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Vector Potential Electromagnetics with Generalized Gauge for Inhomogeneous Media: Formulation

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(*Invited Paper*)

Abstract—The mixed vector and scalar potential formulation is valid from quantum theory to classical electromagnetics. The present rapid development in quantum optics applications calls for electromagnetic solutions that straddle both the quantum and classical physics regimes. The vector potential formulation using $\mathbf{A}$ and Φ (or $\mathbf{A}$-Φ formulation) is a good candidate to bridge these two regimes. Hence, there is a need to generalize this formulation to inhomogeneous media. A generalized gauge is suggested for solving electromagnetics problems in inhomogenous media that can be extended to the anistropic case. An advantage of the resulting equations is their absence of catastrophic breakdown at low-frequencies. Hence, the usual differential equation solvers can be used to solve them over a wide range of scales and bandwidth. It is shown that the interface boundary conditions from the resulting equations reduce to those of classical Maxwell's equations. Also, the classical Green's theorem can be extended to such a formulation, resulting in an extinction theorem and a surface equivalence principle similar to the classical case. Moreover, surface integral equation formulations can be derived for piecewise homogeneous scatterers. Furthermore, the integral equations neither exhibit the low-frequency catastrophe nor the frequency imbalance observed in the classical formulation using $\mathbf{E}$-$\mathbf{H}$ fields. The matrix representation of the integral equation for a PEC (perfect electric conductor) scatterer is given.

1. INTRODUCTION

Electromagnetic theory has been guided by Maxwell's equations for 150 years now [1]. The formulation of electromagnetic theory based on $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$, as simplified by Heaviside [2], offers physical insight that has resulted in the development of a myriad of electromagnetic-related technologies. However, there are certain situations where the $\mathbf{E}$-$\mathbf{H}$ formulation is not ideal. This is in the realm of quantum mechanics where the $\mathbf{A}$-Φ formulation is needed. There exists certain situations in quantum mechanics where $\mathbf{E}$-$\mathbf{H}$ are zero, but $\mathbf{A}$ is not zero, and yet, $\mathbf{A}$ is felt in quantum mechanics. This is true of the Aharonov-Bohm effect [3, 4]. Moreover, the quantization of electromagnetic field can be done more expediently with the vector potential $\mathbf{A}$ than with $\mathbf{E}$ and $\mathbf{H}$. More importantly, when the electromagnetic effect needs to be incorporated in the Schrödinger equation, both vector and scalar potentials are used; this will be important in many quantum optics studies [5–11].

Normally, electromagnetic equations formulated in terms of $\mathbf{E}$-$\mathbf{H}$ have low-frequency breakdown or catastrophe. Hence, many numerical methods based on $\mathbf{E}$-$\mathbf{H}$ formulation are unstable at low frequencies or long wavelengths. Therefore, the $\mathbf{E}$-$\mathbf{H}$ formulation is not truly multi-scale, but exhibits catastrophic breakdown when the dimensions of objects become much smaller than the local wavelength. Different formulations using tree-cotree, or loop-tree decomposition [12–16], have to be sought when the frequency is low or the wavelength is long. Due to the low-frequency catastrophe encountered by $\mathbf{E}$-$\mathbf{H}$ formulation, the vector potential formulation has become popular for solving low frequency problems [17–29].

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This work will arrive at a general theory of vector potential formulation for inhomogeneous anisotropic media, together with the pertinent integral equations. This vector potential formulation does not have the apparent low-frequency catastrophe of the **E**-**H** formulation and it is truly multi-scale. It can be shown that with the proper gauge, which is an extension of the simple Lorenz gauge [33] to inhomogeneous anisotropic media, the scalar potential equation is decoupled from the vector potential equation [17].

Given the capability of computational electromagnetic solvers, and its influence on electromagnetic design problems, there is a pressing need for computational electromagnetics methods that are truly multi-scale. There are many formulations in electromagnetics that can benefit from the vector potential formulation, such as momentum and stress tensors which are important for optical tweezers work or Casimir force calculations [30–32]. We will postpone these discussions to future papers.

Moreover, whenever possible, surface (boundary) integral equations are preferred over volume integral equation due to the curse of dimensionality. When an inhomogeneous medium can be approximated by union of piecewise-homogeneous regions, surface integral equation methods can be invoked to reduce computational cost. For such problems, unknowns only need to be assigned to interfaces or boundaries between regions. In this manner, a 3D problem is reduced to a problem on a 2D manifold, greatly reducing the number of unknowns needed. This is particularly true in recent years where fast algorithms have been developed to solve these surface integral equations rapidly [37–39], greatly underscoring their importance. To this end, we will discuss the derivation of generalized Green's theorem and surface integral equations for the vector potential formulation as well.

The goal of this paper is to arrive at a new formulation that is useful for computational electromagnetics (CEM) for multi-physics and multi-scale calculations. Hence, we study the **A**-Φ formulation (vector potential formulation) to this end. The paper is organized as follows. Section 2 derives the pertinent differential equations for the **A**-Φ formulation for inhomogeneous isotropic media. As boundary conditions are needed to solve inhomogeneous medium problem where the inhomogeneity is piecewise homogeneous, Section 3 derives the relevant boundary conditions. Section 4 generalizes some of the equations and boundary conditions in Sections 2 and 3 to the anisotropic media case. Section 5 compares the differential equations for the **E**-**H** formulation to the **A**-Φ formulation, explaining why the **E**-**H** formulation faces low-frequency catastrophe. Section 6 gives the generalized Green's theorem that can be used for surface equivalence principle and surface integral equations. Section 7 illustrates the use of the results of Section 6 to derive a surface integral equation for a perfect electric conductor (PEC) scatterer, and their resultant matrix representations. Section 8 discusses how the incident plane wave should be derived in the **A**-Φ formulation. Section 9 ends the paper with some discussions and conclusions. Some details of the more involved derivations are given in the Appendices.

2. PERTINENT EQUATIONS-INHOMOGENEOUS ISOTROPIC CASE

The vector potential formulation for homogeneous medium is described in most text books [33–36]. We derive the pertinent equations for the inhomogeneous isotropic medium case first. To this end, we begin with the Maxwell's equations:

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{1}$$
$$\nabla \times \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J}, \tag{2}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$
$$\nabla \cdot \mathbf{D} = \rho. \tag{4}$$

From the above, we let

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{5}$$
$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \Phi \tag{6}$$

so that the first and third of the four Maxwell's equations are automatically satisfied. Then, using $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$ for isotropic, non-dispersive, inhomogeneous media, we obtain from (2) and (4) that

$$-\partial_t \nabla \cdot \varepsilon \mathbf{A} - \nabla \cdot \varepsilon \nabla \Phi = \rho, \tag{7}$$
$$\nabla \times \mu^{-1} \nabla \times \mathbf{A} = -\varepsilon \partial_t^2 \mathbf{A} - \varepsilon \partial_t \nabla \Phi + \mathbf{J}. \tag{8}$$

For homogeneous medium, the above reduces to

$$-\partial_t \nabla \cdot \mathbf{A} - \nabla^2 \Phi = \rho/\varepsilon, \tag{9}$$

$$\mu^{-1}\left(\nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}\right) = -\varepsilon\partial_t^2 \mathbf{A} - \varepsilon\partial_t \nabla\Phi + \mathbf{J}. \tag{10}$$

By using the simple Lorenz gauge

$$\nabla \cdot \mathbf{A} = -\mu\varepsilon\partial_t \Phi \tag{11}$$

we have the usual

$$\nabla^2 \Phi - \mu\varepsilon\partial_t^2 \Phi = -\rho/\varepsilon, \tag{12}$$

$$\nabla^2 \mathbf{A} - \mu\varepsilon\partial_t^2 \mathbf{A} = -\mu\mathbf{J} \tag{13}$$

The Lorenz gauge is preferred because it treats space and time on the same footing as in special relativity [33].

For inhomogeneous media, we can choose the generalized Lorenz gauge. This gauge has been suggested previously, for example in [17, 25].

$$\varepsilon^{-1}\nabla \cdot \varepsilon\mathbf{A} = -\mu\varepsilon\partial_t \Phi. \tag{14}$$

However, we can also decouple (7) and (8) using an even more generalized gauge, namely

$$\nabla \cdot \varepsilon\mathbf{A} = -\chi\partial_t \Phi \tag{15}$$

Then we obtain from (7) and (8) that †

$$\nabla \cdot \varepsilon\nabla\Phi - \chi\partial_t^2 \Phi = -\rho, \tag{16}$$

$$-\nabla \times \mu^{-1}\nabla \times \mathbf{A} - \varepsilon\partial_t^2 \mathbf{A} + \varepsilon\nabla\left(\chi^{-1}\nabla \cdot \varepsilon\mathbf{A}\right) = -\mathbf{J}. \tag{17}$$

It is to be noted that (16) can be derived from (17) by taking the divergence of (17) and then using the generalized gauge and the charge continuity equation $\nabla \cdot \mathbf{J} = -\partial_t \rho$. In general, we can choose

$$\chi = \alpha\varepsilon^2\mu \tag{18}$$

where α can be a function of position $\mathbf{r}$. When $\alpha = 1$, it reduces to the generalized Lorenz gauge used in (14).

For homogeneous medium, (16) and (17) reduce to (12) and (13) when we choose $\alpha = 1$ in (18), which reduces to the simple Lorenz gauge. Unlike the vector wave equations for inhomogeneous electromagnetic fields, the above do not have the apparent low-frequency breakdown difficulty when $\partial_t = 0$. Hence, the above equations can be used for electrodynamics as well as electrostatics where the frequency is zero.

We can rewrite the above as a sequence of three equations, namely,

$$\nabla \cdot \varepsilon\mathbf{A} = -\chi\partial_t \Phi \tag{19}$$

$$\nabla \times \mathbf{A} = \mu\mathbf{H} \tag{20}$$

$$\nabla \times \mathbf{H} + \varepsilon\partial_t^2 \mathbf{A} + \varepsilon\nabla\partial_t \Phi = \mathbf{J} \tag{21}$$

The last equation can be rewritten as

$$\nabla \times \mathbf{H} - \varepsilon\partial_t(-\partial_t \mathbf{A} - \nabla\Phi) = \mathbf{J} \tag{22}$$

which is the same as solving Ampere's law. Hence, solving (17) is similar to solving Maxwell's equations.

It is to be noted that (17) resembles the elastic wave equation in solids where both longitudinal and transverse waves can exist [40, 41]. Furthermore, the two waves can have different velocities in a homogeneous medium if $\alpha \neq 1$ in (18). The longitudinal wave has the same velocity as the scalar potential, which is $1/\sqrt{\alpha\mu\varepsilon}$, while the transverse wave has the velocity of light, or $1/\sqrt{\mu\varepsilon}$. If we choose $\alpha = 0$, or $\chi = 0$, we have the Coulomb gauge where the scalar potential has an infinite velocity. In this case, the vector potential equation is not completely decoupled from the scalar potential equation.

It should be noted that (16) and (17) can be solved by numerical PDE (partial differential equation) solvers such as the finite difference or finite element method without suffering low-frequency catastrophe.

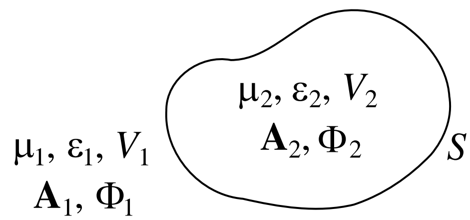

Figure 1. The boundary conditions for the interface between two piecewise homogeneous regions.

3. BOUNDARY CONDITIONS FOR THE POTENTIALS

The above equations, (16) and (17), are the governing equations for the scalar and vector potentials Φ and $\mathbf{A}$ for inhomogeneous media. The boundary conditions at the interface of two homogeneous media are also embedded in these equations. Therefore, when one solves the PDEs directly in an inhomogeneous medium, one need not stipulate the boundary conditions. The solutions naturally obey the boundary conditions if they are arrived at correctly via a numerical method.

However, as mentioned previously, there is an advantage to reducing an inhomogeneous medium to union of piecewise homogeneous regions whenever possible to reduce the computational costs. In this case, the solutions can be sought in each of the piecewise homogenous region, and then sewn (stitched) together using boundary conditions. One method is to write the solution in a piecewise homogeneous region using the generalized Green's theorem (to be derived in Section 6). Then the solutions are stitched together using relevant boundary conditions at the interfaces. More recently, the use of domain decomposition method divides a larger problem into union of smaller problems and the solutions of the smaller problems can be sewn together using boundary conditions (see [42] and references therein).

The boundary conditions can be derived by using pill-box method as expounded in many books, but such derivations can be quite laborious. Alternatively, by eyeballing (observe carefully) Equation (17), we see that $\nabla \times \mathbf{A}$ must be finite at an interface. This induces the boundary condition that

$$\hat{n} \times \mathbf{A}_1 = \hat{n} \times \mathbf{A}_2 \tag{23}$$

across an interface (see Fig. 1). Assuming that $\mathbf{J}$ is finite, we also have

$$\hat{n} \times \frac{1}{\mu_1} \nabla \times \mathbf{A}_1 = \hat{n} \times \frac{1}{\mu_2} \nabla \times \mathbf{A}_2. \tag{24}$$

The above is equivalent to

$$\hat{n} \times \mathbf{H}_1 = \hat{n} \times \mathbf{H}_2 \tag{25}$$

at an interface. When a surface current sheet is present, we have to augment the above with the current sheet as is done in standard electromagnetic boundary conditions. Furthermore, due to the finiteness of $\nabla \cdot \varepsilon \mathbf{A}$ at an interface, it is necessary that

$$\hat{n} \cdot \varepsilon_1 \mathbf{A}_1 = \hat{n} \cdot \varepsilon_2 \mathbf{A}_2. \tag{26}$$

It can be shown that if a surface current dipole layer exists at an interface, we will have to augment the above with the correct discontinuity or jump condition.

By the same token, eyeballing the scalar potential Equation (16), we notice that

$$\hat{n} \cdot \varepsilon_1 \nabla \Phi_1 = \hat{n} \cdot \varepsilon_2 \nabla \Phi_2. \tag{27}$$

The above must be augmented with the necessary jump or discontinuity condition if a surface charge layer exists at an interface. Equations (26) and (27) together imply that

$$\hat{n} \cdot \varepsilon_1 \mathbf{E}_1 = \hat{n} \cdot \varepsilon_2 \mathbf{E}_2. \tag{28}$$

where we have noted that $\mathbf{E} = -\partial_t \mathbf{A} - \nabla \Phi$ from (6). This is the usual boundary condition for the normal component of the electric field.

† The author thanks a reviewer for pointing out Nisbet's [17] work. Nisbet was interested in solving these equations in curvilinear coordinates then.

Equation (16) also implies that

$$\Phi_1 = \Phi_2, \tag{29}$$

or

$$\hat{n} \times \nabla\Phi_1 = \hat{n} \times \nabla\Phi_2. \tag{30}$$

Equations (23) and (30) imply that

$$\hat{n} \times \mathbf{E}_1 = \hat{n} \times \mathbf{E}_2. \tag{31}$$

This is the usual boundary condition for the tangential component of the electric field.

If region 2 is a perfect electric conductor (PEC), $\varepsilon_2 \to \infty$.[‡] From (17), this implies that $\mathbf{A}_2 = 0$, if $\omega \neq 0$ or $\partial_t \neq 0$. Then (23) for a PEC surface becomes

$$\hat{n} \times \mathbf{A}_1 = 0. \tag{32}$$

Also, observing carefully (16), we see that for a PEC, $\Phi_2 = 0$. This together with (29), (30), and (32) imply that $\hat{n} \times \mathbf{E}_1 = 0$ on a PEC surface. For a perfect magnetic conductor (PMC), $\mu_2 \to \infty$, and from (24) and (25)

$$\hat{n} \times \mathbf{H}_1 = 0. \tag{33}$$

When $\omega = 0$ or $\partial_t = 0$, $\mathbf{A}$ does not contribute to $\mathbf{E}$. But from (27), when $\varepsilon_2 \to \infty$, we deduce that $\hat{n} \cdot \nabla\Phi_2 = 0$, implying that $\Phi_2 =$ constant for $\mathbf{r}_2 \in V_2$ for arbitrary S.[††] Hence, from (29) and (30), $\hat{n} \times \mathbf{E}_1 = 0$ on a PEC surface even when $\omega = 0$.

4. GENERAL ANISOTROPIC MEDIA CASE

For inhomogeneous, dispersionless, anisotropic media, the generalized gauge becomes

$$\nabla \cdot \bar{\boldsymbol{\varepsilon}} \cdot \mathbf{A} = -\chi\partial_t\Phi \tag{34}$$

In the above, χ is an arbitrary function of $\mathbf{r}$, but we can choose

$$\chi = \alpha|\bar{\boldsymbol{\varepsilon}} \cdot \bar{\boldsymbol{\mu}} \cdot \bar{\boldsymbol{\varepsilon}}| \tag{35}$$

where the vertical bars indicate a determinant. When the medium is inhomogeneous but isotropic, the above gauge reduces to the generalized gauge previously discussed. When $\alpha = 1$, the above reduces to the generalized Lorenz gauge for inhomogeneous isotropic media. In general, (16) and (17) become [17]

$$\nabla \cdot \bar{\boldsymbol{\varepsilon}} \cdot \nabla\Phi - \chi\partial_t^2\Phi = -\rho \tag{36}$$

$$\nabla \times \bar{\boldsymbol{\mu}}^{-1}\nabla \times \mathbf{A} + \bar{\boldsymbol{\varepsilon}} \cdot \partial_t^2\mathbf{A} - \bar{\boldsymbol{\varepsilon}} \cdot \nabla\left(\chi^{-1}\nabla \cdot \bar{\boldsymbol{\varepsilon}} \cdot \mathbf{A}\right) = \mathbf{J} \tag{37}$$

The above can be rewritten in the manner of (19) to (22), showing that solving the above is the same as solving the original Maxwell's equations. The boundary condition (23) remains the same. Boundary condition (24) becomes

$$\hat{n} \times \bar{\boldsymbol{\mu}}_1^{-1} \cdot \nabla \times \mathbf{A}_1 = \hat{n} \times \bar{\boldsymbol{\mu}}_2^{-1} \cdot \nabla \times \mathbf{A}_2 \tag{38}$$

and boundary condition (25) remains the same. Similarly, boundary condition (26) becomes

$$\hat{n} \cdot \bar{\boldsymbol{\varepsilon}}_1 \cdot \mathbf{A}_1 = \hat{n} \cdot \bar{\boldsymbol{\varepsilon}}_2 \cdot \mathbf{A}_2. \tag{39}$$

The boundary condition (27) becomes

$$\hat{n} \cdot \bar{\boldsymbol{\varepsilon}}_1 \cdot \nabla\Phi_1 = \hat{n} \cdot \bar{\boldsymbol{\varepsilon}}_2 \cdot \nabla\Phi_2 \tag{40}$$

while the other boundary conditions, similar to the isotropic case, can be similarly derived.

[‡] For a more general argument, we can convert the equations into the time-harmonic case. In this case, $\varepsilon_2 = \varepsilon_2' + i\sigma_2/\omega$. $\varepsilon_2 \to \infty$ means $\sigma_2/\omega \to \infty$.

[††] A more elaborate argument shows that if the V_2 spans a region much less than the skin depth $\sqrt{2/\omega\mu\sigma_2}$, then Φ_2 is approximately a constant within V_2.

5. COMPARING WITH E AND H FORMULATION

The vector wave equations for **E**-**H** formulation for inhomogeneous media, assuming $\exp(-i\omega t)$ time dependence, are [43]

$$\nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) = i\omega \mathbf{J}(\mathbf{r}) - \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}(\mathbf{r}), \tag{41}$$

$$\nabla \times \bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}(\mathbf{r}) - \omega^2 \bar{\boldsymbol{\mu}} \cdot \mathbf{H}(\mathbf{r}) = i\omega \mathbf{M}(\mathbf{r}) + \nabla \times \bar{\boldsymbol{\epsilon}}^{-1} \cdot \mathbf{J}(\mathbf{r}), \tag{42}$$

where **J** and **M** are the impressed sources in the medium. It is seen that these equations are not solvable when $\omega \to 0$. Hence, any solutions obtained with $\omega \neq 0$ experience a breakdown when $\omega \to 0$. This is even true of the integral equations derived using **E**-**H** formulation. Moreover, to retrieve the electrostatic and magnetostatic solutions, one has to go back to "the drawing board" to rederive the the equations for them.

In the **A**-Φ formulation, (16) and (17) do not show low frequency catastrophe. However, electrostatic solution is captured in (16) while the magnetostatic solution is in (17) when $\omega = 0$. Therefore, to retrieve both the electroquasistatic and magnetoquasistatic solutions accurately, (16) and (17) should be solved in tandem. This is the low frequency inaccuracy problem when the electroquasistatic solution has to be retrieved from the magnetoquasistatic solution and vice versa. This low frequency inaccuracy problem has been observed in the magnetic field integral equation (MFIE) [53] and the augmented electric field integral equation (A-EFIE) [54]. The above statements also apply to (36) and (37).

An advantage of (41) and (42) is that they are derivable from each other, whereas for the **A**-Φ formulation, only the Φ equation is derivable from the **A** equation. Hence, when the **E** and the **H** fields are equally strong, and strongly coupled to each other, the **E**-**H** formulation is preferred, as it describes wave physics better.

6. GENERALIZED GREEN'S THEOREM, EXTINCTION THEOREM, AND EQUIVALENCE PRINCIPLE — TIME HARMONIC CASE

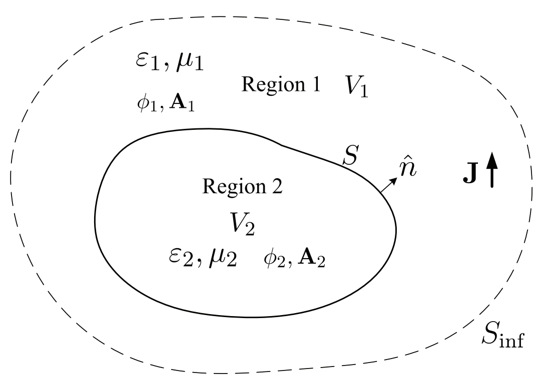

Figure 2. Configuration of media and regions used to derive the Green's theorem for vector potential equation.

As mentioned previously, for inhomogeneous media consisting of piecewise homogeneous regions, it is best to seek the solution in each region first, and then the solutions sewn together by boundary conditions. To this end, we need to derive the equivalence of the Green's theorem for the vector potential formulation. In the following, we assume a simple Lorenz gauge so that the equations for homogeneous regions greatly simplify. These surface integral equation methods have a distinct advantage only if the Green's functions can be found in closed form for each homogeneous region. In other words, we need to derive Green's theorem's equivalence for

$$(\nabla^2 + k^2)\mathbf{A}(\mathbf{r}) = -\mu \mathbf{J}(\mathbf{r}). \tag{43}$$

where $k^2 = \omega^2\mu\varepsilon$ and the time dependence is $\exp(-i\omega t)$. It is more expedient to write the above as[§]

$$\nabla\times\nabla\times\mathbf{A}(\mathbf{r})-\nabla\nabla\cdot\mathbf{A}(\mathbf{r})-k^2\mathbf{A}(\mathbf{r})=\mu\mathbf{J}(\mathbf{r}). \tag{44}$$

We can define a dyadic Green's function that satisfies

$$\nabla\times\nabla\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')-\nabla\nabla\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')-k^2\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}'). \tag{45}$$

The solution to the above is simply

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\overline{\mathbf{I}}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}=\overline{\mathbf{I}}\ g(\mathbf{r},\mathbf{r}'). \tag{46}$$

From the above, using methods outlined in [43, Chapter 8], as well as in Appendix A, we have for region 1 (see Fig. 2),

$$\left.\begin{matrix}\mathbf{r}\in V_1, & \mathbf{A}_1(\mathbf{r})\\ \mathbf{r}\in V_2, & 0\end{matrix}\right\}=\mathbf{A}_{\mathrm{inc}}(\mathbf{r})+\int_S dS'\left\{\mu_1\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\hat{n}'\times\mathbf{H}_1(\mathbf{r}')-\left[\nabla'\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\right]\cdot\hat{n}'\times\mathbf{A}_1(\mathbf{r}')\right\}$$
$$-\int_S dS'\ \hat{n}'\cdot\left\{\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\nabla'\cdot\mathbf{A}_1(\mathbf{r}')-\mathbf{A}_1(\mathbf{r}')\nabla'\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\right\}. \tag{47}$$

We can rewrite the above using a scalar Green's function as

$$\left.\begin{matrix}\mathbf{r}\in V_1, & \mathbf{A}_1(\mathbf{r})\\ \mathbf{r}\in V_2, & 0\end{matrix}\right\}=\mathbf{A}_{\mathrm{inc}}(\mathbf{r})+\int_S dS'\Big\{\mu_1 g_1(\mathbf{r},\mathbf{r}')\hat{n}'\times\mathbf{H}_1(\mathbf{r}')-\nabla' g_1(\mathbf{r},\mathbf{r}')\times\hat{n}'\times\mathbf{A}_1(\mathbf{r}')\Big\}$$
$$+\int_S dS'\Big\{-\hat{n}'g_1(\mathbf{r},\mathbf{r}')\nabla'\cdot\mathbf{A}_1(\mathbf{r}')+\hat{n}'\cdot\mathbf{A}_1(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}')\Big\}. \tag{48}$$

The above can be used to state the equivalence principle for vector potential $\mathbf{A}_1$. They state that the field in region 1 can be reproduced by equivalent sources placed on the surface S. These sources extinct the field in region 2. The vanishing lower parts on the left-hand sides of the above equations constitute what is known as the extinction theorem [43, 44]. A similar equation can be derived for region 2. These equations for the two regions can be used to formulate surface integral equations of scattering of a penetrable scatterer. In the above, the Lorenz gauge can be used to replace $\nabla\cdot\mathbf{A}$ with the scalar potential Φ. It is to be noted that six scalar field quantities are needed in (48) to invoke the equivalence principle, namely, two tangential components of $\mathbf{H}_1$, two tangential components of $\mathbf{A}_1$, Φ and $\hat{n}\cdot\mathbf{A}_1$. This is very much like the augmented equivalence principle algorithm (A-EPA) where six scalar field components are needed to maintain low-frequency stability and accuracy [45–47].

As a side note, one can use the scalar Green's theorem directly on (43) and obtain

$$\left.\begin{matrix}\mathbf{r}\in V_1, & \mathbf{A}_1(\mathbf{r})\\ \mathbf{r}\in V_2, & 0\end{matrix}\right\}=\mathbf{A}_{\mathrm{inc}}(\mathbf{r})-\int_S dS'\Big\{g_1(\mathbf{r},\mathbf{r}')\hat{n}'\cdot\nabla'\mathbf{A}_1(\mathbf{r}')-\hat{n}'\cdot\nabla' g_1(\mathbf{r},\mathbf{r}')\mathbf{A}_1(\mathbf{r}')\Big\}. \tag{49}$$

After some lengthy manipulations, (48) becomes (49), as shown in Appendix A. In the above derivation, there is a surface integral at infinity that can be shown to vanish as in [43, Chapter 8] when the radiation condition is invoked.

It will be interesting to ponder the meaning of the different terms in (48). It is to be noted that the surface charge on the PEC surface is given by

$$\hat{n}\cdot\varepsilon_1\mathbf{E}_1=\hat{n}\cdot\varepsilon_1(i\omega\mathbf{A}_1-\nabla\Phi_1)=\sigma_1 \tag{50}$$

Hence, if a surface charge σ_1 exists on the PEC surface, it causes the coupling of the vector potential to the scalar potential. The scalar potential Φ can be obtained from the vector potential using Lorenz gauge, namely, $\nabla\cdot\mathbf{A}=i\omega\mu\varepsilon\Phi$. Hence, one can view that $\hat{n}\cdot\mathbf{A}$ as the contribution to the surface charge from the vector potential $\mathbf{A}$. In fact, using the Lorenz gauge, and that $\mathbf{E}=i\omega\mathbf{A}-\nabla\Phi$, plus much manipulation, one can derive from (48) that (see Appendix B)

[§] If $\alpha\neq 1$, the ensuing equation is of the form

$$\nabla\times\nabla\times\mathbf{A}(\mathbf{r})-\alpha^{-1}\nabla\nabla\cdot\mathbf{A}(\mathbf{r})-k^2\mathbf{A}(\mathbf{r})=\mu\mathbf{J}(\mathbf{r}).$$

But the dyadic Green's function of such an equation can still be found using methods outlined in [40].

$$\left.\begin{array}{ll}\mathbf{r}\in V_1, & \mathbf{E}_1(\mathbf{r}) \\ \mathbf{r}\in V_2, & 0\end{array}\right\} = \mathbf{E}_{\text{inc}}(\mathbf{r}) + \int_S dS' \left\{ i\omega\mu_1 g_1(\mathbf{r},\mathbf{r}')\hat{n}'\times\mathbf{H}_1(\mathbf{r}') + \nabla' g_1(\mathbf{r},\mathbf{r}')\frac{\sigma_1(\mathbf{r}')}{\epsilon_1} \right\}$$
$$+\nabla\times\int_S dS' g_1(\mathbf{r},\mathbf{r}')\hat{n}'\times\mathbf{E}_1(\mathbf{r}'). \tag{51}$$

In the above, we can define $\mathbf{J}_1(\mathbf{r}') = \hat{n}'\times\mathbf{H}_1(\mathbf{r}')$, and $\nabla'\cdot\mathbf{J}_1(\mathbf{r}') = i\omega\sigma_1(\mathbf{r}')$. The above is just the Green's theorem for the **E**-field formulation. A similar **H**-field formulation can be derived. It is to be noted that when the vector and scalar potential terms are equally important, the equivalence principle can be represented by four scalar field components, two for the tangential components of $\mathbf{H}_1$ and two for the tangential components of $\mathbf{E}_1$. This is more expeditious compared to the **A**-Φ formulation. Hence, the **E**-**H** formulation is preferred for electrodynamics.

For a PEC, the above becomes

$$\mathbf{E}_1 = \mathbf{E}_{\text{inc}} + \int_S dS' \left\{ i\omega\mu_1 g_1(\mathbf{r},\mathbf{r}')\mathbf{J}_1(\mathbf{r}') - \nabla g_1(\mathbf{r},\mathbf{r}')\frac{\sigma_1(\mathbf{r}')}{\varepsilon_1(\mathbf{r}')} \right\} \tag{52}$$

The above is just the traditional relationship between the **E** field in region 1 and the sources on the PEC surface. A similar **H**-field equation can be derived.

It is to be noted that for some problems, (48) should be augmented with the Green's theorem for scalar potential, which has also been shown to be derivable from (48) in Appendix B. This is especially important for low frequency where the electrostatic field is decoupled from the magnetostatic field. Both of them, (48) and (B4), need to be solved in tandem to overcome the low-frequency inaccuracy problem.

7. PEC SCATTERER CASE

For a PEC scatterer, we have proved that $\hat{n}\times\mathbf{A}_1 = 0$. Since $\Phi = 0$ on a PEC surface, we have $\nabla\cdot\mathbf{A}_1 = i\omega\mu_1\varepsilon_1\Phi = 0$. Hence, for surface sources that satisfy the PEC scattering solution, then (48) becomes

$$\left.\begin{array}{ll}\mathbf{r}\in V_1, & \mathbf{A}_1(\mathbf{r}) \\ \mathbf{r}\in V_2, & 0\end{array}\right\} = \mathbf{A}_{\text{inc}}(\mathbf{r}) + \int_S dS' \Big\{ \mu_1 g_1(\mathbf{r},\mathbf{r}')\hat{n}'\times\mathbf{H}_1(\mathbf{r}') + \hat{n}'\cdot\mathbf{A}_1(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}') \Big\}. \tag{53}$$

The first term in the integral comes from the induced surface current flowing on the PEC surface. We can rewrite (53) in terms of two integral expressions

$$\hat{t}\cdot\mathbf{A}_1(\mathbf{r}) = \hat{t}\cdot\mathbf{A}_{\text{inc}}(\mathbf{r}) + \hat{t}\cdot\int_S d\,S' \Big\{ \mu_1 g_1(\mathbf{r},\mathbf{r}')\mathbf{J}_1(\mathbf{r}') + \Sigma_1(\mathbf{r}')\nabla' g(\mathbf{r},\mathbf{r}') \Big\}, \qquad \mathbf{r}\in S^+ \tag{54}$$

$$\Sigma_1(\mathbf{r}) = \Sigma_{\text{inc}}(\mathbf{r}) + \int_S d\,S' \Big\{ \mu_1 g_1(\mathbf{r},\mathbf{r}')\ \hat{n}\cdot\mathbf{J}_1(\mathbf{r}') + \Sigma_1(\mathbf{r}')\ \hat{n}\cdot\nabla' g(\mathbf{r},\mathbf{r}') \Big\}, \qquad \mathbf{r}\in S^+ \tag{55}$$

where $\hat{t}$ is an arbitrary tangential vector, and $\Sigma(\mathbf{r}) = \hat{n}\cdot\mathbf{A}(\mathbf{r})$. Hence, the two equations represent imposing the tangential and normal components of (53) on S^+ where S^+ refers to a surface slightly larger than S.$^{\|}$ Also, the boundary condition is such that $\hat{n}\times\mathbf{A}_1(\mathbf{r}) = 0$ on S. The above can be solved by a subspace projection method such as the Galerkin's [48] or moment methods [49, 50]. The unknowns are $\mathbf{J}_1$ and Σ_1 while $\mathbf{A}_{\text{inc}}$ and Σ_{inc} are known. We expand the unknowns in terms of basis functions $\mathbf{J}_n$ and σ_m that span the subspaces of $\mathbf{J}_1$ and Σ_1, respectively. Namely,

$$\mathbf{J}_1(\mathbf{r}') = \sum_{n=1}^{N} j_n \mathbf{J}_n(\mathbf{r}') \tag{56}$$

$$\Sigma_1(\mathbf{r}') = \sum_{m=1}^{M} s_m \sigma_m(\mathbf{r}') \tag{57}$$

$^{\|}$ If $\mathbf{r}\in S^-$ instead, the left-hand sides of the above would be zero in accordance with the extinction theorem expressed in (53).

We choose $\mathbf{J}_n(\mathbf{r}')$ to be a divergence-conforming tangential current so that the vector potential $\mathbf{A}_1$ that it produces is also divergence conforming [51, 52]. In the above, $\sigma_m(\mathbf{r}')$ should be chosen to well-approximate a surface charge well. After expanding the unknowns, we project the field that they produce onto the subspace spanned by the same unknown set as in the process of testing in the Galerkin's method.¶ Consequently, (54) and (55) become

$$0 = \langle \mathbf{J}_{n'}(\mathbf{r}), \mathbf{A}_{\mathrm{inc}}(\mathbf{r})\rangle + \mu_1 \sum_{n=1}^{N} \langle \mathbf{J}_{n'}(\mathbf{r}), g_1(\mathbf{r},\mathbf{r}'), \mathbf{J}_n(\mathbf{r})\rangle j_n + \sum_{m=1}^{M} s_m \langle \mathbf{J}_{n'}(\mathbf{r}), \nabla' g_1(\mathbf{r},\mathbf{r}'), \sigma_m(\mathbf{r}')\rangle \tag{58}$$

$$\begin{aligned}\sum_{m=1}^{M} s_m \langle \sigma_{m'}(\mathbf{r}), \sigma_m(\mathbf{r})\rangle \;=\; & \langle \sigma_{m'}(\mathbf{r}), \Sigma_{\mathrm{inc}}(\mathbf{r})\rangle + \mu_1 \sum_{n=1}^{N} \langle \sigma_{m'}(\mathbf{r}), \hat{n} g_1(\mathbf{r},\mathbf{r}'), \mathbf{J}_n(\mathbf{r}')\rangle j_n \\ & + \sum_{m=1}^{M} \langle \sigma_{m'}(\mathbf{r}), \hat{n}\cdot\nabla' g_1(\mathbf{r},\mathbf{r}'), \sigma_m(\mathbf{r}')\rangle s_m \end{aligned} \tag{59}$$

The above is a matrix system of the form

$$0 \;=\; \mathbf{a}_{\mathrm{inc}} + \bar{\mathbf{\Gamma}}_{1,\mathbf{J},\mathbf{J}} \cdot \mathbf{j} + \bar{\mathbf{\Gamma}}_{1,\mathbf{J},\sigma} \cdot \mathbf{s} \tag{60}$$

$$\overline{\mathbf{B}} \cdot \mathbf{s} \;=\; \mathbf{\Sigma}_{\mathrm{inc}} + \bar{\mathbf{\Gamma}}_{1,\sigma,\mathbf{J}} \cdot \mathbf{j} + \bar{\mathbf{\Gamma}}_{1,\sigma,\sigma} \cdot \mathbf{s} \tag{61}$$

where $\mathbf{j}$ and $\mathbf{s}$ are unknowns, while $\mathbf{a}_{\mathrm{inc}}$ and $\mathbf{\Sigma}_{\mathrm{inc}}$ are known. In detail, elements of the above matrices and vectors are given by

$$[\mathbf{a}_{\mathrm{inc}}]_{n'} \;=\; \langle \mathbf{J}_{n'}(\mathbf{r}), \mathbf{A}_{\mathrm{inc}}(\mathbf{r})\rangle \tag{62}$$

$$[\bar{\mathbf{\Gamma}}_{1,\mathbf{J},\mathbf{J}}]_{n',n} = \mu \langle \mathbf{J}_{n'}(\mathbf{r}), g_1(\mathbf{r},\mathbf{r}'), \mathbf{J}_n(\mathbf{r}')\rangle \tag{63}$$

$$[\bar{\mathbf{\Gamma}}_{1,\mathbf{J},\sigma}]_{n',m} = \langle \mathbf{J}_{n'}(\mathbf{r}), \nabla' g_1(\mathbf{r},\mathbf{r}'), \sigma_m(\mathbf{r}')\rangle \tag{64}$$

$$[\overline{\mathbf{B}}]_{m',m} \;=\; \langle \sigma_{m'}(\mathbf{r}), \sigma_m(\mathbf{r})\rangle \tag{65}$$

$$[\mathbf{\Sigma}_{\mathrm{inc}}]_{m'} \;=\; \langle \sigma_{m'}(\mathbf{r}), \Sigma_{\mathrm{inc}}(\mathbf{r})\rangle \tag{66}$$

$$[\bar{\mathbf{\Gamma}}_{1,\sigma,\mathbf{J}}]_{m',n} = \mu_1 \langle \sigma_{m'}(\mathbf{r}), \hat{n}\; g_1(\mathbf{r},\mathbf{r}'), \mathbf{J}_n(\mathbf{r}')\rangle \tag{67}$$

$$[\bar{\mathbf{\Gamma}}_{1,\sigma,\sigma}]_{m',m} = \; \langle \sigma_{m'}(\mathbf{r}), \hat{n}\cdot\nabla' g_1(\mathbf{r},\mathbf{r}'), \sigma_m(\mathbf{r}')\rangle \tag{68}$$

$$[\mathbf{j}]_n = j_n, \qquad [\mathbf{s}]_m = s_m \tag{69}$$

Furthermore, in the above,

$$\langle \mathbf{f}(\mathbf{r}), \mathbf{h}(\mathbf{r})\rangle \;=\; \int_S dS \mathbf{f}(\mathbf{r}) \cdot \mathbf{h}(\mathbf{r}) \tag{70}$$

$$\langle \mathbf{f}(\mathbf{r}), \gamma(\mathbf{r},\mathbf{r}'), \mathbf{h}(\mathbf{r})\rangle \;=\; \int_S dS \mathbf{f}(\mathbf{r}) \cdot \int_S dS' \gamma(\mathbf{r},\mathbf{r}') \mathbf{h}(\mathbf{r}') \tag{71}$$

where $\mathbf{f}(\mathbf{r})$ and $\mathbf{h}(\mathbf{r})$ can be replaced by scalar functions, and $\gamma(\mathbf{r},\mathbf{r}')$ can be replaced by a vector function with the appropriate inner products between them.

The $\bar{\mathbf{\Gamma}}$ matrices above are different matrix representations of the scalar Green's function and its derivative. It is to be noted that all the $\bar{\mathbf{\Gamma}}$ matrices above do not contain low-frequency catastrophe property as in the matrix representation of the dyadic Green's function.+ Hence, the above behaves like the augmented electric field integral equation (A-EFIE) [54]. It also bears some similarity to the

¶ The optimal choice of testing function can also be obtained by using dual space concept expounded in [51], or variational method [43], or differential forms [52].

+ This can be shown easily by noting that when $\omega \to 0$, the above matrices exist by Taylor series expansion about $\omega = 0$.

augmented equation introduced in [55]. Some of these pros are also pointed in [56] ♯. It is to be noted that for some problems, when $\omega \to 0$, the $\mathbf{A}$ equation will capture the magnetoquasistatic solutions well. To capture the electroquasistatic solutions well, the scalar potential integral equation has to be derived. The surface integral equations for vector potential, for some problems, can be solved in tandem with the surface integral equation for the scalar potential, such as the equivalence of (B4) for PEC, in order to capture both low-frequency physics well.

When $\omega = 0$, for a PEC object, the non-trivial magnetostatic solution exists without external excitations for objects with genus larger than 0 (a doughnut shape or a doughnut with multiple holes). These are the superconducting current loops that are found naturally and physically in superconductors. They are the natural modes of the equation (see [57] and references therein) giving rise to null spaces at zero frequency. Since they are natural and physical, they cannot be removed, but can be avoided by solving the problem at non-zero frequencies.

8. VECTOR POTENTIAL PLANE WAVE

Since many scattering problems are solved with incident plane wave as the excitation, it is prudent to describe the incident plane wave in terms of vector potentials. A time-harmonic vector potential plane wave is a solution to the equation

$$\left(\nabla^2 + k^2\right)\mathbf{A} = 0 \tag{72}$$

But one may expect from (72) that A_x, A_y, and A_z are decoupled from one another. To dispel this notion, we should think of $\mathbf{A}$ as the solution to

$$\left(\nabla^2 + k^2\right)\mathbf{A} = -\mu\mathbf{J} \tag{73}$$

The vector potential above satisfies the Lorenz gauge via the charge continuity equation. By taking the divergence of the above, we have

$$\left(\nabla^2 + k^2\right)\nabla\cdot\mathbf{A} = i\omega\mu\varepsilon\left(\nabla^2 + k^2\right)\Phi = -\mu\nabla\cdot\mathbf{J} = -i\omega\mu\rho \tag{74}$$

where $\nabla\cdot\mathbf{A} = i\omega\mu\varepsilon\Phi$.

If $\mathbf{J}$ is due to a Hertzian dipole source

$$\mathbf{J}(\mathbf{r}) = I\ell\hat{\ell}\delta(\mathbf{r}) \tag{75}$$

the corresponding vector potential $\mathbf{A}$ is

$$\mathbf{A}(\mathbf{r}) = \mu I\ell\hat{\ell}\frac{e^{ikr}}{4\pi r} \tag{76}$$

We can produce a local plane wave by letting $\mathbf{r} = \mathbf{r}_0 + \mathbf{s}$ where $|\mathbf{r}_0| \gg |\mathbf{s}|$. Then the above spherical wave can be locally approximated by a plane wave

$$\mathbf{A}(\mathbf{r}) \approx \mu I\ell\hat{\ell}\frac{e^{ikr_0}}{4\pi r_0}e^{i\mathbf{k}_0\cdot\mathbf{s}} = \mathbf{a}e^{i\mathbf{k}_0\cdot\mathbf{s}} \tag{77}$$

where $\mathbf{k}_0 = k\hat{r}_0$ and $\hat{r}_0$ is a unit vector that points in the direction of $\mathbf{r}_0$, and $\mathbf{a} = a_0\hat{\ell}$, a vector that points in the $\hat{\ell}$ direction. It is seen that the components of $\mathbf{A}$ generated this way satisfy the gauge condition and are not independent of one another. We have to keep this notion in mind when we generate a vector potential plane wave.

Hence, for a plane wave incident,

$$\mathbf{A}_{\text{inc}}(\mathbf{r}) = \mathbf{a}e^{i\mathbf{k}_i\cdot\mathbf{r}} = (\mathbf{a}_\perp + \mathbf{a}_\parallel)e^{i\mathbf{k}_i\cdot\mathbf{r}} \tag{78}$$

where $\mathbf{a}_\parallel = \hat{k}_i\hat{k}_i\cdot\mathbf{a}$, and $\mathbf{a}_\perp = \hat{\theta}\hat{\theta}\cdot\mathbf{a} + \hat{\phi}\hat{\phi}\cdot\mathbf{a}$. Hence, $\hat{k}_i\cdot\mathbf{a}_\perp = 0$. Therefore

$$\nabla\cdot\mathbf{A}_{\text{inc}} = i\mathbf{k}_i\cdot\mathbf{a}_\parallel e^{i\mathbf{k}_i\cdot\mathbf{r}} = i\hat{k}_i\cdot\hat{\ell}a_0e^{i\mathbf{k}_i\cdot\mathbf{r}} = i\omega\mu\varepsilon\Phi_{\text{inc}} \tag{79}$$

$$\mathbf{B}_{\text{inc}} = \nabla\times\mathbf{A}_{\text{inc}} = i\mathbf{k}_i\times\mathbf{A}_{\text{inc}} = i\mathbf{k}_i\times\mathbf{a}_\perp e^{i\mathbf{k}_i\cdot\mathbf{r}} \tag{80}$$

♯ The author is thankful to the reviewer for pointing out this work.

and

$$\mathbf{E}_{\text{inc}} = \frac{\nabla \times \mathbf{B}_{\text{inc}}}{-i\omega\mu\varepsilon} = \frac{i\mathbf{k}_i \times (\mathbf{k}_i \times \mathbf{A}_{\text{inc}})}{\omega\mu\varepsilon}$$
$$= i\omega\left(\bar{\mathbf{I}} - \hat{k}_i\hat{k}_i\right) \cdot \mathbf{A}_{\text{inc}} = i\omega\left(\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}\right) \cdot \mathbf{A}_{\text{inc}} \tag{81}$$

where $k_i^2 = \mathbf{k}_i \cdot \mathbf{k}_i = k^2$ and $\hat{k}_i = \mathbf{k}_i/k$. It is to be noted that if $\mathbf{A}_{\text{inc}}$ has only a longitudinal component, then both $\mathbf{E}$ and $\mathbf{B}$ are zero even though $\mathbf{A}$ is not zero. This can occur to leading order along the axial direction of a Hertzian dipole.

For some problems, it is convenient to let $\mathbf{a}_\parallel = 0$. This happens, for instance, in the broadside direction of a Hertzian dipole. In this direction, $\Phi = 0$, and hence, such a dipole generates an incident field with zero Φ_{inc}. In this case, the scattering problem governed by (16) yields $\Phi = 0$ everywhere. This is still the Lorenz gauge but with $\Phi = 0$. This is sometimes called the $\Phi = 0$ gauge or radiation gauge. It is useful for scattering problems.

9. DISCUSSIONS AND CONCLUSIONS

We have reviewed the use of $\mathbf{A}$-Φ formulation for general inhomogeneous media in electromagnetics. It is pointed out that the potential formulation has no low-frequency catastrophe compared to the traditional **E**-**H** formulation. This portends well for electromagnetic analysis in multi-scale structures where the wavelength can be much larger than the fine detail of the structures. It also portends well for multi-physics analysis where $\mathbf{A}$ and Φ are directly needed.

We have also derived the pertinent Green's theorem, and derive an integral equation for a PEC scatterer. It is seen that such an integral equation has no low-frequency catastrophe. Low frequency inaccuracy will persist for some of these problems, but they can be salvaged by solving the $\mathbf{A}$-Φ equations in tandem.

From the generalized Green's theorem, we note that the equivalence principle has no low-frequency catastrophe for the $\mathbf{A}$-Φ formulation, but the equivalence principle from the **E**-**H** formulation reigns superior for efficient representation at higher frequencies where wave physics dominates.

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APPENDIX A. DERIVATIONS OF (47), (48), AND (49)

We will ignore the source term $\mathbf{J}$ in order to derive some identities similar to Green's theorem. We begin with the following equations:

$$\nabla \times \nabla \times \mathbf{A} - \nabla\nabla \cdot \mathbf{A} - k^2\mathbf{A} = 0 \tag{A1}$$
$$\nabla \times \nabla \times \overline{\mathbf{G}} - \nabla\nabla \cdot \overline{\mathbf{G}} - k^2\overline{\mathbf{G}} = \bar{\mathbf{I}}\delta\left(\mathbf{r} - \mathbf{r}'\right) \tag{A2}$$

In the above, $\mathbf{A} = \mathbf{A}(\mathbf{r})$ and $\bar{\mathbf{G}} = \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$, but we suppress these spatial dependence for the time being in the following. First, we dot-multiply (A1) from the right by $\bar{\mathbf{G}} \cdot \mathbf{a}$ where $\mathbf{a}$ is an arbitrary vector, and then dot-multiply (A2) from the left by $\mathbf{A}$ and the right by $\mathbf{a}$. We take their difference, and ignoring the $\nabla\nabla$ term for the time being, we obtain

$$\nabla \times \nabla \times \mathbf{A} \cdot \overline{\mathbf{G}} \cdot \mathbf{a} - \mathbf{A} \cdot \nabla \times \nabla \times \overline{\mathbf{G}} \cdot \mathbf{a} = \nabla \cdot \left(\nabla \times \mathbf{A} \times \overline{\mathbf{G}} \cdot \mathbf{a} + \mathbf{A} \times \nabla \times \overline{\mathbf{G}} \cdot \mathbf{a}\right) \tag{A3}$$

Integrating right-hand side of the above over V, we have

$$\begin{aligned}\mathbf{I}_1\cdot\mathbf{a} &= \int_S \hat{n}\cdot\left(\nabla\times\mathbf{A}\times\overline{\mathbf{G}}\cdot\mathbf{a}+\mathbf{A}\times\nabla\times\overline{\mathbf{G}}\cdot\mathbf{a}\right)dS\\ &= \int_S \left[\hat{n}\times(\nabla\times\mathbf{A})\cdot\overline{\mathbf{G}}\cdot\mathbf{a}+(\hat{n}\times\mathbf{A})\cdot\nabla\times\bar{\mathbf{G}}\cdot\mathbf{a}\right]dS\end{aligned}\tag{A4}$$

Including now the $\nabla\nabla$ term gives

$$-\nabla\nabla\cdot\mathbf{A}\cdot\overline{\mathbf{G}}\cdot\mathbf{a}+\mathbf{A}\cdot\nabla\nabla\cdot\overline{\mathbf{G}}\cdot\mathbf{a}=\nabla\cdot\left(-\nabla\cdot\mathbf{A}\,\overline{\mathbf{G}}\cdot\mathbf{a}+\mathbf{A}\nabla\cdot\overline{\mathbf{G}}\cdot\mathbf{a}\right)\tag{A5}$$

Integrating the right-hand side of the above over V, we have

$$\mathbf{I}_2\cdot\mathbf{a}=\int_S \hat{n}\cdot\left(-\nabla\cdot\mathbf{A}\,\overline{\mathbf{G}}\cdot\mathbf{a}+\mathbf{A}\nabla\cdot\overline{\mathbf{G}}\cdot\mathbf{a}\right)dS\tag{A6}$$

Letting $\overline{\mathbf{G}}=g\bar{\mathbf{I}}$, where $g=g(\mathbf{r},\mathbf{r}')$, the scalar Green's function, the above becomes

$$\mathbf{I}_1\cdot\mathbf{a}=\int_S\left[\hat{n}\times\nabla\times\mathbf{A}g\cdot\mathbf{a}+\hat{n}\times\mathbf{A}\cdot\nabla g\times\mathbf{a}\right]dS=\int_S\left[\hat{n}\times(\nabla\times\mathbf{A})\,g+(\hat{n}\times\mathbf{A})\times\nabla g\right]dS\cdot\mathbf{a}\tag{A7}$$

or

$$\mathbf{I}_1=\int_S\left[\hat{n}\times(\nabla\times\mathbf{A})\,g+(\hat{n}\times\mathbf{A})\times\nabla g\right]dS\tag{A8}$$

Similarly, we have

$$\mathbf{I}_2=\int_S\left[-\hat{n}\,(\nabla\cdot\mathbf{A})\,g+\hat{n}\cdot\mathbf{A}\;\nabla g\right]dS\tag{A9}$$

Using the above, we get (48).

To get (49), more manipulations are needed. Using $\hat{n}\times(\nabla\times\mathbf{A})=(\nabla\mathbf{A})\cdot\hat{n}-(\hat{n}\cdot\nabla)\,\mathbf{A}$, $\nabla g\times(\hat{n}\times\mathbf{A})=\hat{n}\,(\mathbf{A}\cdot\nabla g)-(\hat{n}\cdot\nabla g)\,\mathbf{A}$

$$\mathbf{I}_1=\int_S\left[-(\hat{n}\cdot\nabla\mathbf{A})\,g+(\nabla\mathbf{A})\cdot\hat{n}g-\hat{n}\,(\mathbf{A}\cdot\nabla g)+(\hat{n}\cdot\nabla g)\,\mathbf{A}\right]dS\tag{A10}$$

First, we look at

$$\begin{aligned}\mathbf{I}_3 &= \int_S \hat{n}[-\nabla\cdot\mathbf{A}g-\mathbf{A}\cdot\nabla g]\,dS=\int_V dV\;\nabla(-\nabla\cdot\mathbf{A}\,g-\mathbf{A}\cdot\nabla\,g)\\ &= \int_V dV\;[-\nabla\nabla\cdot\mathbf{A}\;\;g-\mathbf{A}\cdot\nabla\nabla\,g-\nabla\mathbf{A}\cdot\;\nabla g-\nabla\cdot\mathbf{A}\;\nabla\,g]\end{aligned}\tag{A11}$$

$$\mathbf{I}_4=\int_S dS\;[g\;\nabla\mathbf{A}\cdot\hat{n}+\hat{n}\cdot\mathbf{A}\;\nabla\,g]=\int_V dV\;\;\nabla\cdot[(g\nabla\mathbf{A})^t+\mathbf{A}\;\nabla\,g]\tag{A12}$$

Furthermore, with the knowledge that

$$\nabla\cdot[(g\nabla\mathbf{A})^t+\mathbf{A}\nabla\,g]=\partial_i[g\partial_k\mathbf{A}_i+\mathbf{A}_i\partial_j g]\tag{A13}$$
$$=(\partial_i g)\partial_k\mathbf{A}_i+g\partial_i\partial_k\mathbf{A}_i+\partial_i\mathbf{A}_i\partial_k g+\mathbf{A}_i\partial_i\partial_k g\tag{A14}$$
$$=\nabla\mathbf{A}\cdot\nabla\,g+g\;\nabla\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{A}\;\nabla\,g+\mathbf{A}\cdot\nabla\nabla\,g\tag{A15}$$

it is seen that $\mathbf{I}_3+\mathbf{I}_4=0$. Using this fact, we can show (49), or that

$$\mathbf{I}_1+\mathbf{I}_2=\int_S dS\;[(\hat{n}\cdot\nabla\,g)\mathbf{A}-g\;\hat{n}\cdot\nabla\mathbf{A}]\tag{A16}$$

APPENDIX B. DERIVATION OF (51) AND (52)

We begin with the upper part of (48), ignoring for the time being the subscript 1, to have

$$\mathbf{A} = \mathbf{A}_{inc} + \int_S dS'\{\mu g\hat{n}' \times \mathbf{H} - \nabla' g \times \hat{n}' \times \mathbf{A}\} + \int_S dS'\{-\hat{n}' g\nabla' \cdot \mathbf{A} + \hat{n}' \cdot \mathbf{A}\nabla' g\} \tag{B1}$$

In the above, we assume that the functions inside the integrand have $\mathbf{r}'$ as the argument, and that outside the integrand, the functions have $\mathbf{r}$ as the argument. The exception is the Green's function $g = g(\mathbf{r}, \mathbf{r}')$. Using the Lorenz gauge condition

$$\nabla \cdot \mathbf{A} = i\omega\mu\epsilon\Phi \tag{B2}$$

we arrive at

$$\begin{aligned} i\omega\mu\epsilon\Phi &= i\omega\mu\epsilon\Phi_{inc} + \int_S dS'\mu g\nabla' \cdot \mathbf{J} + \nabla \cdot \int_S dS'\{-\hat{n}'\nabla' \cdot \mathbf{A}\ g\} + \int_S dS'\{\hat{n}' \cdot \mathbf{A}\ k^2 g\} \\ &= i\omega\mu\epsilon\Phi_{inc} + \int_S dS' i\omega\mu\ g\ \sigma - \nabla \cdot \int_S dS'\{i\omega\mu\epsilon\Phi\hat{n}' g\} + \int_S dS'\{\hat{n}' \cdot \mathbf{A}k^2 g\} \end{aligned} \tag{B3}$$

Consequently,

$$\Phi = \Phi_{inc} + \int_S dS' g\ \frac{\sigma}{\epsilon} - \nabla \cdot \int_S dS'\Phi\hat{n}' g + i\omega \int_S dS'\hat{n}' \cdot \mathbf{A}\ g \tag{B4}$$

Using that

$$\sigma = \hat{n} \cdot \epsilon\mathbf{E} = \hat{n} \cdot \epsilon i\omega\mathbf{A} - \hat{n} \cdot \epsilon\nabla\Phi, \tag{B5}$$

the above becomes

$$\Phi = \Phi_{inc} - \int_S dS' g\ \hat{n}' \cdot \nabla'\Phi + \int_S dS'(\hat{n}' \cdot \nabla' g)\Phi \tag{B6}$$

The above is just the scalar Green's theorem that can be derived directly from the scalar wave equation for the scalar potential, (12).

Since $\mathbf{E} = i\omega\mathbf{A} - \nabla\Phi$, we have, after using (B6) and (B1), that

$$\nabla\Phi = \nabla\Phi_{inc} - \int_S dS'\ (\nabla g)\ \hat{n}' \cdot \nabla'\Phi + \int_S dS'(\hat{n}' \cdot \nabla'\nabla g)\Phi \tag{B7}$$

$$i\omega\mathbf{A} = i\omega\mathbf{A}_{inc} + \int_S dS'\{i\omega\mu g\mathbf{J} - \nabla' g \times \hat{n}' \times i\omega\mathbf{A}\} + \int_S dS'\{k^2\hat{n}' g\Phi + i\omega\hat{n}' \cdot \mathbf{A}\nabla' g\} \tag{B8}$$

Then

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{inc} + \int_S dS' i\omega\mu g\mathbf{J} + \nabla \int_S g \times (\hat{n}' \times i\omega\mathbf{A}) \\ &\quad + \int_S dS'\nabla' g\{i\omega\hat{n}' \cdot \mathbf{A} + \hat{n}' \cdot \nabla'\Phi\} + \int_S dS'\{k^2\hat{n}' g\Phi - \hat{n}' \cdot \nabla'\nabla g\ \Phi\} \end{aligned} \tag{B9}$$

After using the relation in (B5) and the relation between $\mathbf{E}$ and $\mathbf{A}$ and Φ, we have

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{inc} + \int_S dS' \left\{i\omega\mu g\mathbf{J} + \nabla' g\ \frac{\sigma}{\epsilon}\right\} + \nabla \int_S g \times (\hat{n}' \times \mathbf{E}) \\ &\quad + \int_S dS'\{k^2\hat{n}' g\Phi - (\hat{n}' \cdot \nabla'\nabla g)\Phi + \nabla g \times \big(\hat{n}' \times \nabla'\Phi\big)\} \end{aligned} \tag{B10}$$

The above looks like the Green's theorem for the **E-H** formulation if we can show that the last integral is zero. To this end, we let the last integral be I such that

$$I = \int dS'\{k^2\hat{n}' g\Phi + (\hat{n}' \cdot \nabla'\nabla' g)\Phi - \nabla' g \times \big(\hat{n}' \times \nabla'\Phi\big)\} \tag{B11}$$

where we have replaced ∇g with $-\nabla' g$. The last integral above in (B11) is

$$I_1 = -\int dS' \nabla' g \times \left(\hat{n}' \times \nabla' \Phi\right) = -\int [\hat{n}'(\nabla' g \cdot \nabla' \Phi) - (\hat{n}' \cdot \nabla' g)\nabla' \Phi] dS' \tag{B12}$$

Then using the Gauss' divergence theorem, the above becomes

$$\begin{aligned} I_1 &= -\int_V dV' \left\{\nabla'(\nabla' g \cdot \nabla' \Phi) - \nabla' \cdot (\nabla' g \nabla' \Phi)\right\} \\ &= -\int_V dV' \{(\nabla' \nabla' g) \cdot \nabla' \Phi + \nabla' \nabla' \Phi \cdot \nabla' g - \nabla'^2 g \nabla' \Phi - \nabla' \nabla' \Phi \cdot \nabla' g\} \\ &= -\int_V dV' \{(\nabla' \nabla' g) \cdot \nabla' \Phi - \nabla'^2 g \; \nabla' \Phi\} \end{aligned} \tag{B13}$$

The first term in (B11) is

$$\begin{aligned} I_2 &= \int dS' \{k^2 \hat{n}' g \Phi + (\hat{n}' \cdot \nabla' \nabla' g)\Phi\} = \int_V dV' \{k^2 \nabla'(g\Phi) + \nabla' \cdot [(\nabla' \nabla' g)\Phi]\} \\ &= \int_V dV' \{k^2 (\nabla' g)\Phi + k^2 g \nabla' \Phi + (\nabla'^2 \nabla' g)\Phi + \nabla' \Phi \cdot \nabla' \nabla' g\} \end{aligned} \tag{B14}$$

In the above, we have also used Gauss' divergence theorem to convert the surface integral to a volume integral. Consequently, we have

$$\begin{aligned} I = I_2 + I_1 &= \int_V dV' \{\nabla' \left(\nabla'^2 g + k^2 g\right) \Phi + (\nabla'^2 g + k^2 g)\nabla' \Phi\} \\ &= \int_V dV' \{-(\nabla' \delta(\mathbf{r} - \mathbf{r}'))\Phi - \delta(\mathbf{r} - \mathbf{r}')\nabla' \Phi\} = 0 \end{aligned} \tag{B15}$$

where we have used that $\nabla'^2 g(\mathbf{r}, \mathbf{r}') + k^2 g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$. Hence, the last term in (B10) is zero, and (B10) can be used to derive (51).

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