

# An Exactly-Solvable Quasistatic Electricity Inverse Problem: Retrieval of the Complex Permittivity of a Cylinder Taking Account of Nuisance Parameter Uncertainty

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**Abstract**—This study concerns the 2D inverse problem of the retrieval, using external field data, of either one of the two physical parameters, constituted by the real and imaginary parts of the permittivity, of a  $z$ -independent cylindrical dielectric specimen subjected to an external,  $z$ -independent, quasistatic electric field. Six other parameters enter into the inverse problem. They are termed nuisance parameters because: 1) they are not retrieved during the inversion and 2) uncertainty as to their actual values can adversely affect the accuracy of the retrieval of the permittivity. This inverse problem is shown to have an exact, mathematically-explicit, solution, both for continuous and discrete input data, whose properties, with respect to the various nuisance parameter uncertainties, are analyzed for noiseless data. It is found that: a) optimal inversion requires data registered at only a small number of sensors, b) the inverse solution, satisfying pre-existing physical constraints, exists and is unique. Moreover, the inverse solution is shown to be unstable with respect to three nuisance parameter uncertainties, the consequence of which is large retrieval inaccuracy for small nuisance parameter uncertainties, acting either individually or in combination.

## 1. INTRODUCTION

The retrieval of the complex permittivity (or related physical parameters such as the dielectric constant, index of refraction, absorption coefficient, . . .) of a homogeneous, isotropic material is a theoretical and experimental electromagnetic *inverse* (although only relatively-recently recognized as such) problem of considerable importance. The reason for this is that permittivity is a sensitive indicator of the chemical [2] and physical identity of natural and man-made materials [17] and of their state (notably in quality control and health monitoring [21] applications) [19, 22].

The usual techniques (comparison of capacities, resonant circuits, transmission lines, dielectrometry, reflectometry, refractometry, ellipsometry) rely (to match theory to measurement) on the possibility of obtaining homogeneous specimens of prescribed (usually-simple) shape (often blocks, cylinders, spheres, slabs) and size (e.g., films). In naturally-occurring materials, this is not always possible. For instance, in studies of natural phenomena connected with the scattering of light (interstellar dust, air-borne pollution, powders, granular media and other divided matter characterizations, material characterization of living bodies (cells, phytoplankton, etc.), the specimens can have complicated shapes (although they are often considered to be spherical or cylindrical [3, 6]) and are too small (e.g., films so thin that the matter therein appears to be divided) to be examined by the previously-mentioned techniques [3, 6, 14, 20]. It is thus increasingly recognized that discrepancies between the assumed and actual: size, shape and composition of the specimen (divided versus homogeneous, such as in colloids and metamaterials [4, 5, 18]) have to be taken into account in connection with the meaning

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that is attached to the permittivity determined from the response of the specimens to quasistatic or dynamic (wave-like) electric fields (the latter response incorporates diffraction and/or collective effects, not ordinarily accounted-for in methods relying on reflective or refractive response fields). A second trend of permittivity retrieval inverse problems is the recognition of the necessity of taking into account the *uncertainty* (of the experimental results [20]) of certain parameters (and their sensitivity [5]) that enter into the retrieval model, and of the mathematical ingredients of the retrieval model itself [15], in order to evaluate the accuracy of the retrieved parameters, e.g., [7, 9, 10, 20]. The present investigation is inspired by these two trends.

We shall be concerned with an isotropic, generally-inhomogeneous, lossy dielectric medium in which the displacement and the conduction current density are related to the electric field by a dielectric constant  $\varepsilon'(\mathbf{x}, \omega)$  and loss factor  $\varepsilon''(\mathbf{x}, \omega)$  respectively via the constitutive relations:

$$\mathbf{D}(\mathbf{x}, \omega) = \varepsilon'(\mathbf{x}, \omega)\mathbf{E}(\mathbf{x}, \omega), \quad \mathbf{J}^c(\mathbf{x}, \omega) = \omega\varepsilon''(\mathbf{x}, \omega)\mathbf{E}(\mathbf{x}, \omega), \quad (1)$$

wherein:  $\omega$  is the angular frequency (Hz),  $\mathbf{x}$  a position vector, and  $\varepsilon'$  and  $\varepsilon''$  are generally (for passive media) positive (or zero) scalar functions for  $\omega \geq 0$ . In the quasistatic regime and in the absence of impressed bulk charges, it ensues (also from  $\mathbf{E} = -\nabla\psi$ ,  $\psi$  the quasistatic electric potential) that

$$\nabla \cdot [\varepsilon(\mathbf{x}, \omega)\nabla\psi(\mathbf{x}, \omega)] = 0, \quad (2)$$

wherein  $\varepsilon(\mathbf{x}, \omega) = \varepsilon'(\mathbf{x}, \omega) + i\varepsilon''(\mathbf{x}, \omega)$  is the (complex) permittivity. Henceforth, we drop the symbol  $\omega$ , considered to be implicit.

Let  $\mathbb{R}^n$  be divided into two domains  $\mathfrak{D}_0$  and  $\mathfrak{D}_1$ , separated by the interface  $\mathfrak{I}$ , the unit vector normal to which is  $\boldsymbol{\nu}$ . Let  $\mathfrak{M}_0$  and  $\mathfrak{M}_1$  be two homogeneous, isotropic dielectric media (filling  $\mathfrak{D}_0$  and  $\mathfrak{D}_1$  respectively) in which the position-independent permittivities are  $\varepsilon_0$  and  $\varepsilon_1$ , respectively.

If all space (i.e.,  $\mathfrak{D}_0 + \mathfrak{D}_1$ ) is initially occupied solely by  $\mathfrak{M}_0$  and devoid of impressed charges, but subjected to a uniform electric field  $\mathbf{E}^i$  satisfying  $\mathbf{E}^i(\mathbf{x}) = -\nabla\psi^i(\mathbf{x})$ , then the introduction of  $\mathfrak{M}_1$  into  $\mathfrak{D}_1$  induces a potential  $\psi_0^d$  in  $\mathfrak{D}_0$  so that the total potentials are now

$$\psi_0(\mathbf{x}) = \psi^i(\mathbf{x}) + \psi_0^d(\mathbf{x}) \quad \text{in } \mathfrak{D}_0, \quad \psi_1(\mathbf{x}) = \psi_1^d(\mathbf{x}) \quad \text{in } \mathfrak{D}_1. \quad (3)$$

Then the problem of the prediction of  $\psi_l$ ;  $l = 0, 1$ , for given  $\mathbf{E}^i$  or  $\psi^i$ , can be cast in the three-relation form (equivalent to (2))

$$\nabla \cdot (\nabla\psi_l)(\mathbf{x}) = 0 \quad \text{in } \mathfrak{D}_l, \quad l = 0, 1, \quad (4)$$

$$\psi_0(\mathbf{x}) - \psi_1(\mathbf{x}) = 0 \quad \text{on } \mathfrak{I}, \quad (5)$$

$$\varepsilon_0(\mathbf{x})\boldsymbol{\nu}(\mathbf{x}) \cdot \nabla\psi_0(\mathbf{x}) - \varepsilon_1(\mathbf{x})\boldsymbol{\nu}(\mathbf{x}) \cdot \nabla\psi_1(\mathbf{x}) = 0 \quad \text{on } \mathfrak{I}, \quad (6)$$

with uniqueness assured by the condition:

$$\left| \psi_l^d(\mathbf{x}) \right| < \infty \quad \text{in } \mathfrak{D}_l; \quad l = 0, 1. \quad (7)$$

In the preceding lines, the emphasis has been on the *forward problem* of the prediction of the potential field  $\psi$ , assuming that all other ingredients of the configuration and of the solicitation (via  $\psi^i$ ) are known. Actually, the present investigation is more specifically concerned with the *inverse problem* (examples of which can be found in [1, 6, 7, 9–11, 13, 16]) of the retrieval of  $\varepsilon$  (or, more precisely, of  $\varepsilon_1$ ) from data relative to  $\psi$  (more precisely,  $\psi_0$ ), assuming that all other parameters of the configuration as well as of the solicitation (i.e., the nuisance parameters [7]) are more or less well-known (i.e., uncertain to some degree). We qualify a parameter as being uncertain by the fact that its assigned value, resulting from experiment, guessing or borrowing from a published result, is incorrect in a sense akin to systematic measurement error.

The chosen physical configuration (in which  $\mathfrak{D}_1$  is an infinitely-long circular cylinder) will be shown to enable both the forward and inverse problems to be solved in explicit, exact manner so as to make possible a thorough analysis (somewhat in the spirit of [7, 9, 10]) of the influence of nuisance parameter uncertainty on retrieval accuracy. This point merits to be emphasized because it is not often that an other-than-academic inverse problem can be solved exactly, and it is not commonplace in parameter retrieval problems to be able to evaluate analytically the influence of nuisance prior uncertainty on the accuracy of the retrievals.

## 2. DESCRIPTION OF THE PHYSICAL CONFIGURATION

A circular cylinder, occupied by the homogeneous, isotropic medium  $\mathfrak{M}_1$  (in which the permittivity is  $\varepsilon_1$ ) is introduced into another homogeneous, isotropic medium  $\mathfrak{M}_0$  (in which the permittivity is  $\varepsilon_0$ ) of infinite extent and is submitted to an electric field  $\mathbf{E}^i$  whose direction is assumed to be constant at all points of space. The  $z$  axis (of the cartesian coordinate system  $Oxyz$  forms the axis of the cylinder and the circular disk  $\Omega_1$ , with center at  $O$ , constitutes the support of the cylinder in the  $x-y$  plane. The unbounded region exterior to  $\Omega_1$  (in the  $x-y$  plane) is  $\Omega_0$ .

The incident electric field vector  $\mathbf{E}^i$  is assumed to lie in the  $x-y$  plane and independent of  $z$ . The circular boundary of  $\Omega_1$  is  $\Gamma$ , the outward unit vector normal to which is  $\nu$ . Consequently, the incident and induced fields are independent of  $z$ , i.e., the problem is two-dimensional, with  $z$  the ignorable coordinate. In the so-called forward problem, the task is to predict the secondary field  $\mathbf{E}^d = -\nabla\psi^d$  induced by the primary field  $\mathbf{E}^i = -\nabla\psi^i$ , whereas in the inverse problem, the associated potential  $\psi^d$  (combined with  $\psi^i$ ) constitutes the data, which, by means of an inversion scheme, is analyzed to enable the retrieval of the constitutive parameter  $\varepsilon_1$  of  $\mathfrak{M}_1$ . The units of  $\mathbf{E}$ ,  $\psi$ ,  $\varepsilon$ ,  $a$ ,  $\theta^i$ ,  $b$  and  $\theta$  are: volt/m, volt, farad/m, m, ° or rad, m, ° or rad respectively.

With  $r, \theta$  the polar coordinates of a point  $P$  in the  $x-y$  plane and  $\mathbf{x}$  the vector joining  $O$  to  $P$ , the parametric equation of  $\Gamma$  is  $r = a$ ;  $\forall \theta \in [0, 2\pi[$ ,  $a$  being the radius of the circular disk  $\Omega_1$ . The (incident) angle between  $\mathbf{E}^i$  and the  $x$  axis is  $\theta^i$ . The total (primary plus secondary) potential field is sensed at various points on a circle (concentric with  $\Gamma$ ) of radius  $b > a$ . The polar angle at which a generic point-like sensor is located is  $\theta$ .

The first objective is: given  $e^i$  (amplitude of  $\psi^i$ ),  $\theta^i$ ,  $a$  and  $\varepsilon_l$ ;  $l = 0, 1$ , find the total potential fields at one or more positions (starting with  $\theta = \theta^b$ ) on the circle  $r = b$ . It is assumed that the location of the axis of the cylinder is known (and coincides with the  $z$ -axis) and that the number, and angular, positions of the point-like (in the  $x-y$  plane) sensors are perfectly well-known in both the forward and inverse problem contexts.

The second objective is: given the total potential fields registered at one or more sensors located at angular positions starting with  $\theta = \theta^b$  on the circle  $r = B$  (analogous to, but different from,  $b$  due to uncertainty of this parameter), as well as the set of parameters  $E^i$ ,  $\Theta^i$ ,  $A$  and  $\mathcal{E}_0$  (analogous to  $e^i$ ,  $\theta^i$ ,  $a$  and  $\varepsilon_0$ , but integrating uncertainties), find (in separate steps)  $\mathcal{E}'_1$  (analogous to  $\varepsilon'_1$ ) and  $\mathcal{E}''_1$  (analogous to  $\varepsilon''_1$ ). Actually, the *exact* solution (to simulate measured data concerning the total field), of the forward problem, will be employed as the data (the corresponding model is termed the *data simulation model*) to solve the inverse problem in the second part of the document. In addition, we appeal to a *parameter retrieval model*, also based on the aforementioned physical configuration, to recover the constitutive parameter of the cylinder. Some of the fixed (during the inversion) parameters (the so-called *nuisance parameters*) of the retrieval models will be assumed to be not precisely known, and their effect on the accuracy of the retrievals will be studied in depth.

## 3. EXACT SOLUTION OF THE FORWARD PROBLEM

The assumed primary electric potential is

$$\psi^i(r, \theta) = -e^i r \cos(\theta - \theta^i), \quad (8)$$

with  $e^i$  a constant amplitude term. Separating variables in (4) and applying the conditions (5)–(7), gives

$$\psi_0(\mathbf{x}) - \psi^i(\mathbf{x}) = \psi_0^d(\mathbf{x}) = e^i \frac{a^2}{r} \left( \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + \varepsilon_0} \right) \cos(\theta - \theta^i), \quad (9)$$

which is identical to that [12, p. 1185], when  $\theta^i = 0$ .

## 4. EXACT SOLUTION OF THE INVERSE PROBLEM

### 4.1. General Considerations

The inversion consists in obtaining estimates  $\mathcal{E}'_1$ ,  $\mathcal{E}''_1$  of the sought-for constitutive parameters  $\varepsilon'_1$ ,  $\varepsilon''_1$  by minimizing a cost functional which expresses the discrepancy between the data simulation model ( $\psi_0$ )

of the electric potential and the parameter retrieval model ( $\Psi_0$ ) of the electric potential on the circle of radius  $b$  (which becomes  $B$  if it is uncertain) in the angular interval of observation  $[0, 2\pi[$ . This cost functional is:

$$\mathcal{K}(\mathcal{E}'_1, \mathcal{E}''_1) = \frac{\int_0^{2\pi} \left\| \psi_0(b, \theta | e^i, \theta^i, a, \varepsilon_0, \varepsilon'_1, \varepsilon''_1) - \Psi_0(B, \theta | E^i, \Theta^i, A, \mathcal{E}_0, \mathcal{E}'_1, \mathcal{E}''_1) \right\|^2 d\theta}{\int_0^{2\pi} \left\| \psi_0(b, \theta | e^i, \theta^i, a, \varepsilon_0, \varepsilon_1) \right\|^2 d\theta}. \quad (10)$$

$\mathcal{K}$  is actually replaced, in the numerical context, and to account for the discrete nature of the physical sensing process, by another cost functional obtained by adopting a simple quadrature rule for the integrals:

$$\mathcal{K} \approx \mathcal{K}^{(N)}(\mathcal{E}'_1, \mathcal{E}''_1) = \frac{\delta_\theta \sum_{n=1}^N \left\| \psi_0(b, \theta_n | e^i, \theta^i, a, \varepsilon_0, \varepsilon'_1, \varepsilon''_1) - \Psi_0(B, \theta_n | E^i, \Theta^i, A, \mathcal{E}_0, \mathcal{E}'_1, \mathcal{E}''_1) \right\|^2}{\delta_\theta \sum_{n=1}^N \left\| \psi_0(b, \theta_n | e^i, \theta^i, a, \varepsilon_0, \varepsilon'_1, \varepsilon''_1) \right\|^2}, \quad (11)$$

wherein  $\delta_\theta = 2\pi/N$ , and  $\theta_n = \theta^b + \frac{\delta_\theta}{2} + (n-1)\delta_\theta$  are the actual angles (for  $n = 1, 2, \dots, N$ ;  $\theta^b$  a chosen starting angle and  $\theta^e$  a chosen ending angle) at which the electric potential is sensed.

Note that the set of parameters (lower case letters and symbols) is different in the simulated data model from the corresponding set (upper case letters and symbols) in the parameter retrieval model; this expresses the fact that the subset of nuisance parameters (all the parameters except  $\varepsilon_1$ ) may be not well known to us before and during the inversion.

The fact that

$$\psi_0(b, \theta) = f(e^i, a, b, \varepsilon_0, \varepsilon'_1, \varepsilon''_1) \cos(\theta - \theta^i), \quad \Psi_0(B, \theta) = \mathfrak{F}(E^i, A, B, \mathcal{E}_0, \mathcal{E}'_1, \mathcal{E}''_1) \cos(\theta - \Theta^i), \quad (12)$$

leads to:

$$\mathcal{K}(\mathcal{E}_1) = \frac{\int_0^{2\pi} \left\| f \cos(\theta - \theta^i) - \mathfrak{F} \cos(\theta - \Theta^i) \right\|^2 d\theta}{\int_0^{2\pi} \left\| f \cos(\theta - \theta^i) \right\|^2 d\theta}. \quad (13)$$

and

$$\mathcal{K}^{(N)}(\mathcal{E}_1) = \frac{\delta_\theta \sum_{n=1}^N \left\| f \cos(\theta_n - \theta^i) - \mathfrak{F} \cos(\theta_n - \Theta^i) \right\|^2 d\theta}{\delta_\theta \sum_{n=1}^N \left\| f \cos(\theta_n - \theta^i) \right\|^2 d\theta}. \quad (14)$$

With  $G$  any one of the capital-letter parameters and  $\kappa = \cos(\Theta^i - \theta^i)$ , (13) leads to

$$\mathcal{K}(G) = \frac{\|f\|^2 - 2\kappa \Re(f^* \mathfrak{F}) + \|\mathfrak{F}\|^2}{\|f\|^2}, \quad (15)$$

wherein the symbol  $*$  designates the complex conjugate operator. An extremum of the cost functional with respect to the variable  $G$  (henceforth,  $\mathcal{E}'$  or  $\mathcal{E}''$ ) is found for  $\frac{\partial \mathcal{K}(G)}{\partial G} = 0$  or

$$\Re\left(\frac{\partial \mathfrak{F}^*}{\partial G}\right) \Re(\mathfrak{F} - \kappa f) - \Im\left(\frac{\partial \mathfrak{F}^*}{\partial G}\right) \Im(\mathfrak{F} - \kappa f) = 0. \quad (16)$$

## 4.2. Finding $\mathcal{E}'_1$ of the Cylinder by Minimizing $\mathcal{K}$

Equation (16) leads to the *quartic* equation

$$\mathfrak{C}_4 \mathcal{E}'_1{}^4 + \mathfrak{C}_3 \mathcal{E}'_1{}^3 + \mathfrak{C}_2 \mathcal{E}'_1{}^2 + \mathfrak{C}_1 \mathcal{E}'_1 + \mathfrak{C}_0 = 0, \quad (17)$$

whose coefficients are:

$$\mathfrak{C}_3 = 2\mathcal{E}_0 F A^2 - 4\mathcal{E}_0 (F B^2 + f g) + 2\mathcal{E}_1'' f h, \quad \mathfrak{C}_2 = -2\mathcal{E}_0^2 (F B^2 + f g) + 6\mathcal{E}_0 \mathcal{E}_1'' f h, \quad (18)$$

$$\mathfrak{C}_1 = -2\mathcal{E}_0 k_+ F A^2 - 2\mathcal{E}_0 (k_+ + k_-) (F B^2 + f g) + (2k_+ + 4\mathcal{E}_0^2) \mathcal{E}_1'' f h, \quad (19)$$

$$\mathfrak{C}_0 = -k_+^2 F A^2 - k_+ k_- (F B^2 + f g) + 2k_+ \mathcal{E}_0 \mathcal{E}_1'' f h, \quad \mathfrak{C}_4 = F A^2 - (F B^2 + f g), \quad (20)$$

wherein

$$k_{\pm} := \mathcal{E}_0^2 \pm \mathcal{E}_1''^2, \quad F := \frac{E^i}{B}, \quad f := \frac{e^i \kappa}{b}, \quad g := -b^2 + a^2 \left( \frac{\mathcal{E}_1'^2 + \mathcal{E}_1''^2 - \mathcal{E}_0^2}{\|\varepsilon_1 + \varepsilon_0\|^2} \right), \quad h := \frac{2a^2 \varepsilon_1'' \varepsilon_0}{\|\varepsilon_1 + \varepsilon_0\|^2}, \quad (21)$$

It ensues that:

$$\mathfrak{C}_1 = \frac{2\mathcal{E}_0 \mathfrak{C}_0}{k_+} + k_+(\mathfrak{C}_3 - 2\mathcal{E}_0 \mathfrak{C}_4), \quad \mathfrak{C}_2 = \frac{\mathfrak{C}_0}{k_+} + 2\mathcal{E}_0(\mathfrak{C}_3 - 2\mathcal{E}_0 \mathfrak{C}_4) + k_+ \mathfrak{C}_4, \quad (22)$$

so that the quartic equation can re-written as

$$(\mathcal{E}_1'^2 + 2\mathcal{E}_0 \mathcal{E}_1' + k_+) \left[ \mathfrak{C}_4 \mathcal{E}_1'(\mathcal{E}_1' - 2\mathcal{E}_0) + \mathfrak{C}_3 \mathcal{E}_1' + \frac{\mathfrak{C}_0}{k_+} \right] = 0. \quad (23)$$

The solutions of  $( ) = 0$  are:

$$\mathcal{E}_1'^{(1)} = -\mathcal{E}_0 - i\mathcal{E}_1'', \quad \mathcal{E}_1'^{(2)} = -\mathcal{E}_0 + i\mathcal{E}_1'', \quad (24)$$

whereas the two roots of  $[ ] = 0$  are:

$$\mathcal{E}_1'^{(\pm)} = \frac{-k_+(\mathfrak{C}_3 - 2\mathcal{E}_0 \mathfrak{C}_4) \pm \sqrt{k_+^2(\mathfrak{C}_3 - 2\mathcal{E}_0 \mathfrak{C}_4)^2 - 4k_+ \mathfrak{C}_4 \mathfrak{C}_0}}{2k_+ \mathfrak{C}_4}. \quad (25)$$

Eqs. (24)–(25) represent the exact solutions of the inverse problem of the identification of the sole parameter  $\mathcal{E}_1'$ . Recall that it was assumed that  $\mathcal{E}_0$ ,  $\mathcal{E}_1'$  and  $\mathcal{E}_1''$  (of the same nature as  $\varepsilon_0$ ,  $\varepsilon_1'$ ,  $\varepsilon_1''$  respectively) are positive real. Thus,  $\mathcal{E}_1'^{(1)}$  and  $\mathcal{E}_1'^{(2)}$  are not admissible solutions.

By a perturbation analysis, for small  $\mathcal{E}_1''$  and  $\varepsilon_1''$ , we find (neglecting terms of order  $\mathcal{E}_1''$  and  $\varepsilon_1''$ , so that  $g$  becomes  $g^0 = g|_{\varepsilon_1''=0}$ ) that the approximate solution for  $\mathcal{E}_1'$  is

$$\mathcal{E}_1' \approx \mathcal{E}_0 \left( \frac{FA^2 + FB^2 + fg^{(0)}}{FA^2 - FB^2 - fg^{(0)}} \right), \quad (26)$$

which is a consequence of the unique, exact solution for  $\mathcal{E}_1'$

$$\mathcal{E}_1' = \frac{-k_+(\mathfrak{C}_3 - 2\mathcal{E}_0 \mathfrak{C}_4) + \sqrt{k_+^2(\mathfrak{C}_3 - 2\mathcal{E}_0 \mathfrak{C}_4)^2 - 4k_+ \mathfrak{C}_4 \mathfrak{C}_0}}{2k_+ \mathfrak{C}_4}. \quad (27)$$

#### 4.2.1. Comments on the Exact and Approximate Solutions for $\mathcal{E}_1'$

The result embodied in (27) shows that:

- 1- the solution to the inverse problem of the identification of the sole parameter  $\mathcal{E}_1'$  *exists*, even in the presence of nuisance parameter uncertainties;
- 2- it is possible to obtain the *mathematically-explicit and exact solution* to the given inverse problem, even in the presence of nuisance parameter uncertainties;
- 3- this solution is *unique* for a given set of parameters  $\varepsilon_1'$ ,  $\varepsilon_1''$ ,  $\varepsilon_0$ ,  $a$ ,  $e^i$ ,  $\theta^i$ ,  $b$ ,  $\mathcal{E}_1''$ ,  $\mathcal{E}_0$ ,  $A$ ,  $E^i$ ,  $\Theta^i$ ,  $B$ , subject to the assumed physical constraint (i.e., the real part of the permittivity should be positive real);
- 4- the accuracy of the retrieval of the real part of the permittivity ( $\mathcal{E}_1'$ ) is conditioned, albeit in a complex manner, by the uncertainty of the nuisance parameters  $\mathcal{E}_0$ ,  $A$ ,  $E^i$ ,  $\Theta^i$ ,  $B$ ;

The result embodied in (26) shows that

- 5- the accuracy of the retrieval of  $\mathcal{E}_1'$  is weakly-conditioned by  $\varepsilon_1''$  and  $\mathcal{E}_1''$  since the dependence of  $\mathcal{E}_1'$  on these parameters is at least of order  $\mathcal{E}_1''$  or  $\varepsilon_1''$  (and therefore small due to  $\varepsilon_1''$  and  $\mathcal{E}_1''$  having been assumed to be small).

#### 4.2.2. Preliminaries Concerning the Dependence of the Retrieval Error of $\mathcal{E}'_1$ on the Nuisance Parameter Uncertainties

Due to the fifth comment in Section 4.2.1 it is no longer necessary to delve on the issue of the dependence of the retrieval error of  $\mathcal{E}'_1$  on the (assumed-small) nuisance parameter  $\mathcal{E}''_1$ . Thus, from now on, we deal with (26)

$$\mathcal{E}'_1 = \mathcal{E}_0 \left[ \frac{\varepsilon'_1 X_+ + \varepsilon_0 Y_+}{\varepsilon'_1 Y_- + \varepsilon_0 X_-} \right]. \quad (28)$$

wherein

$$X_{\pm} = (FA^2 + fa^2) \pm (FB^2 - fb^2), \quad Y_{\pm} = (FA^2 - fa^2) \pm (FB^2 - fb^2). \quad (29)$$

To unravel the complexity alluded to in the fourth comment in Section 4.2.1, first suppose that the nuisance parameters  $A, B, E^i, \Theta^i$  are known precisely, i.e.,  $E^i = e^i$ ,  $A = a$ ,  $B = b$ ,  $\Theta^i = \theta^i$ , whence  $X_+ = X^-$  and  $Y_{\pm} = 0$ , so that

$$\mathcal{E}'_1 = \tilde{\mathcal{E}}'_1 = \mathcal{E}_0 \frac{\varepsilon'_1}{\varepsilon_0}, \quad (30)$$

which shows that the the relative error of the retrieved parameter, i.e.,

$$\varepsilon_{\varepsilon'_1} = \frac{\mathcal{E}'_1 - \varepsilon'_1}{\varepsilon'_1} = \frac{\mathcal{E}'_1}{\varepsilon'_1} - 1 = \frac{\mathcal{E}_0 - \varepsilon_0}{\varepsilon_0} = \frac{\mathcal{E}_0}{\varepsilon_0} - 1 := \delta_{\varepsilon_0}, \quad (31)$$

depends linearly on the ratio  $\frac{\mathcal{E}_0}{\varepsilon_0}$ .

#### 4.2.3. Properties of $\mathcal{E}'_1$ as a Function of $B$

We can write

$$\mathcal{E}'_1(B) = \mathcal{E}_0 \left[ \frac{E^i b (B^2 + A^2) - e^i B M}{-E^i b (B^2 - A^2) - e^i B M} \right] = -\mathcal{E}_0 \left[ \frac{B^2 - \frac{e^i M}{E^i b} B + A^2}{B^2 - \frac{e^i M}{E^i b} B - A^2} \right] := -\mathcal{E}_0 \frac{\mathcal{N}}{\mathcal{D}}. \quad (32)$$

wherein

$$M := -\kappa g_0 = \kappa \left[ \frac{(\varepsilon'_1 + \varepsilon_0) b^2 - (\varepsilon'_1 - \varepsilon_0) a^2}{\varepsilon'_1 + \varepsilon_0} \right]. \quad (33)$$

Since we assume a relatively-small uncertainty on the nuisance parameter  $\Theta^i$ , it follows that  $|\Theta^i - \theta^i| < \pi/2$ , whence  $\kappa > 0$ . Also, since we assume that  $\varepsilon_0 > 0$ ,  $\varepsilon'_1 > 0$ ,  $b^2 > a^2$ , it follows that

$$M > 0. \quad (34)$$

Finally, recall that we assume  $\mathcal{E}_0 > 0$ .

Now, assume that  $\delta > 0$  and consider  $\mathcal{E}'_1(B_0 \pm \delta)$ . It is easy to show that

$$\mathcal{E}'_1(B_0 + \delta) = \mathcal{E}'_1(B_0 - \delta), \quad (35)$$

entails

$$B_0 = \frac{e^i M}{2E^i b} > 0, \quad (36)$$

so that

$$\mathcal{E}'_1(B) = -\mathcal{E}_0 \left[ \frac{B^2 - 2B_0 B + A^2}{B^2 - 2B_0 B - A^2} \right]. \quad (37)$$

whence

$$\mathcal{E}'_1(B_0) = \mathcal{E}_0. \quad (38)$$

Differentiating gives

$$\frac{d\mathcal{E}'_1(B)}{dB} = \frac{4\mathcal{E}_0(B - B_0)A^2}{B^2 - 2B_0 B - A^2}, \quad (39)$$

whence

$$\frac{d\mathcal{E}'_1(B_0)}{dB} = 0. \quad (40)$$

The equation  $\mathcal{D} = 0$  has two solutions

$$B_{\pm} = \frac{\frac{e^i M}{E^i b} \pm \sqrt{\left(\frac{e^i M}{E^i b}\right)^2 + 4A^2}}{2} = B_0 \pm \sqrt{B_0^2 + A^2}. \quad (41)$$

at which  $\mathcal{E}'_1$  blows up.

It is generally admitted that nonlinear inverse problems are ill posed [8] which means that the solution either does not exist or is not unique (our particular inverse solution was shown to be unique when certain physical constraints are satisfied) and is unstable in the case of existence. *Instability* is usually defined ([1]) by retrievals not being continuously-dependent on variations of the data. We, on the other hand, find that the function  $\mathcal{E}'_1(B)$  *diverges* at  $B_-$  and  $B_+$ , which suggests a new form of instability, first observed, but not explained, in [11]. Borrowing a notion first expressed in [16], we define *retrieval instability induced by nuisance parameter uncertainty* as that which occurs when a very small variation of a nuisance parameter  $B$  (and perhaps other nuisance parameters) produces a very large variation of a retrieved parameter (at present,  $\mathcal{E}'_1$ ).

Now consider  $\mathcal{E}'_1(B_{\pm} \pm \delta)$ , with  $0 < \delta \ll 1$  defined as previously. We find

$$\mathcal{E}'_1(B_{\pm} \pm \delta) = -\mathcal{E}_0 \left[ \frac{A^2 \pm \delta \sqrt{B_0^2 + A^2}}{\pm \delta \sqrt{B_0^2 + A^2}} \right] \approx \mp \mathcal{E}_0 \frac{A^2}{\delta \sqrt{B_0^2 + A^2}}, \quad (42)$$

which shows that for small positive  $\delta$ ,

$$\mathcal{E}'_1(B_+ - \delta) > 0, \quad \mathcal{E}'_1(B_+ + \delta) < 0. \quad (43)$$

Similarly, for  $\varepsilon \gg 1$ ,

$$\mathcal{E}'_1(B_{\pm} \pm \varepsilon) \sim -\mathcal{E}_0; \quad \varepsilon \rightarrow \infty, \quad (44)$$

which shows that  $\mathcal{E}'_1(B)$  tends to the negative value  $-\mathcal{E}_0$  for  $B \rightarrow \pm\infty$ . This result, which at first appears to be surprising since it was assumed at the outset that  $\mathcal{E}_1 > 0$ , actually means that no physically-meaningful result can be obtained for large uncertainty of  $|B|$ . In other words, *the solutions obtained for  $B > B_+$  and  $B < B_+$  must be rejected*.

The analytical properties (symmetries, behavior in the neighborhoods of  $B_0$ ,  $B_-$  and  $B_+$ ) of the function  $\mathcal{E}_1(B)$ , deduced from the preceding formulae, are exhibited in Fig. 1.

It should be noted, in this figure, that it is possible to retrieve a physically-meaningful  $\mathcal{E}'$  (i.e., that is positive real) only for  $B_+ > B \geq 0$ .

#### 4.2.4. Properties of $\mathcal{E}'_1$ as a Function of $A$

We can write

$$\mathcal{E}'_1(A) = \mathcal{E}_0 \left[ \frac{A^2 - \frac{e^i BM}{E^i b} + B^2}{A^2 - \frac{e^i BM}{E^i b} - B^2} \right] = \mathcal{E}_0 \left[ \frac{A^2 - C^2 + B^2}{A^2 - C^2 - B^2} \right] := \mathcal{E}_0 \frac{\mathcal{N}}{\mathcal{D}}, \quad (45)$$

wherein

$$C^2 := \frac{e^i BM}{E^i b} \quad (46)$$

The equation  $\mathcal{D} = 0$  has two solutions

$$A_{\pm} = \pm \sqrt{B^2 - C^2}. \quad (47)$$

at which  $\mathcal{E}'_1$  blows up.

Proceeding as in Section 4.2.3 we are able to show that the analytical properties (symmetries, behavior in the neighborhoods of  $A = 0$ ,  $A_-$  and  $A_+$ ) of the function  $\mathcal{E}'_1(A)$  are as exhibited in Fig. 2. It should be noted, in this figure, that it is possible to retrieve a physically-meaningful  $\mathcal{E}'$  (i.e., that is positive real) only for  $\infty > A > A_+$ .

#### 4.2.5. Properties of $\mathcal{E}'_1$ as a Function of $E^i$

We can write

$$\mathcal{E}'_1(E^i) = -\mathcal{E}_0 \left( \frac{B^2 + A^2}{B^2 - A^2} \right) \left[ \frac{E^i - E_0^i}{E^i - E_+^i} \right], \quad (48)$$

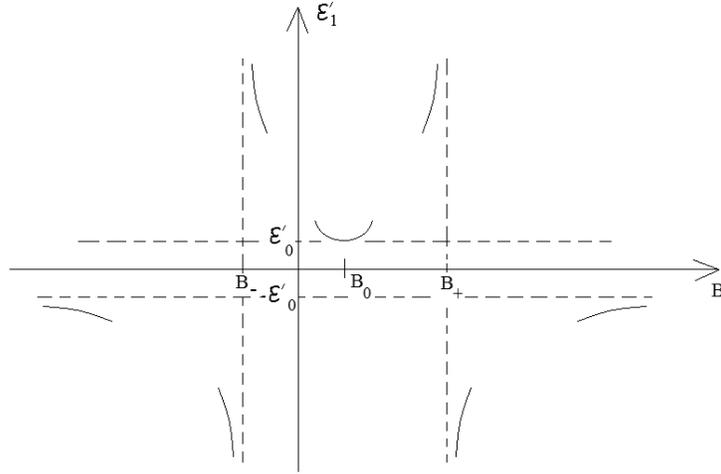
wherein:

$$E_0^i = \frac{e^i BM}{b(B^2 + A^2)}, \quad E_+^i = \frac{e^i BM}{b(B^2 - A^2)} > E_0^i. \quad (49)$$

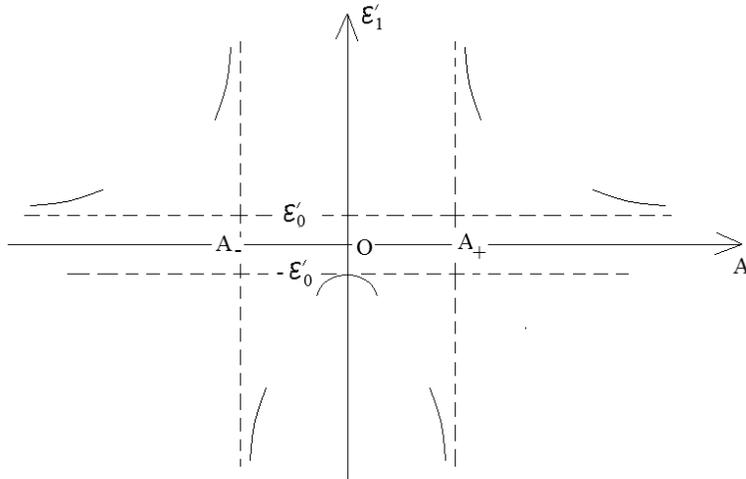
It ensues that  $\mathcal{E}'_1(E^i)$  blows up at  $E^i = E_+^i$ .

Proceeding as in Section 4.2.3, we can show that the analytical properties (behavior in the neighborhoods of  $E_0^i, E_+^i$ ) of the function  $\mathcal{E}'_1(E^i)$  are as exhibited in Fig. 3.

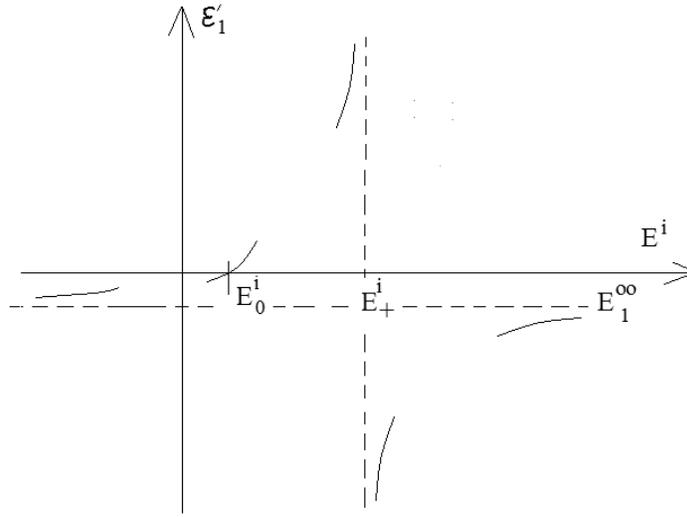
It should be noted, in this figure, that it is possible to retrieve a physically-meaningful  $\mathcal{E}'$  (i.e., that is positive real) only for  $E_+^i > B > E_0^i$ .



**Figure 1.** Graphical representation of the analytical properties of  $\mathcal{E}'_1(B)$ .



**Figure 2.** Graphical representation of the analytical properties of  $\mathcal{E}'_1(A)$ .



**Figure 3.** Graphical representation of the analytical properties of  $\mathcal{E}'_1(E^i)$ .

4.2.6. Comments on the Analytical Properties of  $\mathcal{E}'_1(A)$ ,  $\mathcal{E}'_1(E^i)$ ,  $\mathcal{E}'_1(B)$

In Section 4.2.1, it was stated that the explicit formula for  $\mathcal{E}'_1$  demonstrates the existence of a solution of the inverse problem, even in the presence of nuisance parameter uncertainties. We now see that this statement must be interpreted in the following sense: a physically-admissible solution (i.e., positive and not infinite) for  $\mathcal{E}'_1$  exists only for a range of nuisance parameter uncertainties (i.e., those for which  $\mathcal{E}'_1$  is positive and not infinite).

Furthermore, it was stated in Section 4.2.1, that the explicit formula for  $\mathcal{E}'_1$  constitutes a unique solution for a given set of parameters  $\varepsilon'_1$ ,  $\varepsilon''_1$ ,  $\varepsilon_0$ ,  $a$ ,  $e^i$ ,  $\theta^i$ ,  $b$ ,  $\theta^b$ ,  $\mathcal{E}''_1$ ,  $\mathcal{E}_0$ ,  $A$ ,  $E^i$ ,  $\Theta^i$ ,  $B$ . However, this does not mean that a physically-admissible  $\mathcal{E}'_1$  cannot arise from more than one sets of nuisance parameters, as is illustrated in Fig. 1 for  $B$  in the neighborhood of  $B_0$  (i.e., the two values  $B_0 \pm \delta$  give rise to the same  $\mathcal{E}'_1$ ).

4.3. Finding the Real Part of the Permittivity of the Cylinder by Minimizing  $\mathcal{K}^{(N)}$

The method of obtaining the real part of the permittivity of the cylinder outlined in Section 4.2 is somewhat unrealistic because it supposes that the data are registered at a continuum of points on the sensing circle  $r = b$ . In reality, the data are registered at discrete locations on this circle, and the number  $N$  of these locations is finite. Of course,  $N$  will have an influence on the accuracy of the retrieval, and it is this influence that we shall now examine.

From (14) we obtain

$$\begin{aligned} \mathcal{K}^{(N)}(\mathcal{E}'_1) = & \frac{1}{\|f\|^2 \delta_\theta \sum_{n=1}^N \cos^2(\alpha + (n-1)\delta_\theta)} \\ & \times \delta_\theta \sum_{n=1}^N \left( \|f\|^2 \cos^2(\alpha + (n-1)\delta_\theta) - 2\Re\{f^* \mathfrak{F}\} \cos(\alpha + (n-1)\delta_\theta) \cos(\beta + (n-1)\delta_\theta) \right. \\ & \left. + \|\mathfrak{F}\|^2 \cos^2(\beta + (n-1)\delta_\theta) \right), \end{aligned} \tag{50}$$

wherein  $\alpha = \theta^b + \frac{\delta_\theta}{2} - \theta^i$  and  $\beta = \theta^b + \frac{\delta_\theta}{2} - \Theta^i$ ,  $\delta_\theta = \frac{\theta^e - \theta^b}{N}$ ,  $\theta_n = \theta^b + \frac{\delta_\theta}{2} + (n-1)\delta_\theta$ . Note that since  $\delta_\theta$  depends on  $N$ ,  $\alpha$  and  $\beta$  also depend on  $N$ . Also note that, contrary to what is assumed in  $\mathcal{K}$ , i.e.,

$\theta^e - \theta^b = 2\pi$ , here we admit arbitrary  $\theta^b$  and  $\theta^e$ , with the only restriction that  $\theta^e > \theta^b$  and  $\theta^e$ . Consider

$$\sigma^{(N)}(\alpha, \beta) = \delta_\theta \sum_{n=1}^N \cos(\alpha + (n-1)\delta_\theta) \cos(\beta + (n-1)\delta_\theta), \quad (51)$$

It is straightforward to show that

$$\sigma^{(N)}(\alpha, \beta) = \frac{\delta_\theta}{2} \left[ N \cos(\alpha - \beta) + \cos(\alpha + \beta + (N-1)\delta_\theta) \left( \frac{\sin(N\delta_\theta)}{\sin(\delta_\theta)} \right) \right]. \quad (52)$$

and, on account of (50)

$$\mathcal{K}^{(N)}(\mathcal{E}'_1) = \frac{\|f\|^2 \sigma^{(N)}(\alpha, \alpha) - 2\Re(f^* \mathfrak{F}) \sigma^{(N)}(\alpha, \beta) + \|\mathfrak{F}\|^2 \sigma^{(N)}(\beta, \beta)}{\|f\|^2 \sigma^{(N)}(\alpha, \alpha)}. \quad (53)$$

For  $\theta^e - \theta^b = 2\pi$ , the fact that

$$\lim_{N \rightarrow \infty} \sigma^{(N)}(\alpha, \beta) = \pi \cos(\alpha - \beta) = \pi \cos(\theta^i - \Theta^i) = \pi \kappa, \quad (54)$$

gives rise to the expected result

$$\lim_{N \rightarrow \infty} \mathcal{K}^{(N)}(\mathcal{E}'_1) = \mathcal{K}(\mathcal{E}'_1), \quad (55)$$

Recall that our goal was to obtain an estimation of  $\mathcal{E}'_1$  by minimizing  $\mathcal{K}^{(N)}(\mathcal{E}'_1)$ . The procedure is the same as in Section 4.2 and yields the exact (mathematical) solution

$$\mathcal{E}'_1{}^{(N)} = \frac{-k_+(\mathfrak{C}_3^{(N)} - 2\mathcal{E}_0 \mathfrak{C}_4^{(N)}) + \sqrt{k_+^2 (\mathfrak{C}_3^{(N)} - 2\mathcal{E}_0 \mathfrak{C}_4^{(N)})^2 - 4k_+ \mathfrak{C}_4^{(N)} \mathfrak{C}_0^{(N)}}}{2k_+ \mathfrak{C}_4^{(N)}}, \quad (56)$$

and the approximate solution

$$\mathcal{E}'_1{}^{(N)} \approx \mathcal{E}_0 \left( \frac{FA^2 + FB^2 + f^{(N)}g^{(0)}}{FA^2 - FB^2 - f^{(N)}g^{(0)}} \right), \quad (57)$$

wherein:

$$\mathfrak{C}_4^{(N)} = FA^2 - (FB^2 + f^{(N)}g), \quad \mathfrak{C}_3^{(N)} = 2\mathcal{E}_0 FA^2 - 4\mathcal{E}_0 (FB^2 + f^{(N)}g) + 2\mathcal{E}_1'' f^{(N)}h, \quad (58)$$

$$\mathfrak{C}_0^{(N)} = -k_+^2 FA^2 - k_+ k_- (FB^2 + f^{(N)}g) + 2k_+ \mathcal{E}_0 \mathcal{E}_1'' f^{(N)}h, \quad f^{(N)} = \frac{e^{i\kappa^{(N)}}}{b}, \quad \kappa^{(N)} = \frac{\sigma^{(N)}(\alpha, \beta)}{\sigma^{(N)}(\beta, \beta)}. \quad (59)$$

Eqs. (56)–(59) are the (mathematically) exact and approximate solutions respectively to the inverse problem for discrete data in the interval  $[\theta^b + \frac{\delta_\theta}{2}, \theta^e - \frac{\delta_\theta}{2}]$ . Eqs. (56)–(59) show that the accuracy of the retrieval of  $\mathcal{E}'_1$  is conditioned, not only by the uncertainty of the nuisance parameters  $\mathcal{E}_1''$ ,  $\mathcal{E}_0$ ,  $A$ ,  $E^i$ ,  $B$ ,  $\Theta^i$ , but also by the number  $N$  of data samples.

It is of some interest to see how the choice of  $N$  influences the retrieval of  $\mathcal{E}'_1$ , either via the explicit formula (56) or via minimization of the cost functional  $\mathcal{K}^{(N)}$  in (57). To do this, we consider the case  $\theta^e - \theta^b = 2\pi$  solely, keeping in mind the reference solution obtained by minimization of the cost functional  $\mathcal{K}$ . From (52) it ensues that

$$\kappa^{(N)}(\alpha, \beta) = \frac{\cos \alpha}{\cos \beta}; \quad N = 1, 2, \quad \kappa^{(N)}(\alpha, \beta) = \cos(\alpha - \beta); \quad N \geq 3, \quad (60)$$

wherein

$$\alpha = \frac{\pi}{N} - \theta^i, \quad \beta = \frac{\pi}{N} - \Theta^i. \quad (61)$$

Consequently,

$$\kappa^{(1)} = \frac{\cos \theta^i}{\cos \Theta^i}, \quad \kappa^{(2)} = \frac{\sin \theta^i}{\sin \Theta^i}, \quad \kappa^{(N \geq 3)} = \cos(\theta^i - \Theta^i) = \kappa. \quad (62)$$

This result tells us that the retrieval depends on  $N$  for small  $N$  ( $= 1, 2$ ), but no longer depends on  $N$  for  $N \geq 3$ , which suggests that *the optimal number  $N$  of sensors is  $N = 3$*  separated (in terms of angle  $\theta$ ) by  $2\pi/3$ , on the sensing circle of radius  $b \geq a$ . Note also that owing to the result  $\kappa^{(N \geq 3)} = \kappa$ , it ensues that

$$\mathcal{K}^{(N \geq 3)} = \mathcal{K}, \tag{63}$$

which means that the employment of the term ‘optimal’ is all the more justified that the  $N = 3$  inversion gives rise to the reference retrieval (obtained by minimization of  $\mathcal{K}$ ).

#### 4.4. Finding $\mathcal{E}''$ of the Cylinder by Minimizing $\mathcal{K}$

By proceeding as in Section 4.2 we obtain the fourth-order (in terms of  $\mathcal{E}''$ ) (quartic) algebraic equation

$$\mathfrak{D}_4 \mathcal{E}_1''^4 + \mathfrak{D}_3 \mathcal{E}_1''^3 + \mathfrak{D}_2 \mathcal{E}_1''^2 + \mathfrak{D}_1 \mathcal{E}_1'' + \mathfrak{D}_0 = 0, \tag{64}$$

wherein

$$\begin{aligned} \mathfrak{D}_4 &= fh, \quad \mathfrak{D}_3 = 2 [FA^2(\mathcal{E}_+ - \mathcal{E}_0) - (FB^2 + fg)\mathcal{E}_+], \\ \mathfrak{D}_2 &= 0, \quad \mathfrak{D}_0 = -fh\mathcal{E}_+^4, \quad \mathcal{E}_\pm := \mathcal{E}'_1 \pm \mathcal{E}_0, \quad \varepsilon_\pm := \varepsilon'_1 \pm \varepsilon_0, \\ \mathfrak{D}_1 &= 2\mathcal{E}_+^2 \{ [FA^2\mathcal{E}_- - (FB^2 + fg)\mathcal{E}_+] + FA^2\mathcal{E}_0 \}, \end{aligned} \tag{65}$$

It ensues from these formulae that:

$$\mathfrak{D}_1 = \mathcal{E}_+^2 \mathfrak{D}_3, \quad \mathfrak{D}_0 = -\mathfrak{D}_4 \mathcal{E}_+^4, \tag{66}$$

so that the roots of the quartic equation can be found from

$$\left( \mathcal{E}_1''^2 + \mathcal{E}_+^2 \right) \left[ \mathfrak{D}_4 (\mathcal{E}_1''^2 - \mathcal{E}_+^2) + \mathfrak{D}_3 \mathcal{E}_1'' \right] = 0. \tag{67}$$

The two roots of  $( ) = 0$  are:

$$\begin{aligned} \mathcal{E}_1''^{(1)} &= -i\mathcal{E}_+, \\ \mathcal{E}_1''^{(2)} &= i\mathcal{E}_+ \end{aligned}, \tag{68}$$

and the two roots of  $[ ] = 0$  are:

$$\begin{aligned} \mathcal{E}_1''^{(-)} &= \frac{-\mathfrak{D}_3 - \sqrt{\mathfrak{D}_3^2 + 4\mathfrak{D}_4^2 \mathcal{E}_+^2}}{2\mathfrak{D}_4}, \\ \mathcal{E}_1''^{(+)} &= \frac{-\mathfrak{D}_3 + \sqrt{\mathfrak{D}_3^2 + 4\mathfrak{D}_4^2 \mathcal{E}_+^2}}{2\mathfrak{D}_4}. \end{aligned} \tag{69}$$

Eqs. (68)–(69) represent the exact solutions of the inverse problem of the identification of the sole parameter  $\mathcal{E}_1''$ . Recall that it was assumed that  $\mathcal{E}_0$ ,  $\mathcal{E}'_1$  and  $\mathcal{E}''_1$  (of the same nature as  $\varepsilon_0$ ,  $\varepsilon'_1$ ,  $\varepsilon''_1$  respectively) are positive real. Thus,  $\mathcal{E}_1''^{(1)}$  and  $\mathcal{E}_1''^{(2)}$  are not admissible solutions. By a perturbation analysis, for small  $\mathcal{E}'_1$  and  $\varepsilon'_1$ , we find (neglecting terms of order  $\mathcal{E}'_1$  and  $\varepsilon'_1$ ), that the approximate solution for  $\mathcal{E}_1''$  is

$$\mathcal{E}_1'' \approx \left( \frac{\mathcal{E}_+^2 fh^{(1)}}{FA^2 - FB^2 - fg^{(0)}} \right) \varepsilon'', \tag{70}$$

corresponding to the (only-admissible) exact solution for  $\varepsilon''_1$

$$\mathcal{E}_1'' = \frac{-\mathfrak{D}_3 + \sqrt{\mathfrak{D}_3^2 + 4\mathfrak{D}_4^2 \mathcal{E}_+^2}}{2\mathfrak{D}_4}. \tag{71}$$

#### 4.4.1. Comments on the Exact and Approximate Solutions for $\mathcal{E}_1''$

The same comments apply to the exact  $\mathcal{E}_1''$  of (71) as to the exact  $\mathcal{E}_1'$  in Section 4.2.6. The result embodied in (70) shows:

- 1- that the accuracy of the retrieval of  $\mathcal{E}_1''$  is strongly-conditioned by  $\varepsilon_1''$  and  $\mathcal{E}_1'$  since the dependence of  $\mathcal{E}_1''$  on  $\varepsilon_1''$  is linear and that on  $\mathcal{E}_1'$  is quadratic,
- 2- nevertheless that  $\mathcal{E}_1''$  is usually small due to  $\varepsilon_1''$  having been assumed to be small,
- 3- that, due to the possibility of vanishing denominator in (70), the retrieval of  $\mathcal{E}_1''$ , like that of  $\mathcal{E}_1'$ , can be unstable with respect to the uncertainties of certain nuisance parameters.

#### 4.5. Finding the Imaginary Part of the Permittivity ( $\mathcal{E}''$ ) of the Cylinder by Minimizing $\mathcal{K}^{(N)}$

The goal is to obtain an estimation of  $\mathcal{E}_1''$  by minimizing  $\mathcal{K}^{(N)}(\mathcal{E}_1'')$ . The procedure is the same as in Section 4.4 and yields the exact solution

$$\mathcal{E}_1''^{(N)} = \frac{-\mathfrak{D}_3^{(N)} + \sqrt{\mathfrak{D}_3^{(N)2} + 4\mathfrak{D}_4^{(N)2}\mathcal{E}_+^2}}{2\mathfrak{D}_4^{(N)}}. \quad (72)$$

and the approximate solution

$$\mathcal{E}_1''^{(N)} \approx \left( \frac{\mathcal{E}_+^2 f^{(N)} h^{(1)}}{FA^2 - FB^2 - f^{(N)} g^{(0)}} \right) \varepsilon_1'', \quad (73)$$

wherein:

$$\mathfrak{D}_4^{(N)} = f^{(N)} h, \quad \mathfrak{D}_3^{(N)} = 2\mathcal{E}_0 FA^2 - 4\mathcal{E}_0 (FB^2 + f^{(N)} g). \quad (74)$$

Eqs. (72) and (73) are the (mathematically) exact and approximate solutions to the inverse problem for discrete data in the interval  $[\theta^b + \frac{\delta\theta}{2}, \theta^e - \frac{\delta\theta}{2}]$ . These relations, as well as (74), show that the accuracy of the retrieval of  $\mathcal{E}_1''$  is conditioned, not only by the uncertainty of the nuisance parameters  $\mathcal{E}_1'$ ,  $\mathcal{E}_0$ ,  $A$ ,  $E^i$ ,  $B$ ,  $\Theta^i$ , but also by the number  $N$  of data samples.

As in Section 4.3, we can show that, in the case  $\theta^e - \theta_b = 2\pi$ , the retrieval depends on  $N$  for small  $N$  ( $= 1, 2$ ), but no longer depends on  $N$  for  $N \geq 3$ , which suggests that *the optimal number  $N$  of sensors is  $N = 3$*  separated (in terms of angle  $\theta$ ) by  $2\pi/3$ , on the sensing circle of radius  $b \geq a$ . Moreover, the term ‘optimal’ is all the more justified that the  $N = 3$  inversion gives rise to the reference retrieval (obtained by minimization of  $\mathcal{K}$ ).

## 5. RESULTS CONCERNING THE RETRIEVAL ERROR AS A FUNCTION OF NUISANCE PARAMETER UNCERTAINTIES

### 5.1. Overview of the Evaluation of Retrieval Error for Fixed Nuisance Parameter Uncertainties

All the following numerical results pertain to the choice (the true parameters of which are):  $\varepsilon_1' = 2$ ,  $\varepsilon_1'' = 0.1$ ,  $\varepsilon_0 = 1$ ,  $a = 0.1$ ,  $e^i = 1$ ,  $\theta^i = 2^\circ$ ,  $b = 0.2$ . Moreover,  $\theta^b = 0^\circ$ ,  $\theta^e = 360^\circ$ , and  $N = 3$ .

Six general cases of nuisance parameter uncertainty are possible, of which we retain only the following two:

- 1) one nuisance parameter is uncertain, the five others are equal to their true values;
- 2) five nuisance parameters are uncertain, the sixth is equal to its true value.

The offered tables relate  $\tilde{\mathcal{E}}_1'$  (or  $\tilde{\mathcal{E}}_1''$ ) and  $\mathcal{E}_{\varepsilon_1'} = \frac{\tilde{\mathcal{E}}_1' - \varepsilon_1'}{\varepsilon_1'}$  (or  $\mathcal{E}_{\varepsilon_1''} = \frac{\tilde{\mathcal{E}}_1'' - \varepsilon_1''}{\varepsilon_1''}$ ) to  $\delta_g = \frac{G-g}{g}$  for the fixed nuisance parameter  $G$ .

**5.2. Tables of the Influence of Uncertainty of One Nuisance Parameter, All Other Nuisance Parameters Being Certain, on the Accuracy of the Retrieval of  $\varepsilon'_1$  or  $\varepsilon''_1$**

In Table 1 we give the numerical values of  $\tilde{\mathcal{E}}'_1$  and  $\varepsilon_{\varepsilon'_1}$  in all of the 12 possible cases in which only one nuisance parameter is  $\delta = \pm 10\%$  uncertain at a time.

This table reveals that in six of these cases,  $|\varepsilon_{\varepsilon'_1}| > |\delta|$ , and in four of these cases,  $|\varepsilon_{\varepsilon'_1}| \gg |\delta|$  (due to nuisance parameter uncertainty-induced instability).

In Table 2 we give the numerical values of  $\tilde{\mathcal{E}}''_1$  and  $\varepsilon_{\varepsilon''_1}$  in all of the 12 possible cases in which only one nuisance parameter is  $\delta = \pm 10\%$  uncertain at a time.

This table reveals that in eight of these cases,  $|\varepsilon_{\varepsilon''_1}| > |\delta|$ , and in seven of these cases,  $|\varepsilon_{\varepsilon''_1}| \gg |\delta|$  (due to nuisance parameter uncertainty-induced instability). The comparison of Table 2 with Table 1 shows that the pattern of retrieval error of  $\mathcal{E}''_1$  is substantially the same as that of  $\mathcal{E}'_1$ , as previously predicted in theoretical manner.

**Table 1.** Retrieval of  $\varepsilon'_1$  when one nuisance parameter is  $\delta = \pm 10\%$  uncertain (except  $\Theta^i$ , which when uncertain, deviates from its true value by  $\pm 1^\circ$ ).

$\delta_{\varepsilon''_1}$	$\delta_{\varepsilon_0}$	$\delta_a$	$\delta_{e^i}$	$\Theta^i(^{\circ})$	$\delta_b$	$\tilde{\mathcal{E}}'_1$	$\varepsilon_{\varepsilon'_1}$
-0.1	0	0	0	2	0	2.000	0.0000
0.1	0	0	0	2	0	2.000	0.0000
0	-0.1	0	0	2	0	1.800	-0.1000
0	-0.1	0	0	2	0	2.200	0.1000
0	0	-0.1	0	2	0	2.398	0.1988
0	0	0.1	0	2	0	1.761	-0.1193
0	0	0	-0.1	2	0	0.864	-0.5679
0	0	0	0.1	2	0	5.002	1.5008
0	0	0	0	1	0	2.003	0.0013
0	0	0	0	3	0	2.003	0.0013
0	0	0	0	2	-0.1	0.889	-0.5553
0	0	0	0	2	0.1	9.364	3.6821

**Table 2.** Retrieval of  $\varepsilon''_1$  when one nuisance parameter is  $\delta = \pm 10\%$  uncertain (except  $\Theta^i$ , which when uncertain, deviates from its true value by  $\pm 1^\circ$ ).

$\delta_{\varepsilon'_1}$	$\delta_{\varepsilon_0}$	$\delta_a$	$\delta_{e^i}$	$\Theta^i(^{\circ})$	$\delta_b$	$\tilde{\mathcal{E}}''_1$	$\varepsilon_{\varepsilon''_1}$
-0.1	0	0	0	2	0	0.101	0.0052
0.1	0	0	0	2	0	0.100	0.0039
0	-0.1	0	0	2	0	0.090	-0.0957
0	-0.1	0	0	2	0	0.110	0.1048
0	0	-0.1	0	2	0	0.161	0.6122
0	0	0.1	0	2	0	0.070	-0.2960
0	0	0	-0.1	2	0	0.050	-0.5001
0	0	0	0.1	2	0	3.102	30.0165
0	0	0	0	1	0	0.100	0.0016
0	0	0	0	3	0	0.100	0.0016
0	0	0	0	2	-0.1	0.041	-0.5873
0	0	0	0	2	0.1	34.86	347.60

### 5.3. Tables of the Influence of Uncertainty Regarding Five Nuisance Parameters, the Sixth Nuisance Parameter Being Certain, on the Accuracy of the Retrieval of $\varepsilon'_1$

In Tables 3–4 we give the numerical values of  $\tilde{\mathcal{E}}'_1$  and  $\varepsilon_{\varepsilon'_1}$  in all of the 32 possible cases in which five nuisance parameters are  $\delta = \pm 10\%$  uncertain at a time.

**Table 3.** Retrieval of  $\varepsilon'_1$  when five nuisance parameters (excepting  $\Theta^i$ ) are  $\delta = \pm 10\%$  uncertain.

$\delta_{\varepsilon''_1}$	$\delta_{\varepsilon_0}$	$\delta_a$	$\delta_{e^i}$	$\Theta^i(^{\circ})$	$\delta_b$	$\tilde{\mathcal{E}}'_1$	$\varepsilon_{\varepsilon'_1}$
-0.1	-0.1	-0.1	-0.1	2	-0.1	0.283	-0.8584
0.1	-0.1	-0.1	-0.1	2	-0.1	0.286	-0.8570
-0.1	0.1	-0.1	-0.1	2	-0.1	0.344	-0.8279
0.1	0.1	-0.1	-0.1	2	-0.1	0.346	-0.8269
-0.1	-0.1	0.1	-0.1	2	-0.1	0.435	-0.7825
0.1	-0.1	0.1	-0.1	2	-0.1	0.437	-0.7813
-0.1	0.1	0.1	-0.1	2	-0.1	0.530	-0.7351
0.1	0.1	0.1	-0.1	2	-0.1	0.532	-0.7342
-0.1	-0.1	-0.1	0.1	2	-0.1	1.658	-0.1710
0.1	-0.1	-0.1	0.1	2	-0.1	1.658	-0.1708
-0.1	0.1	-0.1	0.1	2	-0.1	2.027	0.0132
0.1	0.1	-0.1	0.1	2	-0.1	2.027	0.0132
-0.1	-0.1	0.1	0.1	2	-0.1	1.347	-0.3265
0.1	-0.1	0.1	0.1	2	-0.1	1.348	-0.3260
-0.1	0.1	0.1	0.1	2	-0.1	1.646	-0.1772
0.1	0.1	0.1	0.1	2	-0.1	1.646	-0.1768

**Table 4.** Retrieval of  $\varepsilon'_1$  when five nuisance parameters (excepting  $\Theta^i$ ) are  $\delta = \pm 10\%$  uncertain.

$\delta_{\varepsilon''_1}$	$\delta_{\varepsilon_0}$	$\delta_a$	$\delta_{e^i}$	$\Theta^i(^{\circ})$	$\delta_b$	$\tilde{\mathcal{E}}'_1$	$\varepsilon_{\varepsilon'_1}$
-0.1	-0.1	-0.1	-0.1	2	0.1	2.328	0.1638
0.1	-0.1	-0.1	-0.1	2	0.1	2.326	0.1632
-0.1	0.1	-0.1	-0.1	2	0.1	2.846	0.4321
0.1	0.1	-0.1	-0.1	2	0.1	2.845	0.4224
-0.1	-0.1	0.1	-0.1	2	0.1	1.658	-0.1710
0.1	-0.1	0.1	-0.1	2	0.1	1.658	-0.1708
-0.1	0.1	0.1	-0.1	2	0.1	2.027	0.0132
0.1	0.1	0.1	-0.1	2	0.1	2.027	0.0132
-0.1	-0.1	-0.1	0.1	2	0.1	$\infty$	$\infty$
0.1	-0.1	-0.1	0.1	2	0.1	$\infty$	$\infty$
-0.1	0.1	-0.1	0.1	2	0.1	$\infty$	$\infty$
0.1	0.1	-0.1	0.1	2	0.1	$\infty$	$\infty$
-0.1	-0.1	0.1	0.1	2	0.1	59.61	28.806
0.1	-0.1	0.1	0.1	2	0.1	59.59	28.794
-0.1	0.1	0.1	0.1	2	0.1	72.88	35.442
0.1	0.1	0.1	0.1	2	0.1	72.86	35.430

These tables reveal that in sixteen of these cases,  $|\varepsilon_{\varepsilon'_1}| > 5|\delta|$ , and in eight of these cases,  $|\varepsilon_{\varepsilon'_1}| \gg 5|\delta|$  (due to combined nuisance parameter uncertainty-induced instability).

It turns out that the pattern of retrieval error of  $\mathcal{E}'_1$  is substantially the same as that of  $\mathcal{E}''_1$ .

## 6. CONCLUSION

Our inverse problem dealt with the retrieval of one of the seven parameters (either the real ( $\varepsilon'_1$ ) or imaginary ( $\varepsilon''_1$ ) part of the permittivity of a cylinder) that enter into a 2D quasistatic electricity configuration. The exact solution of the forward problem was obtained by separation of variables and employed to furnish the data serving as the input to the inverse problem. The retrieval model also relied on the separation of variables solution, but with one of the seven true parameters thereof replaced by a variable  $\varepsilon'_1$  or  $\varepsilon''_1$  and the remaining six (called nuisance) parameters by more or less well-known values. We solved the inverse problem: 1) by mathematically searching for the minimum of the cost functional  $\mathcal{K}$  relative to continuous data on a measurement circle, and 2) by mathematically searching for the minimum of the cost functional  $\mathcal{K}^{(N)}$  relative to discrete data registered at  $N$  sensors on the measurement circle. This led to exact, mathematically-explicit solutions which lend themselves to a complete mathematical analysis of the manner in which the retrieval error varies as a function of the nuisance parameter uncertainties. In particular, we found that  $N = 3$  sensors, equispaced over the angular range  $[0, 2\pi[$ , are necessary and sufficient to provide the required data for the inversion. It was shown, in addition to the existence and uniqueness of the inverse problem solution satisfying pre-existing physical constraints, that this solution is unstable with respect to uncertainties concerning the nuisance parameters  $A$ ,  $E^i$  and  $B$ , acting individually or in combination. These instabilities manifest themselves by extremely-large retrieval error in the neighborhoods of certain values of these nuisance parameters. It was also shown that, even quite far from these neighborhoods, the retrieval error  $|\varepsilon_{\varepsilon_1}|$  can be much larger than a generic nuisance parameter uncertainty  $|\delta_g|$ . Finally, it was found numerically, in agreement with the theory, that the pattern of retrieval error of  $\varepsilon''_1$  is much the same as that of  $\varepsilon'_1$ , notably as concerns the instability issue; moreover the relative retrieval error for  $M$  uncertain parameters turned out to be roughly proportional to  $M$  (outside of the instability regions, and for both the real and imaginary parts of the permittivity).

This investigation underlines the necessity, in parameter-retrieval inverse problems, to take account of nuisance parameter uncertainty in order to evaluate the accuracy of the retrieved parameter(s). In our study, only one parameter was retrieved at a time, while from one to five parameters were uncertain. It may be possible to reduce the global retrieval error by retrieving two or more parameters at a time while considering the remaining parameters to be uncertain, but this issue is out of the scope of the present study.

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