Elimination of Current Crowding Problem in Flat Conductors Bent at Arbitrary Angles

Tatiana Gerasimenko*, Petr Polyakov, and Igor Frolov

Abstract—This work is devoted to a theoretical investigation of the current crowding problem in flat conductors bent at arbitrary angles. Using conformal mapping techniques, we succeed in obtaining an analytical expression for current density distributions in such conductors. It is shown that the current density increases in a small vicinity of the corner and approaches to infinity at its top. In order to eliminate the infinity, the vertex is replaced by an arc of a circle with a small radius. The method has been developed for an arbitrary angle; several specific examples are considered.

1. INTRODUCTION

The current crowding effect is a current density increase in a conductor. As a rule, it takes place at the vicinity of contacts or various inhomogeneities. Being localized in a small area, this increase may lead to overheating of a conductor and acceleration of electromigration processes [1–3]. It in turn may cause a failure in a chip.

Obtaining measurable parameters of conductors, whether a resistance, magnetic field, or temperature, requires calculating the current density distributions. However, a calculation method applied to conductors with angles leads to the emergence of infinite current density values at angles' vertices [2, 4, 5]. Being non-physical, the emergent singularities cause significant difficulties in further calculations. On the one hand, a numerical algorithm cannot deal with the infinity, on the other hand, one doesn't know what value of the current density should be used instead of an infinite one. One way to cope with this problem is to average the current density over a finite volume near the singularity [3]. This method is very useful yet approximate; and it requires complex numerical investigations.

In this paper, we only deal with conductors whose linear dimensions are much greater than their thickness. It is possible to describe this widespread type of conductors as if they were two-dimensional. We consider conductors with a simple shape, namely, strips of variable width bent at arbitrary angles (Figure 1). The conductors are supposed to be infinite on both sides with a current strength applied at the infinity. On the one hand, such form of conductors allows the application of analytical methods, on the other, it may be used as a part of more complex constructions adopting the technique described in our previous work [6].

An attempt to solve the current density problem for the described type of conductors using a powerful electromagnetic solver (ANSYS Maxwell 2D) was unsuccessful. We have determined that there is no current density convergence at the inner corner with the mesh density increase. The same result was demonstrated in paper [3] for a more complex conductor geometry. This numerical non-convergence doesn't allow us to rely upon the results of the finite element method and requires an analytical analysis of the problem.

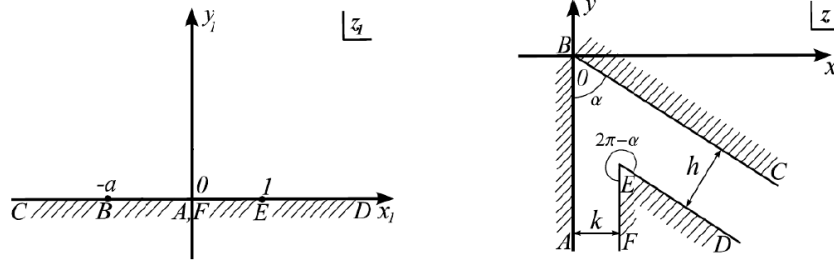## 2. CONFORMAL MAPPING FOR A STRIP BENT AT AN ARBITRARY ANGLE

The geometry of the conductor considered is depicted in Figure 1. α is an arbitrary angle. Only the case of $h > k$ has been investigated (the opposite one may be examined in a similar way). For a direct current, one can reduce Maxwell's equations to the Laplace equation for a scalar potential

$$\Delta\varphi = 0 \tag{1}$$

Figure 1. The points of conformity (plane z_1) and the original geometry of the conductor (plane z).

This equation was solved for a complex potential $W(z) = U(z) + iV(z)$ using a conformal mapping method [5, 7, 8]; $V(z)$ is the scalar potential and $U(z)$ is the stream function. These two functions are related with the Cauchy-Riemann equations, therefore, boundary conditions may be written only for one of them. The boundary conditions require the absence of the current flow through lateral boundaries of the conductor; so the resulting boundary value problem is as follows [6]:

$$\begin{cases} \dfrac{\partial^2 W}{\partial x^2} + \dfrac{\partial^2 W}{\partial y^2} = 0 \\ U(x,y)|_{(x,y)\in\Omega_1} = U_1 = \text{const} \\ U(x,y)|_{(x,y)\in\Omega_2} = U_2 = \text{const} \end{cases} \tag{2}$$

Ω_1 and Ω_2 are the upper and lower boundaries of the conductor respectively.

The solution of this problem can be found by mapping a complex potential of a two-dimensional charge placed at the origin of an upper complex half-plane

$$W(z_1) \sim \ln(z_1) \tag{3}$$

onto the considered area [7].

Thus, the problem is to find the required mapping. It is well known that the mapping of an upper complex half-plane onto a polygon is implemented with the Schwarz-Cristoffel transformation. For the area considered, this transformation is as follows:

$$z = C \int \frac{1}{z_1} \left(\frac{z_1 - 1}{z_1 + a} \right)^{1-\beta} dz_1 \tag{4}$$

$\beta = \alpha/\pi$. The points of conformity are presented in Figure 1. Undefined constants C and a are determined by the conditions of boundary conformity and have the following form:

$$C = \frac{h}{\pi}(\sin\alpha - i\cos\alpha), \quad a = \left(\frac{h}{k} \right)^{\frac{1}{1-\beta}} \tag{5}$$

Expression (4) is representable in terms of elementary functions if $1 - \beta = P/Q$ where $0 < P < Q$, P and Q are integers [9]. From a physical point of view, this condition doesn't impose any restrictions on the angle because any irrational number can be approximated by a rational one with an arbitrary high precision. Thus, (4) can be reduced to

$$z = -(1+a)QC \int \frac{t^{Q-1}}{t^P(t^Q + a)(t^Q - 1)} dt \tag{6}$$

where

$$t = \left(\frac{z_1 + a}{z_1 - 1} \right)^{\frac{1}{Q}} \tag{7}$$

We have integrated expression (6) for some angles.
For the angle of $30°$:

$$z = C \left\{ \frac{i}{b^5} \ln \left(\frac{t + ib}{-t + ib} \right) + \frac{\sqrt{3}}{2b^5} \ln \left(\frac{t^2 + \sqrt{3}bt + b^2}{t^2 - \sqrt{3}bt + b^2} \right) \right.$$

$$+ \frac{i\sqrt{3}}{2} \ln \left(\frac{2t + 1 + i\sqrt{3}}{-2t - 1 + i\sqrt{3}} \right) + \frac{i\sqrt{3}}{2} \ln \left(\frac{2t - 1 + i\sqrt{3}}{-2t + 1 + i\sqrt{3}} \right)$$

$$\left. + \frac{i}{2b^5} \ln \left(\frac{2t + \sqrt{3}b + ib}{-2t - \sqrt{3}b + ib} \right) + \frac{i}{2b^5} \ln \left(\frac{2t - \sqrt{3}b + ib}{-2t + \sqrt{3}b + ib} \right) + \ln \left(\frac{t + 1}{t - 1} \right) + \frac{1}{2} \ln \left(\frac{t^2 + t + 1}{t^2 - t + 1} \right) \right\} + C_1 \tag{8}$$

where

$$C = \frac{h}{2\pi}(1 - i\sqrt{3}), \quad C_1 = k - h\frac{\sqrt{3}}{2} - i\left(\frac{h}{2} + k\sqrt{3} \right), \tag{9}$$

$$t = \left(\frac{z_1 + b^6}{z_1 - 1} \right)^{\frac{1}{6}}, \quad b = \left(\frac{h}{k} \right)^{\frac{1}{5}} \tag{10}$$
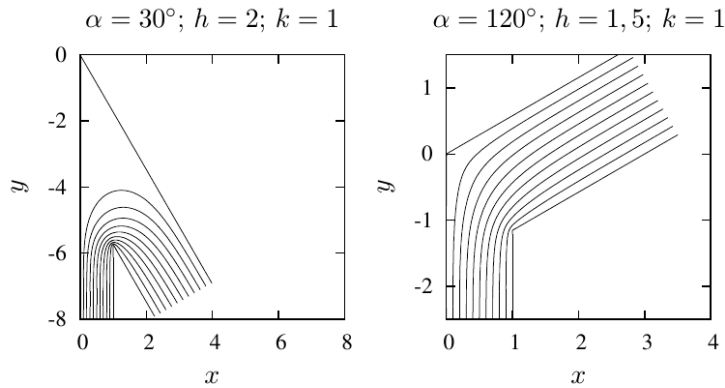
For the angle of $120°$

$$z = C \left\{ -\frac{1}{b} \ln(t + b) - \ln(t - 1) + \frac{1}{2b} \ln(t^2 - bt + b^2) + \frac{1}{2} \ln(t^2 + t + 1) \right.$$

$$\left. - \frac{i\sqrt{3}}{2b} \ln \left(\frac{2t - b - i\sqrt{3}b}{-2t + b - i\sqrt{3}b} \right) + \frac{i\sqrt{3}}{2} \ln \left(\frac{2t + 1 - i\sqrt{3}}{-2t - 1 - i\sqrt{3}} \right) \right\} + C_1 \tag{11}$$

where

$$C = \frac{h}{2\pi}(\sqrt{3} + i), \quad C_1 = \frac{h - k}{4} + i\frac{\sqrt{3}}{12}(7k + h), \tag{12}$$

$$t = \left(\frac{z_1 + b^3}{z_1 - 1} \right)^{\frac{1}{3}}, \quad b = \frac{h}{k} \tag{13}$$

The streamlines of the current density calculated for these angles are shown in Figure 2.

$\alpha = 30°$; $h = 2$; $k = 1$ $\qquad$ $\alpha = 120°$; $h = 1,5$; $k = 1$

Figure 2. The streamlines of the current density.

3. CURRENT DENSITY CALCULATION

The current density is proportional to the derivative of the complex potential $W(z)$:

$$\frac{dW}{dz} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} \sim j_x - i \cdot j_y \tag{14}$$

On the other hand, due to (3),

$$\frac{dW}{dz} = \frac{dW}{dz_1}\frac{dz_1}{dz} \sim \frac{1}{z_1}\frac{dz_1}{dz} \tag{15}$$

So one can obtain an expression for the current density in terms of the implicit function $z_1(z)$:

$$j_x = A\mathrm{Re}\left[\mathfrak{J}(z_1(z))\right], \quad j_y = -A\mathrm{Im}\left[\mathfrak{J}(z_1(z))\right], \quad j = A\left|\mathfrak{J}(z_1(z))\right| \tag{16}$$

where A is a proportionality factor and

$$\mathfrak{J}(z_1) \equiv \frac{1}{z_1}\frac{dz_1}{dz} = \left(\frac{z_1 + a}{z_1 - 1}\right)^{1-\beta} \quad \text{[see (4)]} \tag{17}$$

The dependence $z_1(z)$ is given by Expression (6) in an implicit form and has to be calculated numerically.

The current strength J applied to the conductor is assumed to be specified. The current distribution for $y \to -\infty$ is uniform (Figure 2), therefore, one can assume that the current density at an infinite distance is $j_{-\infty} = J/k$. The line $y \to -\infty$, $x = $ const placed at the infinite distance is a mapping of the half-circle $r_1 \to 0$, $\theta_1 \in [0, \pi]$ where r_1 and θ_1 are the absolute value and the argument of z_1 respectively. The substitution of these values into (16) yields

$$j_{-\infty} = A\left|\frac{\pi}{h}ib\right| = A\frac{\pi}{k} \tag{18}$$

whence it follows that

$$A = \frac{J}{\pi} \tag{19}$$

The top of the angle on the complex plane z corresponds to $z_1 = 1 + i \cdot 0$. Substituting this value into (16), one obtains

$$j(z_1 = 1) = \frac{J}{\pi}\left(\frac{1 + a}{0}\right)^{1-\beta} \to \infty \tag{20}$$

The infinity emerges as a result of the division by zero at the angle's vertex. It should be noted that there are no points of zero size in reality, therefore, there are no point-like vertices. The assumption of vertices having zero sizes doesn't normally cause any trouble. But it is inappropriate for problems of current distributions.

An obvious way to eliminate the infinity is to replace the point-like vertex by a slight arc of a circle. There have been some attempts to do it; but an analytical solution was only found for the angle of 90° with rounding [10]. A technique exists for the Schwarz–Cristoffel mapping that provides a possibility to introduce a small arc rounding to the inner angles of the area considered [11]. Unfortunately, this technique only works well for arcs with a very small radius of curvature. Otherwise an arc turns to be highly asymmetrical.

4. CURRENT DISTRIBUTIONS IN FLAT CONDUCTORS WITH ROUNDED ANGLES

In order to avoid the difficulty described above, we have introduced one more parameter with no smallness restrictions imposed. Thus, the transformation that maps the upper complex half-plain onto the bent strip with a rounded corner (Figure 3) is as follows:

$$z_R = \tilde{C}\int\frac{1}{z_1}\left(\frac{z_1-1}{z_1+\tilde{a}}\right)^{1-\beta}dz_1 + \tilde{C}\gamma\int\frac{1}{z_1}\left(\frac{z_1-1+\delta_1}{z_1+\tilde{a}}\right)^{1-\beta}dz_1 + \tilde{C}\gamma\int\frac{1}{z_1}\left(\frac{z_1-1-\delta_2}{z_1+\tilde{a}}\right)^{1-\beta}dz_1 \tag{21}$$

$$\tilde{C} = \frac{h}{\pi(1 + 2\gamma)}(\sin\alpha - i\cos\alpha) \tag{22}$$
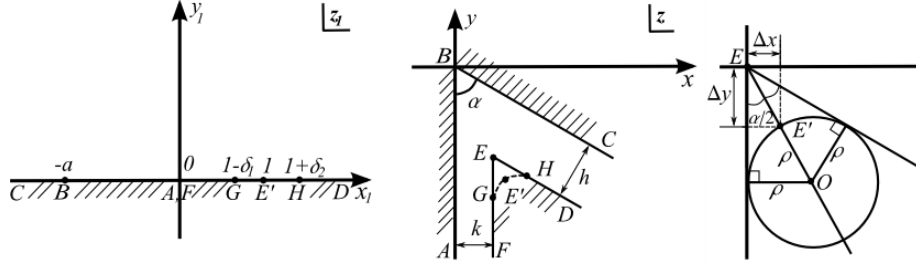
Figure 3. The points of conformity for a conductor with a rounded angle and the schematic diagram of the arc.

$$\tilde{a} = \left(\frac{h}{k}\right)^{\frac{1}{1-\beta}} \left\{\frac{1 + \gamma(1-\delta_1)^{1-\beta} + \gamma(1-\delta_2)^{1-\beta}}{1 + 2\gamma}\right\}^{\frac{1}{1-\beta}} \tag{23}$$

Here δ_1, δ_2, and γ are positive parameters that determine the shape of the arc and its radius of curvature. If they tend to zero, Expression (21) turns into (4). We introduce the following symbol (assuming again that $1 - \beta = P/Q$):

$$I(m) \equiv \tilde{C} \int \frac{1}{z_1}\left(\frac{z_1 - m}{z_1 + \tilde{a}}\right)^{\frac{P}{Q}} dz_1 \tag{24}$$

so that Expression (21) turns into

$$z_R = I(1) + \gamma I(1 - \delta_1) + \gamma I(1 + \delta_2) \tag{25}$$

Utilizing the following change of variables:

$$t_m = \left(\frac{z_1 + \tilde{a}}{z_1 - m}\right)^{\frac{1}{Q}}, \quad b_m = \left(\frac{\tilde{a}}{m}\right)^{\frac{1}{Q}} \tag{26}$$

we reduce integral (24) to a rational one:

$$I(m) = -(1 + b_m^Q)Q\tilde{C} \int \frac{t_m^{Q-1}}{t_m^P(t_m^Q + b_m^Q)(t_m^Q - 1)} dt \tag{27}$$

It in turn is formally equivalent to (6), therefore, all expressions obtained for strips with no rounding can be used here.

By simply varying the values of δ_1, δ_2, and γ, one can obtain various rounding arcs in a relatively wide range of shapes. In practice, however, the curvature radius is important. In order to associate it with the rounding parameters, we use simple geometrical considerations. The first two equations are obtained by expressing the lengths Δx and Δy through the curvature radius ρ. The last one represents the condition of equality between the distances GE' and $E'H$ (Figure 3).

$$\begin{cases} \mathrm{Re}[z_R(1)] \cong k + \rho\left(1 - \sin\frac{\alpha}{2}\right) \\ \mathrm{Im}[z_R(1)] \cong -\frac{h + k\cos\alpha}{\sin\alpha} - \rho\left(1 - \sin\frac{\alpha}{2}\right)\cot\frac{\alpha}{2} \\ |z_R(1) - z_R(1-\delta_1)|^2 = |z_R(1) - z_R(1+\delta_2)|^2 \end{cases} \tag{28}$$

This system has to be solved numerically for each curvature radius value.

The current density for the mapping considered is defined in a similar way to (16) with the following expression for $\mathfrak{J}(z_1)$:

$$\mathfrak{J}(z_1) = \left|\frac{1}{C}\frac{[z_1 + a]^{1-\beta}}{[z_1 - 1]^{1-\beta} + \gamma[z_1 - 1 + \delta_1]^{1-\beta} + \gamma[z_1 - 1 - \delta_2]^{1-\beta}}\right| \tag{29}$$

where $z_1 = z_1(z_R)$.

It is easy to prove that Expression (29) takes finite values all over the conductor.

The streamlines of the current density for conductors with the angles of 30° and 120° are presented in Figure 4.

$\alpha = 30°$; $h = 2$; $k = 1$; $\rho = 0.02k$ $\qquad$ $\alpha = 120°$; $h = 1.5$; $k = 1$; $\rho = 0.1k$
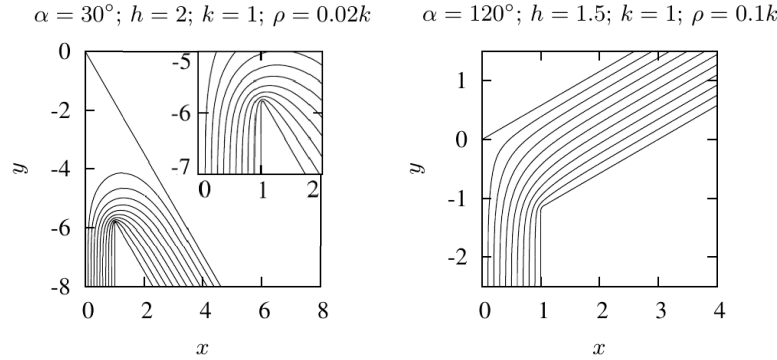
Figure 4. The streamlines of the current density in strips with rounded angles. A region near the corner is depicted with a larger scale to show the rounding more clearly.

5. CONCLUSION

The performed investigation has shown that an infinite current density at the vertex of an angle is a result of an inappropriate model of a conductor. This model assumed a vertex to be a point with a zero size. Therefore, an obvious way to avoid this problem is to replace the point by an arc of a circle with a small radius. We have succeeded in obtaining analytical results for flat conductors with arbitrary angles. The solution has demonstrated that rounding the corner eliminates the emergent infinity. Moreover, a dependence between the radius of an arc and the current density has been found. The obtained results may be used in connection with results of work [6] to calculate current density distributions in flat conductors with complex geometry. The conformal mapping technique utilized in our paper is applicable to two-dimensional problems but we hope that the presented results will also be helpful for numerical investigation of three-dimensional conductors.

REFERENCES

1. Choi, W. J., E. C. C. Yeh, and K. N. Tuc, "Mean-time-to-failure study of flip chip solder joints on Cu/Ni(V)/Al thin-film under-bump-metallization," *J. Appl. Phys.*, Vol. 94, 5665, 2003.
2. Shao, T. L., Y. H. Chen, S. H. Chiu, and C. Chen, "Electromigration failure mechanisms for SnAg3.5 solder bumps on Ti/Cr-Cu/Cu and Ni(P)/Au metallization pads," *J. Appl. Phys.*, Vol. 96, 4518, 2004.
3. Lai, Y.-S. and C.-L. Kao, "Characteristics of current crowding in flip-chip solder bumps," *Microelectronics Reliability*, Vol. 46, 915–922, 2006.
4. Lai, Y.-S. and C.-L. Kao, "Electrothermal coupling analysis of current crowding and Joule heating in flip-chip packages," *Microelectronics Reliability*, Vol. 46, 1357–1368, 2006.
5. Zhang, K., L. Gong, and R. Unbehauen, "Field analysis of a circular thin-film/foil resistor," *Archiv für Elektrotechnik*, Vol. 76, 423–426, 1993.
6. Gerasimenko, T. N., V. I. Ivanov, P. A. Polyakov, and V. Y. Popov, "Application of a conformal-mapping technique to a boundary-value problem of current distribution in plain conductors," *Journal of Mathematical Sciences*, Vol. 172, No. 6, 761–769, 2011.
7. Smythe, W. R., *Static and Dynamic Electricity*, McGraw-Hill Book Company Inc., New York, Toronto, London, 1950.
8. Trefethen, L. N., "Analysis and design of polygonal resistors by conformal mapping," *Zeitschrift für angewandte Mathematik und Physik*, Vol. 35, 692–704, 1984.
9. Kober, H., *Dictionary of Conformal Representations*, Dover, 1957.
10. Hagedorn, F. B. and P. M. Hall, "Right-angle bends in thin strip conductors," *J. Appl. Phys.*, Vol. 34, No. 1, 128–133, 1963.
11. Lavrentiev, M. A. and B. V. Shabat, *Methods of Theory of Functions of Complex Variable*, Nauka, Moscow, 1987 (in russian).