta mn}(\xi_1, \xi_2, \xi_3 + c\tau)\,\Phi_{pq}(\xi_1, \xi_2, \xi_3 + c\tau)\exp\left[\imath(k_3 x_3 - \omega t)\right] \tag{19}$$

where $a_{\mu pq}^{\beta mn}$ is the complex position-dependent vector and $p$ and $q$ are positive integers. In correspondence to the Gaussian beam case, it will be assumed that

$$\sum_p \sum_q \left(|a_{mn1}^{pq}|^2 + |a_{mn2}^{pq}|^2 + |a_{mn3}^{pq}|^2 - |a_{mn0}^{pq}|^2\right) = 1 \tag{20}$$

and

$$a_{1pq}^{\beta mn} = \delta_{mp}\delta_{nq}, \quad a_{2pq}^{\beta mn} = \imath\beta\delta_{mp}\delta_{nq}, \quad a_{3pq}^{\beta mn} = a_{0pq}^{\beta mn} \tag{21}$$

where $\delta_{mp}$ is the Kronecker delta.

Inserting Equation (19) into Maxwell's Equation (1) gives

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 2\imath k_3\left(\frac{\partial}{\partial x_3} + \frac{1}{c}\frac{\partial}{\partial t}\right)\right]\Phi_{mn} = 0 \tag{22}$$

for $\mu = 1$ or $\mu = 2$; and

$$\sum_p \sum_q \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 2\imath k_3\left(\frac{\partial}{\partial x_3} + \frac{1}{c}\frac{\partial}{\partial t}\right)\right]\left(a_{3pq}^{\beta mn}\Phi_{pq}\right) = 0 \tag{23}$$

for $\mu = 3$. Similarly, inserting Equation (19) into the Lorenz gauge condition (2) gives

$$\sum_p \sum_q a_{3pq}^{\beta mn}\Phi_{pq} = \frac{\imath}{2k_3}\left(a_1^\beta \frac{\partial \Phi_{mn}}{\partial x_1} + a_2^\beta \frac{\partial \Phi_{mn}}{\partial x_2}\right) \tag{24}$$

having put $a_1^\beta = a_{1mn}^{\beta mn}$, $a_2^\beta = a_{2mn}^{\beta mn}$ and made use of Equation (21).

Subtracting Equation (22) from Equation (23) gives

$$\sum_p \sum_q \left[\sum_{j=1}^{j=2}\left(\frac{\partial^2}{\partial x_j^2} + \frac{2}{\Phi_{pq}}\frac{\partial \Phi_{pq}}{\partial x_j}\frac{\partial}{\partial x_j}\right) + 2\imath k_3\left(\frac{\partial}{\partial x_3} + \frac{1}{c}\frac{\partial}{\partial t}\right)\right]a_{3pq}^{\beta mn} = 0 \tag{25}$$

The next step is therefore to calculate $A_\mu^{\beta mn}$ from Equations (22), (24) and (25).

The wave Equation (22) is exact but still an analogue of the paraxial wave equation. It can therefore be solved for a complete orthonormal basis set of Hermite-Gaussian functions [4]. These take the form

$$\Phi_{mn} = \frac{C_{mn}w_0}{w}H_m\left(\frac{\sqrt{2}\xi_1}{w}\right)H_n\left(\frac{\sqrt{2}\xi_2}{w}\right)\exp\left[\frac{\imath k_3(\xi_1^2+\xi_2^2)}{\xi_3+c\tau-2\imath L_R} - \imath g_{mn}\right] \tag{26}$$

where

$$w(\xi_3, t) = w_0 \sqrt{1 + \left(\frac{\xi_3 + c\tau}{2L_R}\right)^2} \tag{27}$$

is the radius of the laser spot and

$$g_{mn}(\xi_3, t) = (1 + m + n) \arctan\left(\frac{\xi_3 + c\tau}{2L_R}\right) \tag{28}$$

is the Gouy phase. Additionally, $w_0 = w(0)$ is the radius of the beam waist, $L_R = \frac{1}{2}k_3 w_0^2$ is the Rayleigh range and $H_m$ and $H_n$ are Hermite polynomials.

Inserting Equation (26) into Equation (24) gives

$$\sum_p \sum_q \left(a_{3pq}^{\beta mn}\Phi_{pq}\right) = \frac{-\left(a_1^\beta \xi_1 + a_2^\beta \xi_2\right)\Phi_{mn} - w_0\left(a_1^\beta \sqrt{m}\Phi_{m-1n} + a_2^\beta \sqrt{n}\Phi_{mn-1}\right)}{\xi_3 + c\tau - 2\imath L_R} \tag{29}$$

having used the following expressions [15]:

$$\frac{\partial}{\partial x}H_{mn}(\xi) = 2mH_{m-1n}(\xi), \tag{30}$$

$$\exp\left[-\imath \arctan\left(\frac{\xi_3 + c\tau}{2L_R}\right)\right] = -\imath\frac{\xi_3 + c\tau + 2\imath L_R}{\sqrt{(\xi_3 + c\tau)^2 + 4L_R^2}} \tag{31}$$

and $C_{m-1n} = \sqrt{2m}C_{mn}$. Here, it is understood $\Phi_{m-1n} = 0$ if $m = 0$ and $\Phi_{mn-1} = 0$ if $n = 0$.

Reading off the $a_{3pq}^{\beta mn}$ coefficients from Equation (29) gives

$$a_{3mn}^{\beta mn} = -\frac{a_1^\beta \xi_1 + a_2^\beta \xi_2}{\xi_3 + c\tau - 2\imath L_R} \tag{32}$$

$$a_{3m-1n}^{\beta mn} = -\frac{a_1^\beta w_0 \sqrt{m}}{\xi_3 + c\tau - 2\imath L_R} \tag{33}$$

$$a_{3mn-1}^{\beta mn} = -\frac{a_2^\beta w_0 \sqrt{n}}{\xi_3 + c\tau - 2\imath L_R} \tag{34}$$

It is now readily confirmed that Equation (26) alongside the coefficients (32) through (34) satisfy Equation (25) using the following derivatives:

$$\frac{\partial^2 a_{3pq}^{\beta mn}}{\partial x_1^2} = \frac{\partial^2 a_{3pq}^{\beta mn}}{\partial x_2^2} = \frac{\partial a_{3m-1n}^{\beta mn}}{\partial x_1} = \frac{\partial a_{3mn-1}^{\beta mn}}{\partial x_2} = 0 \tag{35}$$

$$\frac{\partial \Phi_{mn}}{\partial x_1}\frac{\partial a_{3mn}^{\beta mn}}{\partial x_1} = -\frac{2\imath k_3 a_1^\beta \xi_1}{(\xi_3 + c\tau - 2\imath L_R)^2}\Phi_{mn} - \frac{2\imath k_3 a_1^\beta w_0 \sqrt{m}}{(\xi_3 + c\tau - 2\imath L_R)^2}\Phi_{m-1n} \tag{36}$$

$$\frac{\partial \Phi_{mn}}{\partial x_2}\frac{\partial a_{3mn}^{\beta mn}}{\partial x_2} = -\frac{2\imath k_3 a_2^\beta \xi_2}{(\xi_3 + c\tau - 2\imath L_R)^2}\Phi_{mn} - \frac{2\imath k_3 a_2^\beta w_0 \sqrt{n}}{(\xi_3 + c\tau - 2\imath L_R)^2}\Phi_{mn-1} \tag{37}$$

$$\imath k_3 \Phi_{mn}\left(\frac{\partial}{\partial x_3} + \frac{1}{c}\frac{\partial}{\partial t}\right)a_{3mn}^{\beta mn} = \frac{2\imath k_3(a_1^\beta \xi_1 + a_2^\beta \xi_2)}{(\xi_3 + c\tau - 2\imath L_R)^2}\Phi_{mn} \tag{38}$$

$$\imath k_3 \Phi_{m-1n}\left(\frac{\partial}{\partial x_3} + \frac{1}{c}\frac{\partial}{\partial t}\right)a_{3m-1n}^{\beta mn} = \frac{2\imath k_3 a_1^\beta w_0 \sqrt{m}}{(\xi_3 + c\tau - 2\imath L_R)^2}\Phi_{m-1n} \tag{39}$$

$$\imath k_3 \Phi_{mn-1}\left(\frac{\partial}{\partial x_3} + \frac{1}{c}\frac{\partial}{\partial t}\right)a_{3mn-1}^{\beta mn} = \frac{2\imath k_3 a_2^\beta w_0 \sqrt{n}}{(\xi_3 + c\tau - 2\imath L_R)^2}\Phi_{mn-1} \tag{40}$$

This implies the trial solution (19) has been successful giving

$$A_\mu^{\beta mn} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \imath\beta \\ \frac{\xi_1 + \hat{\alpha}_1 + \imath\beta(\xi_2 + \hat{\alpha}_2)}{2\imath L_R - \xi_3 - c\tau} \\ \frac{-\xi_1 - \hat{\alpha}_1 - \imath\beta(\xi_2 + \hat{\alpha}_2)}{2\imath L_R - \xi_3 - c\tau} \end{pmatrix} \Phi_{mn}(\xi_1, \xi_2, \xi_3 + c\tau) \exp(\imath k_3 x_3 - \imath\omega t) \tag{41}$$

where

$$\hat{\alpha}_1 \Phi_{mn} = \left( \frac{\xi_3 + c\tau - 2\imath L_R}{k_3} \frac{\partial}{\partial x_1} - 2\imath\xi_1 \right) \Phi_{mn} = \sqrt{m} w_0 \Phi_{m-1n} \tag{42}$$

$$\hat{\alpha}_2 \Phi_{mn} = \left( \frac{\xi_3 + c\tau - 2\imath L_R}{k_3} \frac{\partial}{\partial x_2} - 2\imath\xi_2 \right) \Phi_{mn} = \sqrt{n} w_0 \Phi_{mn-1} \tag{43}$$

are lowering operators. Equation (41) is therefore an exact solution to Maxwell's Equation (1) and the Lorenz gauge condition (2) for the electromagnetic 4-potential in a Hermite-Gaussian mode laser beam.

## 4. LORENTZ TRANSFORMATIONS

The electromagnetic 4-potential $A_\mu^{\beta mn}(x_\nu)$ has been determined in Equation (41) to be an exact solution of five simultaneous manifestly covariant classical electromagnetic field Equations (1) and (2). The task ahead is to confirm these solutions are form preserving under Lorentz transformations.

A 4-vector $q_\mu$ is relativistic if it preserves its form under the Lorentz transformation equations:

$$q_i' = q_i + \gamma\frac{v_i}{c} \left( \frac{\gamma}{1+\gamma} \frac{v_j q_j}{c} - q_0 \right) \tag{44}$$

$$q_0' = \gamma \left( q_0 - \frac{v_j q_j}{c} \right) \tag{45}$$

[14] where $v_i$ is the relative velocity between any two inertial reference frames and $\gamma = (1 - v^2/c^2)^{-1/2}$. For current purposes $q_\mu$ belongs to the set of position $x_\mu$, relative position $\xi_\mu$, wave vector $k_\mu$ and 4-potential $A_\mu^{\beta mn}$.

To confirm the Hermite-Gaussian beam solutions are fully relativistic it will be necessary to investigate the Lorentz transformation of number of different quantities including the phase factor, the Gaussian function, the Hermite functions, the Gouy phase and the position-dependent complex 4-vector. For brevity, the analysis will be limited to the case of an observer that is moving parallel to the axis of the beam.

The product $k_\mu x_\mu$ is Lorentz covariant implying $k_\mu' x_\mu' = k_\mu x_\mu$. This result can also be expressed as

$$k_3' \left( x_3' - ct' \right) = k_3 \left( x_3 - ct \right) \tag{46}$$

where

$$ck_3' = ck_0' = \omega' \tag{47}$$

$$k_3' = \sqrt{\frac{c-v}{c+v}} k_3 \tag{48}$$

This is the relativistic Doppler effect.

The numerator and denominator in the Gaussian component of Equation (26) can be transformed separately to give

$$k_3' \left( \xi_1'^2 + \xi_2'^2 \right) = \sqrt{\frac{c-v}{c+v}} k_3 \left( \xi_1^2 + \xi_2^2 \right) \tag{49}$$

$$\xi_3' + c\tau' - 2\imath L_R' = \sqrt{\frac{c-v}{c+v}} \left( \xi_3 + c\tau - 2\imath L_R \right) \tag{50}$$

having assumed that

$$L'_R = \sqrt{\frac{c-v}{c+v}} L_R \tag{51}$$

given $L_R = \frac{1}{2} k_3 w_0^2$ and assuming the radius $w_0$ is a Lorentz invariant scalar. Putting these results together leads to

$$\frac{k'_3 \left(\xi_1'^2 + \xi_2'^2\right)}{\xi'_3 + c\tau' - 2\imath L'_R} = \frac{k_3 \left(\xi_1^2 + \xi_2^2\right)}{\xi_3 + c\tau - 2\imath L_R} \tag{52}$$

Equations (49) through (52) can now be used to transform Equation (26) into

$$\Phi_{mn} = \frac{C_{mn} w_0}{w'} H_m \left(\frac{\sqrt{2}\xi'_1}{w'}\right) H_n \left(\frac{\sqrt{2}\xi'_2}{w'}\right) \exp\left[\frac{\imath k'_3 (\xi_1'^2 + \xi_2'^2)}{\xi'_3 + c\tau' - 2\imath L'_R} - \imath g'_{mn}\right] \tag{53}$$

where

$$w' = w_0 \sqrt{1 + \left(\frac{\xi'_3 + c\tau'}{2L'_R}\right)^2} = w_0 \sqrt{1 + \left(\frac{\xi_3 + ct}{2L_R}\right)^2} \tag{54}$$

$$\frac{g'_{mn}}{(1+m+n)} = \arctan\left(\frac{\xi'_3 + c\tau'}{2L'_R}\right) = \arctan\left(\frac{\xi_3 + c\tau}{2L_R}\right) \tag{55}$$

and $C_{mn}$ is a Lorentz invariant scalar.

The components of $A_\mu^{\beta mn}$ transform as

$$A_1^{\beta mn\prime} = A_1^{\beta mn} \qquad A_2^{\beta mn\prime} = A_2^{\beta mn} \tag{56}$$

$$A_3^{\beta mn\prime} = \gamma \left(A_3^{\beta mn} - \frac{v}{c} A_0^{\beta mn}\right) = \sqrt{\frac{c-v}{c+v}} A_3^{\beta mn} \tag{57}$$

$$A_0^{\beta mn\prime} = \gamma \left(A_0^{\beta mn} - \frac{v}{c} A_3^{\beta mn}\right) = \sqrt{\frac{c-v}{c+v}} A_0^{\beta mn} \tag{58}$$

having used the equality $A_0^{\beta mn} = A_3^{\beta mn}$. It follows from taking stock of all of these results that Equation (41) can be rewritten as

$$A_\mu^{\beta mn\prime} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \imath\beta \\ \frac{\xi'_1 + \hat{\alpha}'_1 + \imath\beta\left(\xi'_2 + \hat{\alpha}'_2\right)}{2\imath L'_R - \xi'_3 - c\tau'} \\ \frac{-\xi'_1 + \hat{\alpha}'_1 - \imath\beta\left(\xi'_2 + \hat{\alpha}'_2\right)}{2\imath L'_R - \xi'_3 - c\tau'} \end{pmatrix} \Phi_{mn} \left(\xi'_1, \xi'_2, \xi'_3 + c\tau'\right) \exp\left(\imath k'_3 x'_3 - \imath \omega' t'\right) \tag{59}$$

where

$$\hat{\alpha}'_j = \left(\frac{\xi'_3 + c\tau' - 2\imath L'_R}{k'_3} \frac{\partial}{\partial x'_j} - 2\imath \xi'_j\right) \tag{60}$$

and $j = 1, 2$. On comparing Equations (41) and (59) it is concluded the form of $A_\mu^{\beta mn}$ is form invariant under Lorentz transformations and therefore fully relativistic.

## 5. PARAXIAL BEAMS AND BEYOND

Paraxial beams are well documented in the literature [4, 5]. It is of interest therefore to establish the connection between the foregoing exact solutions and the standard form solutions:

$$F_{\mu\nu}^{\beta mnP} = F_{\mu\nu}^{\beta mn0} \Phi_{mn}^P \left(\xi_1, \xi_2, \xi_3\right) \exp\left[\imath \left(k_3 x_3 - \omega t\right)\right] \tag{61}$$

where $\Phi_{mn}^P$ is the solution of the paraxial wave equation

$$\frac{\partial^2 \Phi_{mn}^P}{\partial x_1^2} + \frac{\partial^2 \Phi_{mn}^P}{\partial x_2^2} + 2\imath k_3 \frac{\partial \Phi_{mn}^P}{\partial x_3} = 0 \tag{62}$$

$F_{\mu\nu}^{\beta mnP}$ is the the electromagnetic field tensor, and the components of $F_{\mu\nu}^{\beta mn0}$ are constants. For comparison to the literature, note it is conventional to choose the waist of the beam to be the origin of the coordinate system such that $\xi_i = x_i$.

Equation (62) gives Hermite-Gaussian solutions of the form

$$\Phi_{mn}^P = \frac{C_{mn}w_0}{w}H_m\left(\frac{\sqrt{2}\xi_1}{w}\right)H_n\left(\frac{\sqrt{2}\xi_2}{w}\right)\exp\left[\frac{ik_3\left(\xi_1^2 + \xi_2^2\right)}{2\left(\xi_3 - iL_R\right)} - ig_{mn}\right] \tag{63}$$

where

$$w(\xi_3) = w_0\sqrt{1 + \left(\frac{\xi_3}{L_R}\right)^2} \tag{64}$$

$$g_{mn}(\xi_3) = (1 + m + n)\arctan\left(\frac{\xi_3}{L_R}\right) \tag{65}$$

It follows that $\Phi_{mn}^P$ is related to the exact solution (26) to the full wave equation through the integral expression

$$\Phi_{mn}^P = \int_{-\infty}^{+\infty}\Phi_{mn}\left(x_\mu\right)\delta\left(\xi_3 - c\tau\right)d\tau \tag{66}$$

where $\delta(\xi_3 - c\tau)$ is a Dirac delta function.

It follows that there are two equivalent methods for arriving at the paraxial approximation for electromagnetic beams. One is the traditional method to neglect the time dependence of $\Phi_{mn}$ and the second order axial derivative in the full wave equation. The other is to start from the exact Bateman-Hillion solution of the full wave equation and restrict it using a delta function that requires phase fronts to propagate at the speed of light along the axis of the beam.

The delta function method can also be applied to the exact 4-potential (41) to give the paraxial approximation

$$A_\mu^{\beta mnP} = \int_{-\infty}^{+\infty}A_\mu^{\beta mn}\delta\left(\xi_3 - c\tau\right)d\tau \tag{67}$$

This leads to

$$A_\mu^{\beta mnP} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\beta\\ \frac{\xi_1 + \hat{\alpha}_1 + i\beta(\xi_2 + \hat{\alpha}_2)}{2(iL_R - \xi_3)}\\ \frac{-\xi_1 - \hat{\alpha}_1 - i\beta(\xi_2 + \hat{\alpha}_2)}{2(iL_R - \xi_3)}\end{pmatrix}\Phi_{mn}^P\left(\xi_1, \xi_2, \xi_3\right)\exp\left(ik_3x_3 - i\omega t\right) \tag{68}$$

where

$$\hat{\alpha}_j = \frac{2\left(\xi_3 - iL_R\right)}{k_3}\frac{\partial}{\partial x_j} - 2i\xi_j \tag{69}$$

and $j = 1, 2$. The relationship between the paraxial field tensor $F_{\mu\nu}^P$ and the paraxial 4-potential $A_\mu^{\beta mnP}$ is

$$F_{\mu\nu}^{\beta mnP} = \frac{\partial A_\nu^{\beta mnP}}{\partial x_\mu} - \frac{\partial A_\mu^{\beta mnP}}{\partial x_\nu} \tag{70}$$

This gives Equation (61) providing many small terms are neglected including $A_3^{\beta mnP}$ and $A_0^{\beta mnP}$.

Finally, it is instructive to compare Equation (68) to the Liénard-Wiechert potentials that describe the electromagnetic field around a point charge. Specifically, the role of the waist of the beam is analogous to the location of the charge in the sense both have retarded coordinates of the form $(x_i^r, t^r)$ that have the property of being related to all other points in the field. In the case of the beam potential, these relationships take the form

$$x_i = x_i^r + \xi_i, \quad t = t^r + \frac{\xi_3}{c} \tag{71}$$

It is therefore concluded that $A_\mu^{\beta mnP}$ for paraxial beams is a retarded potential.

For problems beyond the paraxial limit it is proposed to use

$$\Phi_{mn}^S = \int_{-\infty}^{+\infty} \Phi_{mn}(x_\mu)\,\delta(r - c\tau)\,d\tau \tag{72}$$

where $r(= \sqrt{\xi_1^2 + \xi_2^2 + \xi_2^2})$ replaces $\xi_3$ assuming that the beam evolves from the waist to have spherical rather than planar phase fronts, though this solution has yet to be studied in detail.

## 6. SUMMARY

Exact solutions have been derived to the field equations for the electromagnetic 4-potential in Hermite-Gaussian beams using Bateman-Hillion functions that depend on the relative time between the waist and other points in the beam. These solutions have been shown to preserve their form under Lorentz transformations. It has also been shown that there are two equivalent methods to obtain the 4-potentials in paraxial beams. One is to simplify the field equations into paraxial form and solve them directly. The other is to start from the exact 4-potential in the beams and restrict the relative time between the waist and other points in the beam to equal the axial distance between them divided by the speed of light.

In order to go beyond the paraxial approximation it has been proposed to calculate the relative time between the waist and other points in the field in terms of the total distance between them instead of using just axial component of distance. A detailed investigation of this approach, however, remains as a problem for the future.

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