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Transfer Operator Theory and Inter-Reciprocity of Non-Reciprocal Multiport Antennas

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Abstract—The present article expounds a formalism for the representation of multi-port non-reciprocal antenna structures in an arbitrary surrounding linear medium. In the most general approach the antenna, the waveguides connected to it, as well as the surrounding medium may contain any distribution of anisotropic magneto-electric media. Furthermore, an arbitrary external field is taken into consideration which need not be of plane wave form. A reciprocally adjoint system is introduced to derive relations which describe the antenna under such general conditions. Since the antenna may contain media which prohibit the use of ordinary scattering, admittance or impedance matrices, an approach by means of generalized scattering matrices, or by a generalized admittance and a generalized impedance matrix, is applied. This leads to an n-port description of the whole waveguide-antenna-environment where transfer operators render the interaction between the external field and the state of the ports. These operators are the generalizations of effective length vectors. For its importance the case of reciprocal reflection-symmetric waveguides is treated in detail, including a derivation of the consequences of abstract network reciprocity and complex power relation for voltage-current representations. The formalism is adequate for the description of radar and radio astronomy antennas, in particular when wave polarization plays a crucial role and/or a magnetized plasma environment is present (which is responsible for anisotropy and non-reciprocal conditions).

1. INTRODUCTION

Reciprocity principles for electromagnetic fields play a crucial role in antenna theory, mainly because they provide a relation between transmission and reception properties of antennas. The first reciprocity statements date from the famous work on the theory of sound by Lord Rayleigh [38], and application of reciprocity in electromagnetics followed half a century later [3, 6, 7, 23]. Consequences for the relation between transmitting and receiving properties of antennas were discussed by many authors. Among the early articles were [9, 44]. In the framework of the reaction concept (which is a specific expression of reciprocity) a series of articles ensued, which are mainly aimed at the investigation of antennas, e.g., [10, 39, 41]. Later articles tended to generalize the reciprocity principle more and more, for instance, by extending its validity to anisotropic media [20], in particular to gyrotropic media [16]. This kind of generalization is also a major aim here.

In abstract terms, reciprocity theorems render a link between two independent states (with independent source distributions) of a system and its so-called adjoint system. In the present electromagnetic context such a system is any distribution of linear media which are represented by suitable material tensors (permittivity, permeability etc.) and constitutive relations (giving $\mathbf{D}$ and $\mathbf{B}$ as functions of $\mathbf{E}$ and $\mathbf{H}$). The adjoint system is defined by suitable material tensors which are related to the tensors of the original system in such a way that the reciprocity principle applies. The concrete mathematical expressions are given in Section 3. The literature on reciprocity of electromagnetic phenomena can crudely be divided in two classes. First, formal treatments based on the Maxwell

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equations infer consequences from certain given tensor properties. Second, the pre-conditions of certain properties of the material tensors are studied by means of microscopic quantum mechanical considerations. Macroscopic symmetry properties of matter (as described by the mentioned tensors) are derived from fundamental properties like causality, time-reversal invariance and microscopic spatial symmetries of the crystal structure (see e.g., [17, 29, 32, 33] and references therein).

The present article belongs to the former class, based on a given set of adjointness relations between the material tensors of the original and the adjoint system. Further assumptions (e.g., on the symmetry of these tensors) are not made, except for the premise of linearity of the constitutive relations. On these pre-conditions Kerns [19] derived the following formula for the voltage V° received by an antenna in an external electromagnetic field (electric field strength $\mathbf{E}$):

$$V^\circ = \frac{1}{\tilde{I}} \int \tilde{\mathbf{J}} \cdot \mathbf{E} \, dV \tag{1}$$

The integral is to be taken over the whole antenna volume where the current density $\tilde{\mathbf{J}}$ is induced in transmission mode when the adjoint antenna system is driven with the input (feed) current $\tilde{I}$. Equation (1) is a typical result as obtained by application of reciprocity theorems, relating the reception to the transmission scenario. If the antenna is part of an electronic circuit, its terminals are usually not open and there is a non-zero feed current I, with the applied voltage V being

$$V = ZI - V^\circ \tag{2}$$

where Z is the antenna impedance. The behaviour of the antenna as part of an electronic circuit, including its reaction to external fields, is therefore fully described by its impedance Z and the vector field $\tilde{\mathbf{J}}/\tilde{I}$.

This article presents a generalization of the above theory in several aspects as an extension to multi-port antennas, allowing for formal magnetic currents, and taking account of any linear media distribution in and around the antenna. If several antennas are present, coupling between the antennas occurs. The whole structure of all antennas is regarded as an antenna network with several waveguide connections building an n-port. Any finite number of waveguides may be connected to the antenna. The whole antenna structure, the waveguides connected to it, as well as the environment may contain anisotropic inhomogeneous media. In particular, also negative resistances may occur (see [27] and further references given there). Only little restrictions are made on the waveguides and the antenna, e.g., n-dimensionality of the antenna network and mutual bidirectionality of waveguides, as explained below. With these prerequisites, generalized admittance and impedance matrices, or generalized scattering matrices are needed to represent the whole antenna network as described in Section 4. Transfer matrices are used to represent the effect of incident electromagnetic waves. They were successfully applied to the description of multi-monopole systems aboard spacecraft in vacuo [24], which are examples of a multi-port reciprocal antenna system. The following developments render an extension to non-reciprocal antenna systems, such as antennas aboard spacecraft flying through a magnetized plasma (Section 6.5). The consequences of the reciprocity principle are revisited under the mentioned general conditions in Section 3, resulting in an inter-reciprocity relation for the modal wave coefficients, including the effect of external fields. After introduction of the generalized scattering, admittance and impedance matrices in Section 4, the transfer operators and matrices are defined and analysed in Section 5. They facilitate the calculation of transmission as well as reception properties. Section 6 discusses special cases of the theory which are of most interest in practice.

The following conventions are applied throughout. The harmonic time dependence $e^{j\omega t}$ is assumed, with ω being the angular frequency. As usual the factor $e^{j\omega t}$ is suppressed in all time-dependent quantities, which therefore are complex amplitudes and can be regarded as Fourier or Laplace transforms (in the latter case ω may assume complex values). $\mathbf{0}_n$ and $\mathbf{1}_n$ denote the $n \times n$ zero and identity matrix, respectively. The superscript t is used to indicate the transposition of a matrix, e.g., $\mathbf{Z}^t$ is the transpose of the matrix $\mathbf{Z}$. As already mentioned, all quantities associated with the adjoint system are marked by a tilde, for instance $\tilde{\varepsilon}(\mathbf{r})$ refers to the dielectric tensor field of the adjoint system. On several occasions the electric and magnetic field strengths, $\mathbf{E}$ and $\mathbf{H}$, are subsumed into a concise single 6-element vector

$$\mathbf{f} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \tag{3}$$

and, analogously, for the electromagnetic field in the adjoint system: $\tilde{\mathbf{f}}^t = (\tilde{\mathbf{E}}^t, \tilde{\mathbf{H}}^t)$. The dot product of two n-element vectors $\mathbf{V}$ and $\mathbf{I}$ is $\mathbf{V} \cdot \mathbf{I} = \sum_{m=1}^{n} V_m I_m$. The same dot is used to denote matrix multiplication. The left-multiplication of a matrix $\mathbf{Z}$ by a vector $\mathbf{I}$ is not written as a left-multiplication by a transposed one-column matrix ($\mathbf{I}^t \cdot \mathbf{Z}$), but always as $\mathbf{I} \cdot \mathbf{Z}$, thus $\mathbf{I} \cdot \mathbf{Z} \cdot \mathbf{I} = \sum_{l=1}^{n} \sum_{m=1}^{n} I_l Z_{lm} I_m$. The superscript asterisk denotes the complex conjugate of scalars and the conjugate (Hermitian) transpose of matrices, so $\{\mathbf{S}^*\}_{mq} = (\{\mathbf{S}\}_{qm})^*$.

2. ARRANGEMENT OF ANTENNA(S), WAVEGUIDES AND ELECTRONIC DEVICES

The envisaged arrangement of an antenna, shielded electronics (usually generators or receivers) and a number of connecting waveguides is outlined in Figure 1. The antenna system may contain several antennas and an arbitrary number of parasitic bodies. Several generators and/or receivers or any kind of electronics in the volumes $G_1 \ldots G_W$ interact with the antenna(s) via W waveguides. Since it is immaterial if these regions are separated or connected, only the union $R_G = G_1 \cup \ldots \cup G_W$ is regarded. Though there may be several antennas and parasitic bodies, they are addressed as 'the antenna' in the following, situated in the region R_A, which may (but need not) enclose R_G. The whole space ($\mathbb{R}^3$) is divided into three parts: R_G, R_A, and the remaining external region $R_{\text{ext}} = \mathbb{R}^3 \setminus R_G \setminus R_A$.

Each waveguide is divided into two segments, one lying in R_G and the other in R_A. Let the boundary between R_G and R_A intersect the waveguides at right angles, thereby yielding the cross-sections $S_1 \ldots S_W$. In the w-th waveguide a cartesian coordinate system is defined, with the z-axis parallel to the waveguide and the origin in the center of S_w. The waveguides' z-axes are directed towards the antenna, i.e., $z > 0$ in R_A and $z < 0$ in R_G.

The electromagnetic field in the antenna, in the waveguides as well as in the external region satisfy the Maxwell equations

$$\nabla \times \mathbf{E} = -\mathbf{K} - j\omega \mathbf{B} \tag{4}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D} \tag{5}$$

A magnetic current density $\mathbf{K}$ is allowed for formal purposes, since it is often beneficial in the construction of solutions of boundary value problems, in particular by application of the equivalence principle [15, Section 3-5]. The main assumption on the media in these regions is that they satisfy the constitutive relations

$$\mathbf{D} = \epsilon \cdot \mathbf{E} + \tau \cdot \mathbf{H} \tag{6}$$

$$\mathbf{B} = \nu \cdot \mathbf{E} + \mu \cdot \mathbf{H} \tag{7}$$

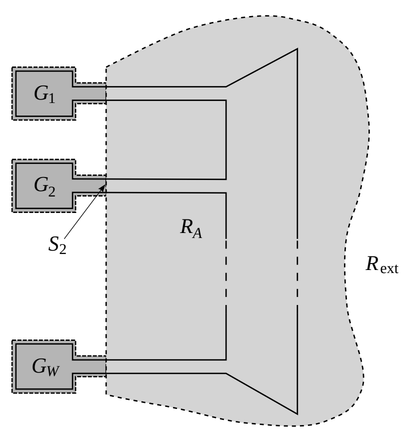

Figure 1. Geometry of an antenna structure and W shielded electronic devices $G_1 \ldots G_W$, with waveguides connecting them. The devices (e.g., generators or receivers) and their waveguide connectors are shown as dark gray regions, the whole antenna system and the main parts of the waveguides are in the light gray region R_A. The surfaces of $G_1 \ldots G_W$ and R_A intersect the waveguides at the cross-sections $S_1 \ldots S_W$.

These are the most general linear relations, which permit also the description of linear magnetoelectric effects [13]. In general, the material tensors ϵ, μ, τ and ν depend on $\mathbf{r}$, permitting not only anisotropic but also inhomogeneous media.

The electromagnetic fields in the waveguides are expanded in terms of travelling waves $(\mathbf{e}^{\sigma}_{Tm}(x,y), \mathbf{h}^{\sigma}_{Tm}(x,y))e^{-\gamma^{\sigma}_m z}$. In general, the propagation constants γ^{σ}_m and the transverse vector fields $\mathbf{e}^{\sigma}_{Tm}$ and $\mathbf{h}^{\sigma}_{Tm}$ depend on the waveguide number w. The superscript σ identifies the branch of the mode m, where $\sigma = \pm 1$ stands for propagation in $\pm z$ direction. In the sequel the superscripts ± 1 will be abbreviated as $\pm$. Let $\mathbf{f}_T(x,y,z)$ be the 6-element vector containing the total transverse electric and magnetic field strength in the waveguide, which is to be expanded in modes, and let $\mathbf{f}^{\sigma}_m(x,y)$ be the corresponding juxtaposed vectors of the σ-branch of mode m, i.e.,

$$\mathbf{f}_T = \begin{pmatrix} \mathbf{E}_T \\ \mathbf{H}_T \end{pmatrix}, \quad \mathbf{f}^{\sigma}_m = \begin{pmatrix} \mathbf{e}^{\sigma}_{Tm} \\ \mathbf{h}^{\sigma}_{Tm} \end{pmatrix} \tag{8}$$

If the cross-sections S_w are far enough from the waveguide ends, the majority of evanescent modes is negligible and only a finite number of modes is to be considered for each waveguide. With n_w being the number of modes in waveguide w, the expansion of the transverse components of the electromagnetic field in this waveguide reads [12, 30]

$$\mathbf{f}_T(x,y,z) = \sum_{p=1}^{n_w} \left[\alpha^{+}_m \, \mathbf{f}^{+}_m(x,y)\, e^{-\gamma^{+}_m z} + \alpha^{-}_m \, \mathbf{f}^{-}_m(x,y)\, e^{-\gamma^{-}_m z} \right] \tag{9}$$

Here the dependence of $\mathbf{f}_T$, γ^{σ}_m, $\mathbf{f}^{\sigma}_m$ and the expansion coefficients α^{σ}_m on the waveguide number w is suppressed for brevity. Each mode is regarded as an antenna port, and the two branches $\sigma = +1$ and $\sigma = -1$ represent the waves travelling towards and away from the antenna through the respective port. The coefficients $\alpha^{+}_m(w)$ of the incident waves of all ports (all modes m of all waveguides w) are collected in the vector $\mathbf{a}^{+}$. Similarly, $\mathbf{a}^{-}$ denotes the coefficient vector of all outgoing waves. Thus,

$$\mathbf{a}^{\sigma} = \left(\alpha^{\sigma}_1(1) \ldots \alpha^{\sigma}_{n_1}(1),\, \alpha^{\sigma}_1(2) \,\ldots\, \alpha^{\sigma}_{n_2}(2),\, \ldots,\, \alpha^{\sigma}_1(W) \,\ldots\, \alpha^{\sigma}_{n_W}(W) \right)^t \tag{10}$$

The port notation enables us to treat the coupling of different modes via the antenna by the same formalism, no matter if they belong to the same or different waveguides. Let $n = \sum_{w=1}^{W} n_w$ denote the total number of ports. The $2n$ elements of the 'port state vector'

$$\mathbf{s}_{+-} = \begin{pmatrix} \mathbf{a}^{+} \\ \mathbf{a}^{-} \end{pmatrix} = \left(a^{+}_1 \,\ldots\, a^{+}_n,\, a^{-}_1 \,\ldots\, a^{-}_n \right)^t \tag{11}$$

can be used to completely represent the state of the antenna ports.

3. INTER-RECIPROCITY

The excitation of the antenna is established, on the one hand, by the port state $\mathbf{s}_{+-}$ and, on the other hand, by a potential incident electromagnetic field ($\mathbf{E}^i$, $\mathbf{H}^i$) generated by current densities $\mathbf{J}$ and $\mathbf{K}$ in the external region R_{ext}. The aim of this section is to derive a reciprocity principle for the antenna system expressed in terms of both, the modal coefficients $\mathbf{a}^{+}$ and $\mathbf{a}^{-}$ (i.e., the state vector $\mathbf{s}_{+-}$) and the incident field.

3.1. Expression in Terms of Fields and Currents

The whole antenna system including the waveguides and the external region (in the following called the original system Σ) is compared with the reciprocally adjoint system $\tilde{\Sigma}$. The distribution of matter in $\tilde{\Sigma}$ is characterized by the material tensors $\tilde{\epsilon}$, $\tilde{\tau}$, $\tilde{\nu}$, and $\tilde{\mu}$, which depend, in general, on position. An electromagnetic field maintained in $\tilde{\Sigma}$ satisfies the Maxwell Equations (4) and (5), as well as the constitutive relations (6) and (7), with all quantities replaced by their respective adjoint ones, indicated by a tilde: $\tilde{\mathbf{E}}$, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{J}}$, $\tilde{\mathbf{K}}$, $\tilde{\epsilon}$, $\tilde{\tau}$, $\tilde{\mu}$ and $\tilde{\nu}$. The material tensors of the adjoint system are related to the tensors of the original system by

$$\tilde{\epsilon}(\mathbf{r}) = \epsilon^t(\mathbf{r}), \tag{12}$$

$$\tilde{\mu}(\mathbf{r}) = \mu^t(\mathbf{r}), \tag{13}$$

$$\tilde{\nu}(\mathbf{r}) = -\tau^t(\mathbf{r}), \tag{14}$$

$$\tilde{\tau}(\mathbf{r}) = -\nu^t(\mathbf{r}). \tag{15}$$

so that the reciprocity principle (see, e.g., [15, Section 3-8])

$$\int_v \left(\tilde{\mathbf{E}} \cdot \mathbf{J} - \tilde{\mathbf{H}} \cdot \mathbf{K} - \mathbf{E} \cdot \tilde{\mathbf{J}} + \mathbf{H} \cdot \tilde{\mathbf{K}}\right) dv = \oint_{\partial v} \mathbf{n} \cdot \left(\mathbf{E} \times \tilde{\mathbf{H}} - \tilde{\mathbf{E}} \times \mathbf{H}\right) da \tag{16}$$

is satisfied. The volume integral in (16) can be taken over any volume v where the Maxwell equations, the constitutive equations and (12)–(15) hold. The surface integral is over the boundary ∂v of the volume v, with da indicating the differential area element and $\mathbf{n}$ the outward pointing unit normal. Equation (16) relates the electromagnetic field $(\mathbf{E}, \mathbf{H})$ generated by the current densities $\mathbf{J}$ and $\mathbf{K}$ in Σ to the electromagnetic field $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ generated by the current densities $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{K}}$ in $\tilde{\Sigma}$. The sources and fields in Σ can be chosen independently of $\tilde{\Sigma}$, but the media of the two systems must satisfy (12)–(15). The notion inter-reciprocity is used by many authors to emphasize that two systems are involved which are not generally the same [5, 36]. In the present context a system is named reciprocal if it agrees with its adjoint system in an electromagnetic sense, that is ϵ and μ are symmetric and $\tau = -\nu^t$.

Here the reciprocity principle is applied to the whole region outside the devices $G_1 \ldots G_W$, that is $v = \mathbb{R}^3 \setminus R_G = R_A \cup R_{\text{ext}}$. The surface ∂v is composed of the boundaries $\partial G_1 \ldots \partial G_W$ of the devices and the surface S_∞ of a large sphere containing the whole scene, with its boundary infinitely receding to the distance.

3.2. The Surface Integral

For the sake of clarity and short notation we introduce the bilinear skew-symmetric form

$$\langle \mathbf{f}, \mathbf{f}' \rangle_S = \int_S \mathbf{n} \cdot \left(\mathbf{E} \times \mathbf{H}' - \mathbf{E}' \times \mathbf{H}\right) da \tag{17}$$

which acts on the restrictions of electromagnetic fields to the surface S. The notation $\mathbf{f} = (\mathbf{E}^t, \mathbf{H}^t)^t$ and $\mathbf{f}' = (\mathbf{E}'^t, \mathbf{H}'^t)^t$ is used in accordance with (3). Actually, only the tangential components $\mathbf{E}_T = -\mathbf{n} \times (\mathbf{n} \times \mathbf{E})$ and $\mathbf{H}_T = -\mathbf{n} \times (\mathbf{n} \times \mathbf{H})$ play a role in the above integral, as the dot product with $\mathbf{n}$ appears in the definition. To make this definition unique, the direction of $\mathbf{n}$ has to be fixed. In the following we apply (17) to parts of ∂v as they appear in (16), so it is natural to assume that $\mathbf{n}$ points outwards, i.e., away from v, in all these cases. With the respective forms for $S = \partial R_G$ and $S = S_\infty$, the surface integral in (16) can be written

$$\left\langle \mathbf{f}, \tilde{\mathbf{f}} \right\rangle_{\partial R_G} + \left\langle \mathbf{f}, \tilde{\mathbf{f}} \right\rangle_{S_\infty} = \sum_{w=1}^{W} \left\langle \mathbf{f}, \tilde{\mathbf{f}} \right\rangle_{\partial G_w} + \left\langle \mathbf{f}, \tilde{\mathbf{f}} \right\rangle_{S_\infty} \tag{18}$$

The contribution $\langle \mathbf{f}, \tilde{\mathbf{f}} \rangle_{S_\infty}$ vanishes in the limit $r_\infty \to \infty$, with r_∞ being the radius of the spherical surface S_∞. A formal proof can be based on the condition of a slightly dissipative nature of the medium outside an arbitrarily large region [11], if at least one of the fields is generated by local sources (which applies in the following). Similarly, the surface integral over the shielding of the devices is negligible. The reason is that the integrand vanishes on good conductors due to the skin effect, where $\mathbf{E} = -\zeta_S \mathbf{n} \times \mathbf{H}$ with surface impedance ζ_S. So only the contributions of the integrals over the waveguide cross-sections $S_1 \ldots S_W$ remain, $\langle \mathbf{f}, \tilde{\mathbf{f}} \rangle_{\partial G_w} = \langle \mathbf{f}, \tilde{\mathbf{f}} \rangle_{S_w}$, with [27]

$$\left\langle \mathbf{f}, \tilde{\mathbf{f}} \right\rangle_{S_w} = -\int_{S_w} \hat{\mathbf{z}} \cdot \left(\mathbf{E}_T \times \tilde{\mathbf{H}}_T - \tilde{\mathbf{E}}_T \times \mathbf{H}_T\right) da \tag{19}$$

Here $\hat{\mathbf{z}}$ is the unit vector of the z-axis in the w-th waveguide (parallel to waveguide axis at S_w), which is directed towards the antenna. The minus sign in front of the integral appears because $\hat{\mathbf{z}} = -\mathbf{n}$ (the same definition of $\langle \mathbf{f}, \tilde{\mathbf{f}} \rangle_{S_w}$ was used in [27], but with the wrong sign in front of the integrals in (54) and consequentially (95) of this article).

The terminology for the electromagnetic fields in the waveguides of Σ applies, in an analogous way, to the adjoint waveguides of $\tilde{\Sigma}$. So the Expressions (9)–(11) are formally the same but with tildes over the adjoint quantities: $\tilde{\mathbf{f}}_T$, $\tilde{\alpha}_m^\sigma$, $\tilde{\mathbf{f}}_m^\sigma$, $\tilde{\gamma}_m^\sigma$, $\tilde{\mathbf{a}}^\sigma$ and $\tilde{\mathbf{s}}_{+-}$. As mentioned above the only assumption on the waveguides and their adjoint ones is — in addition to the linearity of the media according to (6), (7) and (12)–(15) — that they are mutually bidirectional [30]. In consequence, the modes of the adjoint waveguide w can be rearranged in such a way that [27, 30]

$$\left\langle \mathbf{f}_m^\sigma, \tilde{\mathbf{f}}_q^\tau \right\rangle_{S_w} = N_m^{(\sigma)}\, \delta_{mq}(1 - \delta_{\sigma\tau}) \tag{20}$$

where $N_m^{(\sigma)}(w)$ are normalization constants for mode m (branch σ) of waveguide w. It is shown in [27] that this bi-orthogonalization of the modes can be utilized to express $\langle \mathbf{f}, \tilde{\mathbf{f}} \rangle_{S_w}$ in terms of the expansion coefficients α_m^σ and $\tilde{\alpha}_m^\sigma$, with the following sum over all waveguides

$$\left\langle \mathbf{f}, \tilde{\mathbf{f}} \right\rangle_{\partial v} = \sum_{w=1}^{W} \left\langle \mathbf{f}, \tilde{\mathbf{f}} \right\rangle_{S_w} = \mathbf{a}^+ \cdot \mathbf{N}_+ \cdot \tilde{\mathbf{a}}^- + \mathbf{a}^- \cdot \mathbf{N}_- \cdot \tilde{\mathbf{a}}^+ \tag{21}$$

Here the diagonal matrices $\mathbf{N}^\pm$ contain the normalization constants $N_m^\pm(w)$ of the respective branch in their diagonal, sorted in the order specified by (10).

3.3. The Volume Integral

Taking advantage of the fact that the currents of the original and the adjoint system appearing in the reciprocity principle (16) can be chosen independently, the following assumptions are made. In Σ external sources are allowed: part of the electromagnetic field, that is the incident field ($\mathbf{E}^i$, $\mathbf{H}^i$), is generated by currents ($\mathbf{J}$, $\mathbf{K}$) in the external region R_{ext}. Due to the relations (6) and (7) all currents in the antenna and waveguides in R_A are represented as polarization currents, so in R_A no explicit current sources have to be accounted for.

In contrast to Σ, a pure transmission scenario is considered for $\tilde{\Sigma}$, which means that there are no explicit current sources outside the devices. Because of the linear constitutive relations for $\tilde{\Sigma}$, which are analogous to (6) and (7), all conduction currents in the antenna and in the external region are part of the polarization currents. Therefore $\tilde{\mathbf{J}} = 0$ and $\tilde{\mathbf{K}} = 0$ outside the devices (in $v = R_A \cup R_{\text{ext}}$). The volume integral in (16) becomes

$$\int_{R_{\text{ext}}} \left(\tilde{\mathbf{E}} \cdot \mathbf{J} - \tilde{\mathbf{H}} \cdot \mathbf{K} \right) dv \tag{22}$$

where the integration volume can be reduced to the external region R_{ext}, actually to the support of the current fields $\mathbf{J}$ and $\mathbf{K}$ (the smallest closed region containing the current sources), $R_{\mathbf{JF}} = \operatorname{supp}(\mathbf{J}) \cup \operatorname{supp}(\mathbf{K})$.

3.4. Second Application of Reciprocity Principle

The last two sections show that the application of the reciprocity principle (16) to the region $v = R_A \cup R_{\text{ext}}$ leads to the equality of (21) and (22). In this subsection another application of (16) is made (this time to the whole space $v = \mathbb{R}^3$), so that the volume integral (22) can be rearranged expediently. For that purpose a 'surrogate' system Σ' and the surrogate's adjoint $\tilde{\Sigma}'$ are considered. They are defined on the basis of the original system Σ and its adjoint $\tilde{\Sigma}$ by omitting the antenna, waveguides and devices. Instead, the surrounding medium spreads into $R_A \cup R_G$, with suitable material tensors. The quantities associated with the surrogate configurations are marked by a prime, e.g., the electric field strength $\mathbf{E}'$ for Σ', and $\tilde{\mathbf{E}}'$ for $\tilde{\Sigma}'$. The region $R_A \cup R_G$ (which contains the devices, waveguides and antenna in Σ) is filled with linear media in Σ', described by the material tensor fields ϵ', τ', μ' and ν'. A similar substitution is made for the adjoint system, so that $\tilde{\epsilon}' = \epsilon'^t$, $\tilde{\mu}' = \mu'^t$, $\tilde{\nu}' = -\tau'^t$, and $\tilde{\tau}' = -\nu'^t$ hold in $R_A \cup R_G$. Σ' contains the same material as Σ in the external region (precisely speaking the material tensor fields of Σ' and Σ coincide in R_{ext}). The same obtains for the adjoint systems $\tilde{\Sigma}'$ and $\tilde{\Sigma}$.

We consider the following currents and fields in the surrogate configurations: The currents in Σ' are confined to the external region, where they agree with those in Σ, that is $\mathbf{J}'(\mathbf{r}) = 0$ and $\mathbf{K}'(\mathbf{r}) = 0$ for $\mathbf{r} \in R_A \cup R_G$; $\mathbf{J}'(\mathbf{r}) = \mathbf{J}(\mathbf{r})$ and $\mathbf{K}'(\mathbf{r}) = \mathbf{K}(\mathbf{r})$ for $\mathbf{r} \in R_{\text{ext}}$. The currents in $\tilde{\Sigma}'$ are confined to $R_A \cup R_G$ (assumed to be a closed set), with $(\tilde{\mathbf{J}}', \tilde{\mathbf{K}}')$ determined in such a way that the field generated in R_{ext} coincides with that appearing in $\tilde{\Sigma}$ where the port state $\tilde{\mathbf{s}}$ is maintained, i.e., $\tilde{\mathbf{E}}'(\mathbf{r}) = \tilde{\mathbf{E}}(\mathbf{r})$ and $\tilde{\mathbf{H}}'(\mathbf{r}) = \tilde{\mathbf{H}}(\mathbf{r})$ for $\mathbf{r} \in R_{\text{ext}}$. There is an infinite number of current distributions which generate this same external field. In the following we regard any currents $(\tilde{\mathbf{J}}', \tilde{\mathbf{K}}')$ in $\tilde{\Sigma}'$ which satisfy this condition as 'surrogate currents' associated to the port state $\tilde{\mathbf{s}}$ of the adjoint system. The application of the equivalence principle makes it even possible to confine the currents to the boundary of $R_G \cup R_A$. If we request a restriction of currents to this surface, in general, both an electric and a magnetic surface current density, $\tilde{\mathbf{J}}'_S$ and $\tilde{\mathbf{K}}'_S$, are needed to generate the field in the exterior region [15, Sections 3–5]. However, another valid condition on the currents is that there are no formal magnetic currents at all, $\tilde{\mathbf{K}}' = 0$. Such a solution exists, because the actual currents which generate the external field are, in fact, of electric type. Of course, they are not generally confined to the boundary of $R_A \cup R_G$ but extend into its interior.

Applying the reciprocity principle (16) to the surrogate configurations, with $v = \mathbb{R}^3$, the surface integral over $\partial v = S_\infty$ vanishes in the limit $R_\infty \to \infty$, leaving

$$\int_{R_A \cup R_G} \left(\mathbf{E}' \cdot \tilde{\mathbf{J}}' - \mathbf{H}' \cdot \tilde{\mathbf{K}}'\right) dv = \int_{R_{\text{ext}}} \left(\tilde{\mathbf{E}}' \cdot \mathbf{J}' - \tilde{\mathbf{H}}' \cdot \mathbf{K}'\right) dv \tag{23}$$

The right side equals the integral (22), because in the integration volume the occurring fields and current densities of the surrogate systems Σ' and $\tilde{\Sigma}'$ agree with the respective ones of Σ and $\tilde{\Sigma}$ ($\tilde{\mathbf{E}}' = \tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}' = \tilde{\mathbf{H}}$, $\mathbf{J}' = \mathbf{J}$ and $\mathbf{K}' = \mathbf{K}$ in R_{ext}).

Since in the left integral of (23) $\mathbf{E}'$ and $\mathbf{H}'$ signify the electromagnetic field incident to the region of the antenna when the antenna has been removed from the scene (transition $\Sigma \to \Sigma'$), the symbols $\mathbf{E}^i$ and $\mathbf{H}^i$ are rather used from now on. Hence, the identity of (21), (22) and (23) yields

$$\tilde{\mathbf{a}}^- \cdot \mathbf{N}_+ \cdot \mathbf{a}^+ + \tilde{\mathbf{a}}^+ \cdot \mathbf{N}_- \cdot \mathbf{a}^- = \int_{R_{\text{ext}}} \left(\tilde{\mathbf{E}} \cdot \mathbf{J} - \tilde{\mathbf{H}} \cdot \mathbf{K}\right) dv = \int_{R_A \cup R_G} \left(\tilde{\mathbf{J}}' \cdot \mathbf{E}^i - \tilde{\mathbf{K}}' \cdot \mathbf{H}^i\right) dv \tag{24}$$

This is an extension of the inter-reciprocity relation given in [27], which was obtained for n-ports without independent sources (which means absent external fields in the present context). (24) provides a reciprocity principle as well as a means to describe the interaction of external electromagnetic fields with the antenna from a circuit-theoretical point of view, with the antenna regarded as an active n-port. The currents in the external region (or the fields incident to the antenna) act as independent internal sources of this n-port. The next section discusses this issue in more detail. In summary, (24) relates the excitation state of the original antenna, which is established by the ports state $\mathbf{s}_{+-}$ and the incident external field ($\mathbf{E}^i$, $\mathbf{H}^i$), to any state of the adjoint antenna in transmission mode given by the port state $\tilde{\mathbf{s}}_{+-} = ((\tilde{\mathbf{a}}^+)^t, (\tilde{\mathbf{a}}^-)^t)^t$ and the associated current densities ($\tilde{\mathbf{J}}'$, $\tilde{\mathbf{K}}'$). Here $\tilde{\mathbf{J}}'$ and $\tilde{\mathbf{K}}'$ comprise formal electric and magnetic currents in $\mathrm{R_A} \cup \mathrm{R_G}$ which, in the absence of the devices-waveguides-antenna structure (i.e., in $\tilde{\Sigma}'$), generate the same field throughout R_{ext} that is radiated by the adjoint antenna in $\tilde{\Sigma}$ when the state $\tilde{\mathbf{s}}_{+-}$ is maintained at the ports. As mentioned above, the freedom in the choice of these currents can be an aid to solving the boundary value problem associated with the transmission scenario.

4. NETWORK FORMULATION

The focus of this section is on the mathematical representation of the antenna network for the transmission case ($\mathbf{J} = 0$ and $\mathbf{K} = 0$ in (24)). As part of an electronic circuit the antenna acts as a linear n-port without independent internal sources. The representation of n-ports by means of generalized scattering matrices or generalized admittance and impedance matrices is described in [27]. In Sections 4.1 and 4.2 the most important definitions and relations which are needed in the following are summarized. Since voltages and currents are defined in a more general way here, the implications are discussed in Section 4.3.

4.1. Modal Wave Coefficients

There are instances where the standard impedance, admittance and scattering matrices are not sufficient to describe an n-port. Some of these matrices may be singular at certain frequencies or not even exist (in rare cases there may be no hybrid matrix at all [1]). As a minimum precondition it is assumed here that the antenna acts as an n-port of dimensionality n when there are no external fields. This means that the space of possible ports state vectors $\mathbf{s}_{+-}$ is of dimension n. In other words, the antenna network can be represented by n linearly independent equations for the modal coefficients $\mathbf{a}^+$ and $\mathbf{a}^-$. These equations are written by means of the so-called generalized scattering matrices:

$$\mathbf{Q}_{+-} \cdot \mathbf{s}_{+-} = \mathbf{Q}_+ \cdot \mathbf{a}^+ + \mathbf{Q}_+ \cdot \mathbf{a}^+ = 0 \tag{25}$$

Here $\mathbf{Q}_{+-} = (\mathbf{Q}_+, \mathbf{Q}_-)$ is an $n \times 2n$ matrix which is split in two $n \times n$ matrices $\mathbf{Q}_+$ and $\mathbf{Q}_-$. The matrix $\mathbf{Q}_{+-}$ is of rank n, so there are n linearly independent solutions $\mathbf{s}_{+-}^{(1)} \ldots \mathbf{s}_{+-}^{(n)}$, which are juxtaposed (as columns) to build the matrix

$$\mathcal{B}_{+-} = \left(\mathbf{s}_{+-}^{(1)}, \ldots, \mathbf{s}_{+-}^{(n)}\right) \tag{26}$$

which satisfies $\mathbf{Q}_{+-} \cdot \mathcal{B}_{+-} = 0$. Since the columns of $\mathcal{B}_{+-}$ are a basis of the space $\{\mathbf{s}_{+-}\}$ of possible states at the antenna ports, every possible excitation $\mathbf{s}_{+-}$ of the ports (of a transmitting antenna) can be expanded by $\mathbf{s}_{+-} = \mathcal{B}_{+-} \cdot \mathbf{x}$, where $\mathbf{x}$ is a vector of expansion coefficients.

By analogy to (25) and (26) we define scattering matrices $\tilde{\mathbf{Q}}_{+-} = (\tilde{\mathbf{Q}}_+, \tilde{\mathbf{Q}}_-)$ and a basis matrix $\tilde{\mathcal{B}}_{+-}$ for the adjoint system, with $\tilde{\mathbf{Q}}_{+-} \cdot \tilde{\mathcal{B}}_{+-} = 0$. In accordance with [27], the $2n \times 2n$ matrices

$$\mathcal{S}_n = \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} \mathbf{N}_+ & \mathbf{0}_n \\ \mathbf{0}_n & -\mathbf{N}_- \end{pmatrix} \tag{27}$$

are introduced, where $\mathbf{0}_n$ and $\mathbf{1}_n$ denote the $n \times n$ zero and identity matrix, respectively. In terms of these matrices, the 'ports state side' of equation (24) reads

$$\tilde{\mathbf{a}}^- \cdot \mathbf{N}_+ \cdot \mathbf{a}^+ + \tilde{\mathbf{a}}^+ \cdot \mathbf{N}_- \cdot \mathbf{a}^- = -\mathbf{s}_{+-} \cdot \mathbf{N} \cdot \mathcal{S}_n \cdot \tilde{\mathbf{s}}_{+-} \tag{28}$$

So $\mathbf{s}_{+-} \cdot \mathbf{N} \cdot \mathcal{S}_n \cdot \tilde{\mathbf{s}}_{+-} = 0$ describes the pure transmission case where external fields are absent from the original and the adjoint system, which implies $\mathcal{B}_{+-}^t \cdot \mathbf{N} \cdot \mathcal{S}_n \cdot \tilde{\mathcal{B}}_{+-} = \mathbf{0}_n$.

4.2. General Port Variables

In many applications quantities other than the modal coefficients are used to describe the state of the ports, for instance port voltages $\mathbf{V} = (V_1 \ldots V_n)^t$ and currents $\mathbf{I} = (I_1 \ldots I_n)^t$. Let $\mathbf{s}$ denote any set of parameters which represent the ports state uniquely, that is, there is a $2n \times 2n$ regular matrix $\mathcal{U}^{+-}$ mapping the linear space of vectors $\mathbf{s}$, denoted by $\{\mathbf{s}\}$, onto the space $\{\mathbf{s}_{+-}\}$,

$$\mathbf{s}_{+-} = \mathcal{U}^{+-} \cdot \mathbf{s} \tag{29}$$

The columns of the matrix

$$\mathcal{B} = \left(\mathcal{U}^{+-}\right)^{-1} \cdot \mathcal{B}_{+-} = \left(\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(n)}\right) \tag{30}$$

build a basis of $\{\mathbf{s}\}$. Similar definitions apply to the adjoint antenna, indicated by a tilde over the respective quantities: $\tilde{\mathbf{s}}_{+-} = \tilde{\mathcal{U}}^{+-} \cdot \tilde{\mathbf{s}}$, and $\tilde{\mathcal{B}} = (\tilde{\mathcal{U}}^{+-})^{-1} \cdot \tilde{\mathcal{B}}_{+-}$. In terms of the new parameters the network Equation (25) reads

$$\mathbf{Q} \cdot \mathbf{s} = 0, \quad \mathbf{Q} = \mathbf{Q}_{+-} \cdot \mathcal{U}^{+-} \tag{31}$$

and analogously for the adjoint system. The reciprocity principle (24) adopts the form $\mathbf{s} \cdot \chi \cdot \tilde{\mathbf{s}} = 0$ for absent external fields, where χ is given below in (33). So the basis matrices $\mathcal{B}$ and $\tilde{\mathcal{B}}$ fulfil the relation

$$\mathcal{B}^t \cdot \chi \cdot \tilde{\mathcal{B}} = 0 \tag{32}$$

$$\chi = (\mathcal{U}^{+-})^t \cdot \mathbf{N} \cdot \mathcal{S}_n \cdot \tilde{\mathcal{U}}^{+-} \tag{33}$$

All results for the states $\mathbf{s}$ can be readily applied to the modal wave coefficients $\mathbf{s}_{+-}$ by putting $\mathcal{U}^{+-} = \tilde{\mathcal{U}}^{+-} = \mathbf{1}_{2n}$, e.g., the χ-matrix becomes

$$\chi_{+-} = \mathbf{N} \cdot \mathcal{S}_n \tag{34}$$

Because $\mathbf{Q}\cdot\mathcal{B}=0$ and $\tilde{\mathbf{Q}}\cdot\tilde{\mathcal{B}}=0$, a reciprocity relation for the $\mathbf{Q}$-matrices can be inferred from (32), i.e., (a proof is given in [27])

$$\tilde{\mathbf{Q}}\cdot\chi^{-1}\cdot\mathbf{Q}^t=0 \tag{35}$$

The consequences of the above equations for the description of receiving antennas, where external fields impact on the antenna, are treated in Section 5.

4.3. Voltage and Current Representation

Voltages and currents are associated with each port and collected in the vectors $\mathbf{V}$ and $\mathbf{I}$, respectively. The complete ports state is denoted by $\mathbf{s_{VI}}=(\mathbf{V}^t,\mathbf{I}^t)^t$ and the matrix mapping the voltages and currents on the modal wave coefficients by $\mathcal{U}^{+-}_{\mathbf{VI}}$. Also other quantities associated with the voltage-current representation are marked by the subscript $_{\mathbf{VI}}$, for instance $\chi_{\mathbf{VI}}$ and $\mathcal{B}_{\mathbf{VI}}$.

The most common relation between, on the one hand, port voltages and currents and, on the other hand, wave coefficients, is of the form $\mathbf{a}^\pm=\xi\cdot(\mathbf{V}\pm\zeta\cdot\mathbf{I})/2$ where ξ and ζ are $n\times n$ diagonal matrices. ζ contains the characteristic impedances of the waveguides in its diagonal, and $\xi_{ii}=(\sqrt{\zeta_{ii}})^{-1}$ for all diagonal elements $i=1\dots n$. Other relations are used as well, for instance $\xi=\zeta=\mathbf{1}_n$. Power waves cannot be described by the above relation [21]. In order to cover the great variety of commonly used definitions, the quite general relations

$$\mathbf{a}^\pm=\frac{1}{2}\,\xi^\pm\cdot\left(\mathbf{V}\pm\zeta^\pm\cdot\mathbf{I}\right) \tag{36}$$

$$\mathbf{s}_{+-}=\mathcal{U}^{+-}_{\mathbf{VI}}\cdot\mathbf{s_{VI}} \tag{37}$$

$$\mathcal{U}^{+-}_{\mathbf{VI}}=\frac{1}{2}\begin{pmatrix}\xi^+ & \xi^+\cdot\zeta^+\\ \xi^- & -\xi^-\cdot\zeta^-\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\xi^+ & \mathbf{0}_n\\ \mathbf{0}_n & \xi^-\end{pmatrix}\cdot\begin{pmatrix}\mathbf{1}_n & \zeta^+\\ \mathbf{1}_n & -\zeta^-\end{pmatrix} \tag{38}$$

are applied. Here $\zeta^\pm$ and $\xi^\pm$ are $n\times n$ matrices which, in general, may even be non-diagonal. Analogous definitions hold for the adjoint system. These equations also comprise the relation between voltages/currents and coefficients of power waves [21] (however, power wave coefficients are regarded as the derived quantities, whereas in the present context the voltages and currents are derived from the modal wave coefficients). The case $\xi^+=\xi^-$, $\zeta^+=\zeta^-$, $\tilde{\xi}^+=\tilde{\xi}^-$, and $\tilde{\zeta}^+=\tilde{\zeta}^-$, is investigated in detail in [27]. There an explicit formula for the matrix $\chi_{\mathbf{VI}}$ is given, which governs the inter-reciprocity of voltage-current representations. To treat the more complex case (38) in a concise way the symbols

$$\underline{\zeta}=\frac{1}{2}\left(\zeta^++\zeta^-\right),\quad \underline{\tilde{\zeta}}=\frac{1}{2}\left(\tilde{\zeta}^++\tilde{\zeta}^-\right) \tag{39}$$

$$\underline{\mathbf{N}}_\pm=(\xi^\pm)^t\cdot\mathbf{N}_\pm\cdot\tilde{\xi}^\mp \tag{40}$$

are introduced. The determinant of the transform $\mathcal{U}^{+-}_{\mathbf{VI}}$ is calculated by means of the product rule. Since the determinant of the last matrix in (38) equals $(-1)^n\det(\zeta^++\zeta^-)=(-2)^n\det(\underline{\zeta})$, we obtain

$$\det(\mathcal{U}^{+-}_{\mathbf{VI}})=\det(\xi^+)\det(\xi^-)\det(\underline{\zeta})/(-2)^n \tag{41}$$

In order that the transform renders a one-to-one correspondence of states, the determinant must not be zero, which is equivalent to the regularity of the matrices ξ^+, ξ^- and $\underline{\zeta}$. On this condition, the inverse of $\mathcal{U}^{+-}_{\mathbf{VI}}$ can be written

$$(\mathcal{U}^{+-}_{\mathbf{VI}})^{-1}=\begin{pmatrix}\zeta^-\cdot\underline{\zeta}^{-1} & \zeta^+\cdot\underline{\zeta}^{-1}\\ \underline{\zeta}^{-1} & -\underline{\zeta}^{-1}\end{pmatrix}\cdot\begin{pmatrix}\xi^+ & \mathbf{0}_n\\ \mathbf{0}_n & \xi^-\end{pmatrix}^{-1} \tag{42}$$

The same considerations apply to the respective adjoint matrices. After these preparations we can use (33) to calculate $\chi_{\mathbf{VI}}$ and its inverse

$$\begin{aligned}\chi_{\mathbf{VI}}=(\mathcal{U}^{+-}_{\mathbf{VI}})^t\cdot\mathbf{N}\cdot\mathcal{S}_n\cdot\tilde{\mathcal{U}}^{+-}_{\mathbf{VI}}&=-\frac{1}{4}\begin{pmatrix}\mathbf{1}_n & \zeta^+\\ \mathbf{1}_n & -\zeta^-\end{pmatrix}^t\cdot\begin{pmatrix}\mathbf{0}_n & \underline{\mathbf{N}}_+\\ \underline{\mathbf{N}}_- & \mathbf{0}_n\end{pmatrix}\cdot\begin{pmatrix}\mathbf{1}_n & \tilde{\zeta}^+\\ \mathbf{1}_n & -\tilde{\zeta}^-\end{pmatrix}\\ &=\frac{1}{4}\begin{pmatrix}-\underline{\mathbf{N}}_+-\underline{\mathbf{N}}_- & \underline{\mathbf{N}}_+\cdot\tilde{\zeta}^--\underline{\mathbf{N}}_-\cdot\tilde{\zeta}^+\\ -(\zeta^+)^t\cdot\underline{\mathbf{N}}_++(\zeta^-)^t\cdot\underline{\mathbf{N}}_- & (\zeta^+)^t\cdot\underline{\mathbf{N}}_+\cdot\tilde{\zeta}^-+(\zeta^-)^t\cdot\underline{\mathbf{N}}_-\cdot\tilde{\zeta}^+\end{pmatrix}\end{aligned} \tag{43}$$

$$
\begin{aligned}
(\chi_{\mathbf{VI}})^{-1} &= -\begin{pmatrix} \tilde{\zeta}^{-} \cdot \underline{\tilde{\zeta}}^{-1} & \tilde{\zeta}^{+} \cdot \underline{\tilde{\zeta}}^{-1} \\ \underline{\tilde{\zeta}}^{-1} & -\underline{\tilde{\zeta}}^{-1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{0}_n & \underline{\mathbf{N}}_{-}^{-1} \\ \underline{\mathbf{N}}_{+}^{-1} & \mathbf{0}_n \end{pmatrix} \cdot \begin{pmatrix} \zeta^{-} \cdot \underline{\zeta}^{-1} & \zeta^{+} \cdot \underline{\zeta}^{-1} \\ \underline{\zeta}^{-1} & -\underline{\zeta}^{-1} \end{pmatrix}^{t} \\
&= -\begin{pmatrix} \tilde{\zeta}^{+} \cdot \bar{\mathbf{N}}_{+}^{-1} \cdot (\zeta^{-})^{t} + \tilde{\zeta}^{-} \cdot \bar{\mathbf{N}}_{-}^{-1} \cdot (\zeta^{+})^{t} & \tilde{\zeta}^{+} \cdot \bar{\mathbf{N}}_{+}^{-1} - \tilde{\zeta}^{-} \cdot \bar{\mathbf{N}}_{-}^{-1} \\ -\bar{\mathbf{N}}_{+}^{-1} \cdot (\zeta^{-})^{t} + \bar{\mathbf{N}}_{-}^{-1} \cdot (\zeta^{+})^{t} & -\bar{\mathbf{N}}_{+}^{-1} - \bar{\mathbf{N}}_{-}^{-1} \end{pmatrix}
\end{aligned}
\tag{44}
$$

with $\bar{\mathbf{N}}_{\pm} = \underline{\zeta}^{t} \cdot \underline{\mathbf{N}}_{\pm} \cdot \underline{\tilde{\zeta}} = \underline{\zeta}^{t} \cdot (\xi^{\pm})^{t} \cdot \mathbf{N}_{\pm} \cdot \tilde{\xi}^{\mp} \cdot \underline{\tilde{\zeta}}$.

It is shown in [27] that the inter-reciprocity (35) is equivalent to the abstract network inter-reciprocity (as commonly assumed in network theory)

$$\tilde{\mathbf{V}} \cdot \mathbf{I} - \tilde{\mathbf{I}} \cdot \mathbf{V} = 0 \tag{45}$$

if and only if $\chi_{\mathbf{VI}}$ is a multiple of $\mathcal{S}_n$. The multiplication factor is written $-\lambda/2$ for later convenience and concurrence with [27], i.e.,

$$\chi_{\mathbf{VI}} = -\frac{\lambda}{2}\,\mathcal{S}_n \tag{46}$$

with a scalar $\lambda \neq 0$. To find the conditions under which this equivalence holds, $\mathcal{S}_n$ and (43) are substituted into (46). The $2n \times 2n$ equations split into four $n \times n$ matrix equations

$$\mathbf{0}_n = \underline{\mathbf{N}}_{+} + \underline{\mathbf{N}}_{-} \tag{47}$$

$$2\lambda \mathbf{1}_n = \underline{\mathbf{N}}_{+} \cdot \tilde{\zeta}^{-} - \underline{\mathbf{N}}_{-} \cdot \tilde{\zeta}^{+} \tag{48}$$

$$2\lambda \mathbf{1}_n = (\zeta^{+})^{t} \cdot \underline{\mathbf{N}}_{+} - (\zeta^{-})^{t} \cdot \underline{\mathbf{N}}_{-} \tag{49}$$

$$\mathbf{0}_n = (\zeta^{+})^{t} \cdot \underline{\mathbf{N}}_{+} \cdot \tilde{\zeta}^{-} + (\zeta^{-})^{t} \cdot \underline{\mathbf{N}}_{-} \cdot \tilde{\zeta}^{+} \tag{50}$$

With the replacement $\underline{\mathbf{N}}_{-} = -\underline{\mathbf{N}}_{+}$ and (39) taken into account, (48)–(50) become

$$\underline{\mathbf{N}}_{+} = \lambda \underline{\tilde{\zeta}}^{-1} \tag{51}$$

$$\underline{\mathbf{N}}_{+} = \lambda (\underline{\zeta}^{t})^{-1} \tag{52}$$

$$(\zeta^{+})^{t} \cdot \underline{\mathbf{N}}_{+} \cdot \underline{\tilde{\zeta}} = \underline{\zeta}^{t} \cdot \underline{\mathbf{N}}_{+} \cdot \tilde{\zeta}^{+} \tag{53}$$

The first two equations show that $\underline{\tilde{\zeta}} = \underline{\zeta}^{t}$. Further, they can be used to transform the third equation into $\tilde{\zeta}^{+} = (\zeta^{+})^{t}$, and so also $\tilde{\zeta}^{-} = (\zeta^{-})^{t}$. Thus, (46) implies

$$\tilde{\zeta}^{\pm} = (\zeta^{\pm})^{t} \tag{54}$$

$$\pm\,\underline{\mathbf{N}}_{\pm} = \lambda\,(\underline{\zeta}^{t})^{-1} \tag{55}$$

We can also go the way of conclusions back from these results to (46), so it is revealed that (45) is valid if and only if (54) and (55) hold. For later use we write the last equation also in terms of $\mathbf{N}_{\pm}$ and $\bar{\mathbf{N}}_{\pm}$:

$$\bar{\mathbf{N}}_{\pm} = \pm\lambda \underline{\zeta}^{t}, \quad \mathbf{N}_{\pm} = \pm 2\lambda \left[\xi^{\pm} \cdot (\zeta^{+} + \zeta^{-}) \cdot (\tilde{\xi}^{\mp})^{t}\right]^{-1} \tag{56}$$

where the derivation of the second equation makes use of $\mathbf{N}_{\pm} = \mathbf{N}_{\pm}^{t}$. In the particular case $\xi^{+} = \xi^{-} =: \xi$ and $\zeta^{+} = \zeta^{-} =: \zeta$ we obtain that (45) is valid if and only if

$$\tilde{\zeta} = \zeta^{t} \tag{57}$$

$$\pm\,\mathbf{N}_{\pm} = \lambda\,[\xi \cdot \zeta \cdot \tilde{\xi}^{t}]^{-1} \tag{58}$$

These equations are equivalent to (92)–(94) in [27].

Although it was not a precondition of the above derivations, the matrices $\xi^{\pm}$, $\tilde{\xi}^{\pm}$, $\zeta^{\pm}$ and $\tilde{\zeta}^{\pm}$ are usually of diagonal form. In this case each element of the vectors $\mathbf{V}$ and $\mathbf{I}$ is characteristic of a single port, defined on the basis of the travelling wave coefficients of the same port without dependence on the other ports. The diagonals of $\zeta^{\pm}$ and $\tilde{\zeta}^{\pm}$ contain the characteristic impedances associated with the respective waveguide modes. (54) states that the characteristic impedance associated with a mode of an adjoint waveguide has to be the same as that for the associated mode of the original waveguide. Once the characteristic impedances are defined, (55) can be satisfied by suitable normalization of the waveguide modes (represented by the elements of the diagonal matrices $\mathbf{N}_{\pm}$).

5. TRANSFER OPERATORS AND TRANSFER MATRICES

For a transmitting antenna the reciprocity relation (24) is bilinear with regard to the ports states, i.e., linear with regard to $\mathbf{s}$ as well as linear with regard to $\tilde{\mathbf{s}}$. External currents/fields introduce an inhomogeneity in (24) which can be described expediently by transfer operators or matrices. It is shown how they have to be linked to the network description of the antenna by $\mathbf{Q}$-matrices.

5.1. Transfer Operators and Antenna Network Equations

The integral over the volume $\mathrm{R_A} \cup \mathrm{R_G}$ appearing in (24) constitutes a linear functional which associates a complex number with each external electromagnetic field $\mathbf{f}^i = ((\mathbf{E}^i)^t, (\mathbf{H}^i)^t)^t$, i.e., each field generated by sources in R_{ext}. Let $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathbf{s}}]$ denote this functional and $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathbf{s}}]\mathbf{f}^i$ its value when applied to the field $\mathbf{f}^i$. Here the tilde over the functional emphasizes that it is established by currents in the adjoint system. The symbol $[\tilde{\mathbf{s}}]$ indicates that the currents (and so the functional) are fully determined by the ports state $\tilde{\mathbf{s}}$ at the adjoint antenna. Further, let $\tilde{\mathsf{T}}_{\mathsf{E}}$ and $\tilde{\mathsf{T}}_{\mathsf{H}}$ stand for the functional part acting on the electric and magnetic field strength, respectively. That is,

$$\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathbf{s}}]\mathbf{f}^i = \tilde{\mathsf{T}}_{\mathsf{E}}[\tilde{\mathbf{s}}]\mathbf{E}^i + \tilde{\mathsf{T}}_{\mathsf{H}}[\tilde{\mathbf{s}}]\mathbf{H}^i \tag{59}$$

$$\tilde{\mathsf{T}}_{\mathsf{E}}[\tilde{\mathbf{s}}]\mathbf{E}^i = \int_{R_A \cup R_G} \tilde{\mathbf{J}}' \cdot \mathbf{E}^i \, dv \tag{60}$$

$$\tilde{\mathsf{T}}_{\mathsf{H}}[\tilde{\mathbf{s}}]\mathbf{H}^i = -\int_{R_A \cup R_G} \tilde{\mathbf{K}}' \cdot \mathbf{H}^i \, dv \tag{61}$$

These functionals are called transfer operators since they relate the external conditions (fields generated by sources outside the antenna-device region $R_A \cup R_G$) to the internal state at the antenna ports. Applying (28), (29) and (33) we have $\tilde{\mathbf{a}}^- \cdot \mathbf{N}_+ \cdot \mathbf{a}^+ + \tilde{\mathbf{a}}^+ \cdot \mathbf{N}_- \cdot \mathbf{a}^- = -\mathbf{s} \cdot \chi \cdot \tilde{\mathbf{s}}$. Hence, with the notation for transfer operators and ports states, (24) can be written in the short form

$$\mathbf{s} \cdot \chi \cdot \tilde{\mathbf{s}} + \tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathbf{s}}]\mathbf{f}^i = 0 \tag{62}$$

In an analogous way the transfer operators $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}]$, $\tilde{\mathsf{T}}_{\mathsf{E}}[\tilde{\mathcal{B}}]$ and $\tilde{\mathsf{T}}_{\mathsf{H}}[\tilde{\mathcal{B}}]$ are defined which map the vector space of external fields to an n dimensional vector space as follows:

$$\tilde{\mathsf{T}}_{\mathsf{f}}\left[\tilde{\mathcal{B}}\right]\mathbf{f}^i = \tilde{\mathsf{T}}_{\mathsf{E}}\left[\tilde{\mathcal{B}}\right]\mathbf{E}^i + \tilde{\mathsf{T}}_{\mathsf{H}}\left[\tilde{\mathcal{B}}\right]\mathbf{H}^i \tag{63}$$

$$\tilde{\mathsf{T}}_{\mathsf{E}}\left[\tilde{\mathcal{B}}\right]\mathbf{E}^i = \int_{R_A \cup R_G} \tilde{\mathcal{J}}'^t \cdot \mathbf{E}^i \, dv \tag{64}$$

$$\tilde{\mathsf{T}}_{\mathsf{H}}\left[\tilde{\mathcal{B}}\right]\mathbf{H}^i = -\int_{R_A \cup R_G} \tilde{\mathcal{K}}'^t \cdot \mathbf{H}^i \, dv \tag{65}$$

where the current matrices

$$\tilde{\mathcal{J}}' = \left(\tilde{\mathbf{J}}'^{(1)}, \ldots, \tilde{\mathbf{J}}'^{(n)}\right), \quad \tilde{\mathcal{K}}' = \left(\tilde{\mathbf{K}}'^{(1)}, \ldots, \tilde{\mathbf{K}}'^{(n)}\right) \tag{66}$$

contain the surrogate current densities $\tilde{\mathbf{J}}'^{(m)}$ and $\tilde{\mathbf{K}}'^{(m)}$, $m = 1 \ldots n$, as columns. They generate in R_{ext} of $\tilde{\Sigma}'$ the fields which are radiated by the adjoint system in $\tilde{\Sigma}$ when the respective ports states $\tilde{\mathbf{s}}^{(m)}$ are maintained (no external field is present). In fact, $\tilde{\mathcal{J}}'$ and $\tilde{\mathcal{K}}'$ need not agree with the actual currents generated in the real adjoint system but generate the same fields in R_{ext}. Taking into consideration

$$\tilde{\mathsf{T}}_{\mathsf{f}}\left[\tilde{\mathcal{B}}\right]\mathbf{f}^i = \left(\tilde{\mathsf{T}}_{\mathsf{f}}\left[\tilde{\mathbf{s}}^{(1)}\right]\mathbf{f}^i, \ldots, \tilde{\mathsf{T}}_{\mathsf{f}}\left[\tilde{\mathbf{s}}^{(n)}\right]\mathbf{f}^i\right)^t \tag{67}$$

the application of (62) to each $\tilde{\mathbf{s}}^{(m)}$ and arrangement of the results in a column vector yields

$$\mathbf{Q} \cdot \mathbf{s} + \tilde{\mathsf{T}}_{\mathsf{f}}\left[\tilde{\mathcal{B}}\right]\mathbf{f}^i = 0 \tag{68}$$

with

$$\mathbf{Q} = \tilde{\mathcal{B}}^t \cdot \chi^t \tag{69}$$

The obtained antenna network Equation (68) describes the port states of the antenna including the effect of external fields. From a circuitry point of view the fields can be regarded as independent sources internal to the antenna n-port. The n rows of $\tilde{\mathcal{B}}^t$ are linearly independent and χ is regular, so the $n \times 2n$ matrix $\tilde{\mathcal{B}}^t \cdot \chi^t$ is of rank n and provides a network matrix $\mathbf{Q}$ which fully represents the transmitting antenna as an n-port. Vice versa, if a network matrix $\mathbf{Q}$ of rank n represents the antenna at transmission by satisfying $\mathbf{Q} \cdot \mathbf{s} = 0$ for all possible ports states, we can find a basis matrix $\tilde{\mathcal{B}}$ for the adjoint antenna system by means of $\tilde{\mathcal{B}} = \chi^{-1} \cdot \mathbf{Q}^t$.

5.2. Application to the Adjoint System

Since the adjoint of the adjoint is equal to the original system, the identification $\tilde{\tilde{\mathcal{B}}} = \mathcal{B}$ and $\tilde{\tilde{\mathbf{Q}}} = \mathbf{Q}$ is possible. So we can interchange the roles of the original and the adjoint system in the above discussion. In analogy to (66) current matrices $\mathcal{J}' = \left(\mathbf{J}'^{(1)}, \ldots, \mathbf{J}'^{(n)}\right)$ and $\mathcal{K}' = \left(\mathbf{K}'^{(1)}, \ldots, \mathbf{K}'^{(n)}\right)$ are defined which contain the surrogate currents of Σ' which are associated to the ports states $\mathbf{s}^{(1)} \ldots \mathbf{s}^{(n)}$ of Σ (i.e., $(\mathbf{J}'^{(m)}, \mathbf{K}'^{(m)})$ radiate into R_{ext} of Σ' the same field as the original antenna in Σ when driven with $\mathbf{s}^{(m)}$). Let $\mathsf{T}_\mathsf{f}[\mathcal{B}]$, $\mathsf{T}_\mathsf{E}[\mathcal{B}]$ and $\mathsf{T}_\mathsf{H}[\mathcal{B}]$ signify the transfer operators for the original system, i.e. the operators defined in analogy with (63)–(65) on the basis of $\mathcal{J}'$ and $\mathcal{K}'$. Applying the above discussion with interchanged roles of original and adjoint system thus implies

$$\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{s}} + \mathsf{T}_\mathsf{f}\left[\mathcal{B}\right] \tilde{\mathbf{f}}^i = 0 \tag{70}$$

where $\tilde{\mathbf{f}}^i$ is the electromagnetic field incident to the adjoint antenna (generated by sources in R_{ext}), and

$$\tilde{\mathbf{Q}} = \mathcal{B}^t \cdot \tilde{\chi}^t \tag{71}$$

$$\tilde{\chi} = \left(\tilde{\mathcal{U}}^{+-}\right)^t \cdot \tilde{\mathbf{N}} \cdot \mathcal{S}_n \cdot \mathcal{U}^{+-} \tag{72}$$

To investigate the relation between χ and $\tilde{\chi}$ we have to take into account that $\mathbf{N}_+$ and $\mathbf{N}_-$ are diagonal matrices and $\tilde{\mathbf{N}}_\pm = -\mathbf{N}_\mp$ [27, equ. (63)], so

$$\tilde{\mathbf{N}} \cdot \mathcal{S}_n = \begin{pmatrix} \tilde{\mathbf{N}}_+ & \mathbf{0}_n \\ \mathbf{0}_n & -\tilde{\mathbf{N}}_- \end{pmatrix} \cdot \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} = \begin{pmatrix} \mathbf{0}_n & -\tilde{\mathbf{N}}_+ \\ -\tilde{\mathbf{N}}_- & \mathbf{0}_n \end{pmatrix} = \begin{pmatrix} \mathbf{0}_n & \mathbf{N}_- \\ \mathbf{N}_+ & \mathbf{0}_n \end{pmatrix} = -(\mathbf{N} \cdot \mathcal{S}_n)^t \tag{73}$$

After substitution of this result into (72), the comparison with the transposition of (33) proves

$$\tilde{\chi} = -\chi^t \tag{74}$$

For the interpretation of (70) we have to note that the adjoint antenna is impacted by an external field, whereas the original system is to be considered in pure transmission mode, with its ports states composing the n-dimensional vector space spanned by the columns of the matrix $\mathcal{B}$. Since interpretations of (68) and (70) run in parallel, the following discussions focus on the former.

5.3. Discussion of Solutions

The solution of (68) for a given external field $\mathbf{f}^i$ depends on the electronic devices connected to the antenna. For instance, receivers connected via the waveguides act as loads to the antenna n-port. Let $\mathbf{s}_0$ be any particular solution of (68), e.g., when a certain admissible set of loads is connected. Here 'admissible' means that the loads impose conditions which are not in contradiction with the network Equation (68), allowing its solution. Any other admissible conditioning imposed by the devices leads to a solution $\mathbf{s}$ which can be decomposed into the particular solution $\mathbf{s}_0$ and a supplement $\Delta\mathbf{s}$ solving the linear equations system $\mathbf{Q} \cdot \Delta\mathbf{s} = 0$. Since $\Delta\mathbf{s}$ can be written as a linear combination of basis vectors in the matrix $\mathcal{B}$, the general solution of (68) has the form

$$\mathbf{s} = \mathbf{s}_0 + \mathcal{B} \cdot \mathbf{x} \tag{75}$$

$$\mathbf{Q} \cdot \mathbf{s}_0 = -\tilde{\mathsf{T}}_\mathsf{f}\left[\tilde{\mathcal{B}}\right] \mathbf{f}^i \tag{76}$$

with an n-element coefficient vector $\mathbf{x}$. In order to determine $\mathbf{x}$, further n equations are needed which are fixed by the connected devices (and waveguides).

The network equation system (68) is determined by, on the one hand, the homogeneous part established by the matrix $\mathbf{Q}$ of the original system, and on the other hand, by the inhomogeneity $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}]\mathbf{f}^i$ which depends on the external field. The latter is a function of the basis matrix $\tilde{\mathcal{B}}$ associated with the adjoint system. However, this fact does not mean that $\tilde{\mathcal{B}}$ is independent of the original system. Quite the contrary is true: $\tilde{\mathcal{B}}$ and $\mathbf{Q}$ as appearing in (68) must fulfil (69). This means that, after having determined an appropriate matrix $\mathbf{Q}$ representing the antenna n-port at transmission, the adjoint basis matrix has to be calculated by $\tilde{\mathcal{B}} = \chi^{-1} \cdot \mathbf{Q}^t$, which defines the possible ports states of the adjoint antenna at transmission. From the ports states given by the columns of $\tilde{\mathcal{B}}$ we can calculate the corresponding surrogate current densities $\tilde{\mathcal{J}}'$ and $\tilde{\mathcal{K}}'$ in $R_A \cup R_G$ of $\tilde{\Sigma}'$, which determine $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}]$. This is actually the procedure how an antenna can be studied in practice, for instance by computer simulations.

It is of high practical importance that all quantities used to represent the antenna system by (68) are obtained for transmission conditions ($\mathbf{Q}$ for the original, $\tilde{\mathsf{T}}_{\mathsf{f}}[\mathcal{B}]$ for the adjoint system). Nevertheless, (68) is a general formula allowing the description of the antenna's response to external fields as well as any kind of operation of the antenna as part of an electronic circuit (no matter if the antenna is used for reception or transmission or for both purposes). This kind of reception-transmission relation is a typical consequence of the application of (inter-)reciprocity principles.

5.4. Transfer Matrices for Incident Plane Waves

Here an external electromagnetic field is considered which takes the form of a plane wave with wave vector $\mathbf{k}$ and electric and magnetic field strength amplitudes $\mathbf{E}_0$ and $\mathbf{H}_0$. Thus,

$$\mathbf{f}^i(\mathbf{r}) = \mathbf{f}_0 \, e^{-j\mathbf{k}\cdot\mathbf{r}} = \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{77}$$

In this case the transfer operator $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}]$ maps the vector field $\mathbf{f}^i(\mathbf{r})$ to the following n-element vector

$$\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}]\mathbf{f}^i = \int_{R_A \cup R_G} \left(\tilde{\mathcal{J}}'^t, -\tilde{\mathcal{K}}'^t \right) \cdot \mathbf{f}_0 \, e^{-j\mathbf{k}\cdot\mathbf{r}} \, dv \tag{78}$$

The constant vector $\mathbf{f}_0$ can be drawn out of the integral. The remaining integral depends only on the wave vector $\mathbf{k}$ of the incident wave, but not on its wave amplitude $\mathbf{f}_0$. It is therefore an $n \times 6$ matrix, which can be split in two $n \times 3$ matrices, one acting on $\mathbf{E}_0$, the other on $\mathbf{H}_0$. These so-called transfer matrices are defined by

$$\tilde{\mathbf{T}}_{\mathsf{f}} = \left(\tilde{\mathbf{T}}_{\mathsf{E}}, \tilde{\mathbf{T}}_{\mathsf{H}} \right) \tag{79}$$

$$\tilde{\mathbf{T}}_{\mathsf{E}}(\mathbf{k}) = \int_{R_A \cup R_G} \tilde{\mathcal{J}}'^t \, e^{j\mathbf{k}\cdot\mathbf{r}} \, dv \tag{80}$$

$$\tilde{\mathbf{T}}_{\mathsf{H}}(\mathbf{k}) = -\int_{R_A \cup R_G} \tilde{\mathcal{K}}'^t \, e^{j\mathbf{k}\cdot\mathbf{r}} \, dv \tag{81}$$

Hence (78) can be written

$$\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}]\mathbf{f}^i = \tilde{\mathbf{T}}_{\mathsf{f}} \, (-\mathbf{k}) \cdot \mathbf{f}_0 = \tilde{\mathbf{T}}_{\mathsf{E}}(-\mathbf{k}) \cdot \mathbf{E}_0 + \tilde{\mathbf{T}}_{\mathsf{H}}(-\mathbf{k}) \cdot \mathbf{H}_0 \tag{82}$$

In general, transfer matrices depend on $\mathbf{k}$, that is on the direction of incidence and on frequency. The directional dependence is an effect of the appearance of $\mathbf{k}$ in the exponents, whereas the frequency dependence is caused by both the exponents and the current densities. If $\tilde{\mathbf{T}}_{\mathsf{f}}$ does not vanish in the limit $\omega \to 0$, it converges to a direction-independent matrix. This is the case for antennas which behave as capacitors in the quasi-static range (for example dipoles).

In case there is only one port, the transfer matrices shrink to one row which is often treated as a vector [26], the so-called effective length vector $\mathbf{h}$ (the relation to the effective length vector $\mathbf{h}_S$ introduced by Sinclair [42] is given in Section 5.5). In general, each row of a transfer matrix can be regarded as an effective length vector associated with the respective port.

In Section 3.4 it was pointed out that the surrogate currents ($\tilde{\mathcal{J}}'$, $\tilde{\mathcal{K}}'$) associated to a given ports state $\tilde{\mathbf{s}}$ are not uniquely determined. On the same ground there is a variety of valid transfer matrices $\tilde{\mathbf{T}}_{\mathsf{f}}$ that can be obtained by (79)–(81). This ambiguity is also realized when applying (68) to determine the transfer matrices from the antenna's response to incident plane waves. For instance, in a homogeneous and isotropic medium, there are generally two independent wave polarizations for each direction of incidence ($\mathbf{E}_0$ and $\mathbf{H}_0$ orthogonal to $\mathbf{k}$). For a given direction of incidence and a given frequency (68) provides two independent equations for each effective length vector (for each row of $\tilde{\mathbf{T}}_{\mathsf{f}}$). Hence, the antenna's response to plane waves do not establish the transfer matrices uniquely unless additional conditions are provided for their rows. The vanishing of $\tilde{\mathbf{T}}_{\mathsf{H}}$ is the most often used condition, which is feasible because $\tilde{\mathcal{K}}' = 0$ is a valid choice for the surrogate currents at transmission (as explained in Section 3.4). However, $\tilde{\mathbf{T}}_{\mathsf{H}} = 0$ is not yet sufficient to make the transfer matrix $\tilde{\mathbf{T}}_{\mathsf{E}}$ unique. There are two important examples for additional conditions on $\tilde{\mathbf{T}}_{\mathsf{E}}$. First, the rows of $\tilde{\mathbf{T}}_{\mathsf{E}}$ may be postulated to be orthogonal to the direction of incidence. The effective length vector defined by Sinclair satisfies this condition [42]. Second, the condition of direction-independence of the effective length vectors is applicable to capacitive antennas in the quasi-static frequency range [24]. Each of these two additional conditions uniquely determines a matrix $\tilde{\mathbf{T}}_{\mathsf{E}}$, which yields a valid description $\mathbf{Q} \cdot \mathbf{s} = -\tilde{\mathbf{T}}_{\mathsf{E}}(-\mathbf{k}) \cdot \mathbf{E}$ of the antenna.

An example for an experimental method to determine $\tilde{\mathbf{T}}_{\mathbf{E}}$ for the quasi-static limit ($\omega \to 0$) is the so-called rheometry [24, 40]. This technique employs an electrolytic tank, in which a homogeneous electric field of some kHz is maintained. The whole antenna system (or a downscaled model of it) is immersed in the electrolyte. The voltages induced at the antenna ports are observed as a function of the antenna orientation, from which the effective length vectors can be inferred. The technique avoids the influence of parasitic objects around the measurement setup, which would play a significant role without the electrolyte. Further, the electrolyte decreases the antenna impedance significantly, which is of particular importance for the quasi-static frequency range where capacitive antennas possess very high impedances. Rheometry is not applicable to higher frequencies above the quasi-static range, where the condition of direction-independence of the transfer matrix is not applicable. Therefore similar experiments are typically performed in anechoic chambers to minimize radiation coupling with parasitic bodies and the ground. Since the transfer matrix is direction dependent for such frequencies, one has to repeat the measurement for each direction of wave incidence and each value of the wave number k (which may vary with polarization in anisotropic media). Computer simulation can be performed in close analogy [25]. The determination of the antenna's response to incident waves is achieved by solution of the underlying boundary value problem (electric and magnetic field integral equations) and has to be done for each direction of incidence and each of two independent polarizations of the wave.

5.5. Relation to Far Field Transmitted in $\tilde{\Sigma}$

The adjoint transfer matrices do not only govern the reception properties of the original antenna system, but also the structure of the far field transmitted by the adjoint antenna. The very definition of transfer matrices is restricted to plane waves but, in principle, does not need any other assumption about the external medium (provided that plane waves can propagate). However, the following expressions for the far field transmitted by the adjoint antenna requires the homogeneity and isotropy of the external medium. Thus, in this subsection let $\epsilon = \tilde{\epsilon}$ and $\mu = \tilde{\mu}$ be scalar constants and $\tau = \tilde{\tau} = \nu = \tilde{\nu} = 0$ throughout R_{ext}. Nonetheless the media in the waveguides and in the antenna may be anisotropic and inhomogeneous. If there are anisotropic media around the antenna, the antenna region R_A is extended suitably so as to contain these media. The formal surrogate current densities $\tilde{\mathcal{J}}'$ and $\tilde{\mathcal{K}}'$ appearing in the definitions of transfer matrices are confined to the region $R_A \cup R_G$, which, in $\tilde{\Sigma}'$, contains the same medium as R_{ext}. Of course, this does not apply to the real adjoint system $\tilde{\Sigma}$, where the region $R_A \cup R_G$ contains the adjoint waveguides and the adjoint antenna (generally composed of media different from the medium in R_{ext}).

Let an arbitrary ports state $\tilde{\mathbf{s}}$ of the transmitting adjoint system be expanded into the basis states $\tilde{\mathcal{B}}$, so $\tilde{\mathbf{s}} = \tilde{\mathcal{B}} \cdot \tilde{\mathbf{x}}$ with the vector $\tilde{\mathbf{x}}$ containing the expansion coefficients. This state leads to the surrogate current densities $\tilde{\mathbf{J}}' = \tilde{\mathcal{J}}' \cdot \tilde{\mathbf{x}}$ and $\tilde{\mathbf{K}}' = \tilde{\mathcal{K}}' \cdot \tilde{\mathbf{x}}$. The fields radiated into R_{ext} are expressed by electric and magnetic vector potentials ($\tilde{\mathbf{A}}$ and $\tilde{\mathbf{F}}$) [15, Section 3-2]

$$\tilde{\mathbf{A}}(\mathbf{r}) = \tilde{\mathbf{x}} \cdot \frac{\mu}{4\pi} \int \tilde{\mathcal{J}}'^t(\mathbf{r}') \frac{e^{-jkR}}{R} \, dv', \quad \tilde{\mathbf{E}}(\mathbf{r}) = \frac{1}{j\omega\epsilon\mu} \nabla \times (\nabla \times \tilde{\mathbf{A}}) - \frac{1}{\epsilon} \nabla \times \tilde{\mathbf{F}} \tag{83}$$

$$\tilde{\mathbf{F}}(\mathbf{r}) = \tilde{\mathbf{x}} \cdot \frac{\epsilon}{4\pi} \int \tilde{\mathcal{K}}'^t(\mathbf{r}') \frac{e^{-jkR}}{R} \, dv', \quad \tilde{\mathbf{H}}(\mathbf{r}) = \frac{1}{j\omega\epsilon\mu} \nabla \times (\nabla \times \tilde{\mathbf{F}}) + \frac{1}{\mu} \nabla \times \tilde{\mathbf{A}} \tag{84}$$

where $k = \omega\sqrt{\epsilon\mu}$, $R = |\mathbf{r} - \mathbf{r}'|$, and $\mathbf{r}'$ is the radius vector to the source points in $R_A \cup R_G$, over which the integration is to be performed. The lowest order asymptotic expansion for $r \to \infty$ (the far field) reads [18, Chapter 9]

$$\tilde{\mathbf{A}}(\mathbf{r}) = \tilde{\mathbf{x}} \cdot \frac{\mu \, e^{-jkr}}{4\pi r} \int \tilde{\mathcal{J}}'^t(\mathbf{r}') e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'} \, dv', \quad \tilde{\mathbf{E}}(\mathbf{r}) = j\omega \left[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \tilde{\mathbf{A}}) + \sqrt{\frac{\mu}{\epsilon}} \, \hat{\mathbf{r}} \times \tilde{\mathbf{F}} \right] \tag{85}$$

$$\tilde{\mathbf{F}}(\mathbf{r}) = \tilde{\mathbf{x}} \cdot \frac{\epsilon \, e^{-jkr}}{4\pi r} \int \tilde{\mathcal{K}}'^t(\mathbf{r}') e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'} \, dv', \quad \tilde{\mathbf{H}}(\mathbf{r}) = j\omega \left[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \tilde{\mathbf{F}}) - \sqrt{\frac{\epsilon}{\mu}} \, \hat{\mathbf{r}} \times \tilde{\mathbf{A}} \right] \tag{86}$$

with $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/r$. For the derivation of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ the asymptotic relation $\nabla \times \tilde{\mathbf{A}} \sim -j\mathbf{k} \times \tilde{\mathbf{A}}$ and analogous ones for the other curl operations are utilized. A comparison with (80) and (81) proves that, under the given conditions, the far zone fields can be expressed in terms of the transfer matrices, in particular

$$\tilde{\mathbf{A}}(\mathbf{r}) = \frac{\mu \, e^{-jkr}}{4\pi r} \, \tilde{\mathbf{x}} \cdot \tilde{\mathbf{T}}_{\mathsf{E}}(k\hat{\mathbf{r}}), \quad \tilde{\mathbf{F}}(\mathbf{r}) = -\frac{\epsilon \, e^{-jkr}}{4\pi r} \, \tilde{\mathbf{x}} \cdot \tilde{\mathbf{T}}_{\mathsf{H}}(k\hat{\mathbf{r}}) \tag{87}$$

These equations facilitate the determination of transfer matrices via the transmission properties of antennas, for instance by means of computer programs: First, one calculates the currents induced in the antenna (or in the surrogate system) when the ports are in the state $\tilde{\mathbf{s}}^{(m)}$. This amounts to the solution of a boundary value problem. From the resulting current distribution (often formal surface current densities $\tilde{\mathbf{J}}'^{(m)}_S$ and $\tilde{\mathbf{K}}'^{(m)}_S$ defined on the boundary of $\mathrm{R_A} \cup \mathrm{R_G}$) the far field radiated by the antenna is calculated, giving the vector potentials $\tilde{\mathbf{A}}^{(m)}$ and $\tilde{\mathbf{F}}^{(m)}$. Since $\tilde{x}_q = \delta_{qm}$ in this case, each vector potential is proportional to the m-th row of the respective transfer matrix. Thus, the matrix elements are $\{\tilde{\mathbf{T}}_{\mathsf{E}}\}_{mq} = \tilde{A}^{(m)}_q \, 4\pi r \exp(jkr)/\mu$ and $\{\tilde{\mathbf{T}}_{\mathsf{H}}\}_{mq} = \tilde{F}^{(m)}_q \, 4\pi r \exp(jkr)/\epsilon$. Note that, at a give frequency, the last step (calculation of transfer matrix rows via far zone vector potentials) has to be repeated for each direction $\hat{\mathbf{r}}$, whereas the solution of the boundary value problem needs only be performed once for each ports state. Since the solution of the boundary value problem is most time consuming, this approach reduces the computation time significantly in comparison with the determination of the adjoint transfer matrices via the response of the original antenna to incident waves as discussed in Section 5.4.

It was mentioned in Section 5.4 that each row of a transfer matrix is considered as an effective length vector $\tilde{\mathbf{h}}$ associated with the respective port. This vector is extensively used in radio astronomy (pertinent principles are described in [22, 28], specific applications to antenna analysis in [24, 26, 40] and in-flight calibration in [34, 35, 45]). In this context $\tilde{\mathbf{T}}_{\mathsf{H}} = 0$ is usually assumed, describing the surrogate system by electric currents alone. In consequence, there is only one effective length vector per port. In accordance with (85) Sinclair introduced the effective length vector $\tilde{\mathbf{h}}_S$ on the basis of the transmitted $\mathbf{E}$-field [37, 42]. So $\tilde{\mathbf{h}}_S$ does not agree with the definition by means of the respective row $\tilde{\mathbf{h}}$ of the transfer matrix $\tilde{\mathbf{T}}_{\mathsf{E}}$. The two different definitions are related by

$$\tilde{\mathbf{h}}_S = -\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \tilde{\mathbf{h}} \right) \tag{88}$$

which proves $\tilde{\mathbf{h}}_S$ to be the projection of $\tilde{\mathbf{h}}$ on the wave plane. As discussed above the freedom of choice stems from the indeterminacy of the component in the direction of wave incidence.

6. SPECIAL CASES

6.1. Scattering Parameter Representation

According to the derivations in Section 3, the ports state representation by modal wave coefficients $\mathbf{s}_{+-}$ is most fundamental. It leads to an antenna description by scattering parameters.

$$\mathbf{Q}_{+-} \cdot \mathbf{s}_{+-} + \tilde{\mathsf{T}}_{\mathsf{f}} \left[\tilde{\mathcal{B}}_{+-} \right] \tilde{\mathbf{f}}^i = 0 \tag{89}$$

By means of (34) and (69) the adjoint basis matrix is obtained as a function of the generalized scattering matrices $\mathbf{Q}_{\pm}$:

$$\tilde{\mathcal{B}}_{+-} = \chi_{+-}^{-1} \cdot \mathbf{Q}_{+-}^t = -\mathcal{S}_n \cdot \mathbf{N}^{-1} \cdot \begin{pmatrix} \mathbf{Q}_+^t \\ \mathbf{Q}_-^t \end{pmatrix} = - \begin{pmatrix} \mathbf{N}_-^{-1} \cdot \mathbf{Q}_-^t \\ \mathbf{N}_+^{-1} \cdot \mathbf{Q}_+^t \end{pmatrix} \tag{90}$$

The case where $\mathbf{Q}_-$ is regular is of particular importance, because all abstractly passive antenna networks are of this kind (even more the concretely passive ones — meaning that no part of the antenna network generates energy [4, Section 2.34]). In this case the antenna n-port can be represented by a scattering matrix $\mathbf{S}$ and the identifications $\mathbf{Q}_- = \mathbf{1}_n$ and $\mathbf{Q}_+ = -\mathbf{S}$ are valid. Similarly the adjoint antenna can be characterized by a scattering matrix $\tilde{\mathbf{S}}$ and the identifications $\tilde{\mathbf{Q}}_- = \mathbf{1}_n$ and $\tilde{\mathbf{Q}}_+ = -\tilde{\mathbf{S}}$ obtain. With these specifications the antenna network equation can be written

$$\mathbf{a}^- = \mathbf{S} \cdot \mathbf{a}^+ - \tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}_+]\mathbf{f}^i \tag{91}$$

where the adjoint basis matrix is denoted by $\tilde{\mathcal{B}}_+$ to distinguish this special form from the general (89). $\tilde{\mathcal{B}}_+$ is calculated by using the inter-reciprocity rule for scattering matrices [27, Equation (99)], which can be put in the form

$$\tilde{\mathbf{S}} \cdot \mathbf{N}_-^{-1} = -\mathbf{N}_+^{-1} \cdot \mathbf{S}^t \tag{92}$$

Substitution of (92) and $\mathbf{N}_{\pm} = -\tilde{\mathbf{N}}_{\mp}$ in (90) gives an expression of $\tilde{\mathcal{B}}_+$ in terms of the scattering matrix $\mathbf{S}$ or its adjoint $\tilde{\mathbf{S}}$,

$$\tilde{\mathcal{B}}_+ = - \begin{pmatrix} \mathbf{N}_-^{-1} \\ -\mathbf{N}_+^{-1} \cdot \mathbf{S}^t \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n \\ \tilde{\mathbf{S}} \end{pmatrix} \cdot \tilde{\mathbf{N}}_+^{-1} \tag{93}$$

In the above derivations we have utilized the fact that $\mathbf{N}_{\pm}$ and $\tilde{\mathbf{N}}_{\pm}$ are diagonal matrices. For the same reason (93) can be interpreted in connection with (91) as follows. When terminating each antenna port by a matched load (so that $\mathbf{a}^+ = 0$) an external field $\mathbf{f}^i$ induces waves of strength $\mathbf{a}^- = -\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}_+]\mathbf{f}^i$ travelling in the waveguides from the antenna to the receivers (loads). The transfer operator $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}_+]$ can be split into transfer operators $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathbf{s}}_{+-}^{(m)}]$ for each port $m = 1 \dots n$ in accordance with (67). Since $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathbf{s}}_{+-}^{(m)}]$ maps $\mathbf{f}^i$ on the wave coefficient a_m^-, it represents the sensitivity of the wave emergent from the m-th port to the external field. On the basis of (59)–(61) the operator $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathbf{s}}_{+-}^{(m)}]$ is defined by the current densities $(\tilde{\mathbf{J}}'^{(m)}, \tilde{\mathbf{K}}'^{(m)})$ induced in the surrogate adjoint system when the state $\tilde{\mathbf{s}}_{+-}^{(m)}$ is maintained at the ports. According to (93) $\tilde{\mathbf{s}}_{+-}^{(m)}$ signifies the state where a travelling wave of strength $\tilde{a}_m^{+(m)} = 1/\{\tilde{\mathbf{N}}_+\}_{mm}$ is sent to the adjoint antenna via port m, and the other ports are terminated by matched loads (so that no waves travel towards the antenna there, $\tilde{a}_q^{+(m)} = 0$ for $q \neq m$). When the canonical normalization $\mathbf{N}_+ = \tilde{\mathbf{N}}_+ = \lambda' \mathbf{1}_n$ with a scalar $\lambda' \neq 0$ is applied (i.e., the travelling wave normalization constants are the same for all modes), the waves incident to the driven ports must be of the same 'strength', $\tilde{a}_m^{+(m)} = 1/\lambda'$. Only in this case the scattering matrices satisfy the simple reciprocity relation $\tilde{\mathbf{S}} = \mathbf{S}^t$.

6.2. Arbitrary Pairs $(\mathbf{c}, \mathbf{d})$ of Ports State Variables

Before the inter-reciprocity relations for voltages and currents are discussed a general change of the ports state variables in accordance with Section 4.2 is considered. The variables are divided in two sets of n parameters each, which are represented by the vectors $\mathbf{c}$ and $\mathbf{d}$ and juxtaposed in the new ports state vector $\mathbf{s}_{\mathbf{cd}} = (\mathbf{c}^t, \mathbf{d}^t)^t$. A regular $2n \times 2n$ matrix $\mathcal{U}_{\mathbf{cd}}^{+-}$ maps $\mathbf{s}_{\mathbf{cd}}$ on $\mathbf{s}_{+-}$. Similar definitions hold for the adjoint antenna. So we have the following relations between the new representation and the travelling wave description

$$\mathbf{s}_{\mathbf{cd}} = \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = (\mathcal{U}_{\mathbf{cd}}^{+-})^{-1} \cdot \mathbf{s}_{+-} \tag{94}$$

$$\mathbf{Q}_{\mathbf{cd}} = (\mathbf{Q}_{\mathbf{c}}, \mathbf{Q}_{\mathbf{d}}) = \mathbf{Q}_{+-} \cdot \mathcal{U}_{\mathbf{cd}}^{+-} \tag{95}$$

$$\chi_{\mathbf{cd}} = \left(\mathcal{U}_{\mathbf{cd}}^{+-}\right)^t \cdot \mathbf{N} \cdot \mathcal{S}_n \cdot \tilde{\mathcal{U}}_{\mathbf{cd}}^{+-} \tag{96}$$

By putting a tilde over the quantities appearing in (94) and (95) the analogous relations for the adjoint antenna are obtained. The inter-reciprocity principle (68) reads

$$\mathbf{Q}_{\mathbf{cd}} \cdot \mathbf{s}_{\mathbf{cd}} + \tilde{\mathsf{T}}_{\mathsf{f}} \left[\tilde{\mathcal{B}}_{\mathbf{cd}}\right] \mathbf{f}^i = 0 \tag{97}$$

where

$$\tilde{\mathcal{B}}_{\mathbf{cd}} = \chi_{\mathbf{cd}}^{-1} \cdot \mathbf{Q}_{\mathbf{cd}}^t \tag{98}$$

contains the ports state basis vectors of the adjoint system which define the transfer operator $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}_{\mathbf{cd}}]$ via (63)–(65) as explained there. Substitution of (95) and (96) in (98) yields an expression of $\tilde{\mathcal{B}}_{\mathbf{cd}}$ in terms of $\tilde{\mathcal{U}}_{\mathbf{cd}}^{+-}$ and the travelling wave quantities:

$$\tilde{\mathcal{B}}_{\mathbf{cd}} = -(\tilde{\mathcal{U}}_{\mathbf{cd}}^{+-})^{-1} \cdot \mathcal{S}_n \cdot \mathbf{N}^{-1} (\mathbf{Q}_{+-})^t \tag{99}$$

For the discussion of special cases an expedient subdivision of $\chi_{\mathbf{cd}}^{-1}$ in $n \times n$ block matrices is used,

$$\chi_{\mathbf{cd}}^{-1} = \begin{pmatrix} \chi'_{11} & \chi'_{12} \\ \chi'_{21} & \chi'_{22} \end{pmatrix} \tag{100}$$

which enables us to write $\tilde{\mathcal{B}}_{\mathbf{cd}}$ as two stacked $n \times n$ matrices:

$$\tilde{\mathcal{B}}_{\mathbf{cd}} = \begin{pmatrix} \chi'_{11} \cdot \mathbf{Q}_{\mathbf{c}}^t + \chi'_{12} \cdot \mathbf{Q}_{\mathbf{d}}^t \\ \chi'_{21} \cdot \mathbf{Q}_{\mathbf{c}}^t + \chi'_{22} \cdot \mathbf{Q}_{\mathbf{d}}^t \end{pmatrix} \tag{101}$$

This expression can be simplified essentially when $\mathbf{Q}_{\mathbf{c}}$ or $\mathbf{Q}_{\mathbf{d}}$ is regular. In this case the $n \times n$ matrix ${\mathbf{Q}_{\mathbf{c}}}^{-1} \cdot \mathbf{Q}_{\mathbf{d}}$ is representative of the antenna n-port (at transmission).

First, let us consider the case where the elements of $\mathbf{d}$ can be prescribed independently. Because of the n-dimensionality of the space of possible ports states for the antenna at transmission, this means that $\mathbf{c}$ is uniquely determined by $\mathbf{d}$. In other words there is an $n \times n$ matrix $\mathcal{H}_{\mathbf{d}}$ which is associated with the parameter pair ($\mathbf{c}$, $\mathbf{d}$) and describes the antenna by

$$\mathbf{c} = \mathcal{H}_{\mathbf{d}} \cdot \mathbf{d} - \tilde{\mathsf{T}}_{\mathsf{f}} \left[\tilde{\mathcal{B}}_{\mathbf{d}}\right] \mathbf{f}^i \tag{102}$$

The calligraphy font is used for the matrix symbol ($\mathcal{H}$) in order to avoid confusion with magnetic field vectors ($\mathbf{H}$). The existence of $\mathcal{H}_{\mathbf{d}}$ means that the choices $\mathbf{Q}_{\mathbf{c}} = \mathbf{1}_n$ and $\mathbf{Q}_{\mathbf{d}} = -\mathcal{H}_{\mathbf{d}}$ are valid. To refer to this special representation, $\tilde{\mathcal{B}}_{\mathbf{d}}$ is used instead of $\tilde{\mathcal{B}}_{\mathbf{cd}}$ to signify the corresponding basis matrix of adjoint ports states. Substitution into (35) yields

$$\tilde{\mathbf{Q}}_{\mathbf{c}} \cdot \left(\chi'_{11} - \chi'_{12} \cdot \mathcal{H}_{\mathbf{d}}^t\right) + \tilde{\mathbf{Q}}_{\mathbf{d}} \cdot \left(\chi'_{21} - \chi'_{22} \cdot \mathcal{H}_{\mathbf{d}}^t\right) = 0 \tag{103}$$

This equation is analysed on the basis of a Lemma which is proven in [27]:

Let $\mathbf{Q}$ and $\mathbf{Q}'$ be $n \times 2n$ matrices of rank n, each of which being composed of two $n \times n$ matrices as follows: $\mathbf{Q} = (\mathbf{Q}_1, \mathbf{Q}_2)$ and $\mathbf{Q}' = (\mathbf{Q}'_1, \mathbf{Q}'_2)$. Then the matrix $\mathbf{Q}_1$ is regular if and only if $\mathbf{Q}'_2$ is regular (and similarly for $\mathbf{Q}_2$ and $\mathbf{Q}'_1$).

Application of this Lemma to (103) shows that $\tilde{\mathbf{Q}}_{\mathbf{c}}$ is regular if and only if $\chi'_{21} - \chi'_{22} \cdot \mathcal{H}_{\mathbf{d}}^t$ is regular. Only on this condition does the adjoint antenna permit a representation by an $n \times n$ matrix $\tilde{\mathcal{H}}_{\mathbf{d}}$ which defines the possible ports states at transmission via $\tilde{\mathbf{c}} = \tilde{\mathcal{H}}_{\mathbf{d}} \cdot \tilde{\mathbf{d}}$. We assume that the above condition is satisfied, which requires a suitable definition of $\tilde{\mathcal{U}}_{\mathbf{cd}}^{+-}$. It amounts to the validity of the identification

$\tilde{\mathbf{Q}}_{\mathbf{c}} = \mathbf{1}_n$ and $\tilde{\mathbf{Q}}_{\mathbf{d}} = -\tilde{\mathcal{H}}_{\mathbf{d}}$. In consequence, (103) becomes an inter-reciprocity relation for the matrix $\mathcal{H}_{\mathbf{d}}$ and its adjoint counterpart $\tilde{\mathcal{H}}_{\mathbf{d}}$:

$$\chi'_{11} - \chi'_{12} \cdot \mathcal{H}^t_{\mathbf{d}} = \tilde{\mathcal{H}}_{\mathbf{d}} \cdot (\chi'_{21} - \chi'_{22} \cdot \mathcal{H}^t_{\mathbf{d}}) \tag{104}$$

Taking this equation into account, (101) can be rearranged to

$$\tilde{\mathcal{B}}_{\mathbf{d}} = \begin{pmatrix} \tilde{\mathcal{H}}_{\mathbf{d}} \\ \mathbf{1}_n \end{pmatrix} \cdot (\chi'_{21} - \chi'_{22} \cdot \mathcal{H}^t_{\mathbf{d}}) \tag{105}$$

This result is of particular importance for the voltage-current representation as explained in Section 6.3.

In a similar way the case where $\mathbf{Q}_{\mathbf{d}}$ and $\tilde{\mathbf{Q}}_{\mathbf{d}}$ are regular can be treated. The matrices $\mathcal{H}_{\mathbf{c}}$ and $\tilde{\mathcal{H}}_{\mathbf{c}}$ are introduced such that $\mathbf{d} = \mathcal{H}_{\mathbf{c}} \cdot \mathbf{c}$ and $\tilde{\mathbf{d}} = \tilde{\mathcal{H}}_{\mathbf{c}} \cdot \tilde{\mathbf{c}}$ describe the original and the adjoint antenna system at transmission. The full representation of the original antenna system in an external field $\mathbf{f}^i$ is rendered by

$$\mathbf{d} = \mathcal{H}_{\mathbf{c}} \cdot \mathbf{c} - \tilde{\mathsf{T}}_{\mathsf{f}} \left[\tilde{\mathcal{B}}_{\mathbf{c}} \right] \mathbf{f}^i \tag{106}$$

where the basis matrix $\tilde{\mathcal{B}}_{\mathbf{c}}$ of adjoint ports states defining the transfer operator $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}_{\mathbf{c}}]$ is given by

$$\tilde{\mathcal{B}}_{\mathbf{c}} = \begin{pmatrix} \mathbf{1}_n \\ \tilde{\mathcal{H}}_{\mathbf{c}} \end{pmatrix} \cdot (\chi'_{12} - \chi'_{11} \cdot \mathcal{H}^t_{\mathbf{c}}) \tag{107}$$

The simplest application of this formula is to the travelling wave amplitudes: $\mathbf{c} = \mathbf{a}_+$, $\mathbf{d} = \mathbf{a}_-$, $\mathcal{H}_{\mathbf{c}} = \mathbf{S}$ and $\tilde{\mathcal{H}}_{\mathbf{c}} = \tilde{\mathbf{S}}$. By inversion of (34) one obtains $\chi'_{21} = -\mathbf{N}_+^{-1}$, $\chi'_{12} = -\mathbf{N}_-^{-1}$, $\chi'_{11} = \chi'_{22} = \mathbf{0}_n$, and $\chi'_{12} - \chi'_{11} \cdot \mathcal{H}^t_{\mathbf{c}} = \tilde{\mathbf{N}}_+^{-1}$. So, in fact, (106) and (107) turn into (91) and (93).

6.3. Voltage-current Representation

With voltages and currents related to the travelling wave coefficients as specified in (36), Equations (97) and (101) can be applied with $\mathbf{c} = \mathbf{V}$ and $\mathbf{d} = \mathbf{I}$. If the abstract inter-reciprocity (45) holds, we obtain

$$\mathbf{Q}_{\mathbf{VI}} \cdot \mathbf{s}_{\mathbf{VI}} + \tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}_{\mathbf{VI}}]\mathbf{f}^i = 0 \tag{108}$$

with

$$\tilde{\mathcal{B}}_{\mathbf{VI}} = \frac{2}{\lambda} \begin{pmatrix} -\mathbf{Q}^t_{\mathbf{I}} \\ \mathbf{Q}^t_{\mathbf{V}} \end{pmatrix} \tag{109}$$

where $\chi'_{11} = \chi'_{22} = \mathbf{0}_n$ and $\chi'_{21} = -\chi'_{12} = (2/\lambda)\mathbf{1}_n$ has been used in accordance with (46).

Next let us assume that the currents can be prescribed independently in the original as well as the adjoint system, which amounts to the existence of the respective impedance matrices $\mathbf{Z}$ and $\tilde{\mathbf{Z}}$. So (102)–(105) apply with $\mathcal{H}_{\mathbf{d}} = \mathbf{Z}$ and $\tilde{\mathcal{H}}_{\mathbf{d}} = \tilde{\mathbf{Z}}$. So the representation of the antenna system by means of its impedance matrix becomes

$$\mathbf{V} = \mathbf{Z} \cdot \mathbf{I} - \tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}_{\mathbf{I}}]\mathbf{f}^i \tag{110}$$

$$\tilde{\mathcal{B}}_{\mathbf{I}} = \frac{2}{\lambda} \begin{pmatrix} \tilde{\mathbf{Z}} \\ \mathbf{1}_n \end{pmatrix} \tag{111}$$

The transfer operator $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}_{\mathbf{I}}]$ is established via (63)–(65) on the basis of the current densities induced in the antenna system by driving the current $2/\lambda$ through one port with the other ports left open (no currents through the non-driven ports). The form of Equation (110) provides evidence of $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}_{\mathbf{I}}]$ being an open port transfer operator: it maps the external electromagnetic field $\mathbf{f}^i$ on the vector $\mathbf{V}^\circ$ the m-th element of which is the voltage drop (sign opposite to applied voltage) at port m when all ports are open. The initial Equations (1) and (2) are special expressions of (111) and (110) for a 1-port antenna operated via a conventional waveguide or transmission line, where standard voltage/current definitions imply $\lambda = 2$ (on the condition (45) and the requirement that the complex power delivered to the antenna is $\mathbf{I}^* \cdot \mathbf{V}/2$; a proof is provided in Section 6.4).

In a similar way the admittance representation can be derived when the port voltages can be prescribed arbitrarily in the original and adjoint antenna. Let $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ be the respective admittance matrices. Hence, the Formulas (106) and (107) obtain with $\mathcal{H}_{\mathbf{c}} = \mathbf{Y}$ and $\tilde{\mathcal{H}}_{\mathbf{c}} = \tilde{\mathbf{Y}}$. The admittance matrix representation of the antenna system reads

$$\mathbf{I} = \mathbf{Y} \cdot \mathbf{V} - \tilde{\mathsf{T}}_{\mathsf{f}} \left[\tilde{\mathcal{B}}_{\mathbf{V}}\right] \mathbf{f}^{i} \tag{112}$$

$$\tilde{\mathcal{B}}_{\mathbf{V}} = -\frac{2}{\lambda} \begin{pmatrix} \mathbf{1}_n \\ \tilde{\mathbf{Y}} \end{pmatrix} \tag{113}$$

The transfer operator $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}_{\mathbf{V}}]$ is determined via (63)–(65) on the basis of the current densities induced by applying the voltage $-2/\lambda$ at one port, with the other ports short-circuited (zero voltage at the other ports). Putting $\mathbf{V} = 0$ in (112) we see that $\tilde{\mathsf{T}}_{\mathsf{f}}[\tilde{\mathcal{B}}_{\mathbf{V}}]$ is the transfer operator which maps any external field on the currents driven through the short-circuited ports (as seen from the antenna according to the minus sign in (112)).

Next let us drop the assumption of abstract inter-reciprocity (45), i.e., the general inter-reciprocity (35) prevails, with $\chi = \chi_{\mathbf{VI}}$ from (44). So we have to cope with more complex expressions for $\tilde{\mathcal{B}}_{\mathbf{I}}$ and $\tilde{\mathcal{B}}_{\mathbf{V}}$, which can be calculated by using the corresponding submatrices of (44) in (105) and (107). In this way one proves that the factors $2/\lambda$ and $-2/\lambda$ in (111) and (113) have to be replaced with the following terms, respectively:

$$\chi'_{21} - \chi'_{22} \cdot \mathcal{H}^t_{\mathbf{d}} = \left(\underline{\zeta}^t \cdot \underline{\mathbf{N}}_+ \cdot \underline{\tilde{\zeta}}\right)^{-1} \cdot \left[\zeta^- - \mathbf{Z}\right]^t - \left(\underline{\zeta}^t \cdot \underline{\mathbf{N}}_- \cdot \underline{\tilde{\zeta}}\right)^{-1} \cdot \left[\zeta^+ + \mathbf{Z}\right]^t \tag{114}$$

$$\chi'_{12} - \chi'_{11} \cdot \mathcal{H}^t_{\mathbf{c}} = -\tilde{\zeta}^+ \cdot \left(\underline{\zeta}^t \cdot \underline{\mathbf{N}}_+ \cdot \underline{\tilde{\zeta}}\right)^{-1} \cdot \left[\mathbf{1}_n - \mathbf{Y} \cdot \zeta^-\right]^t + \tilde{\zeta}^- \cdot \left(\underline{\zeta}^t \cdot \underline{\mathbf{N}}_- \cdot \underline{\tilde{\zeta}}\right)^{-1} \cdot \left[\mathbf{1}_n + \mathbf{Y} \cdot \zeta^+\right]^t \tag{115}$$

6.4. Reciprocal Waveguides with Reflection Symmetry

As an example of great practical importance we consider reciprocal waveguides with reflection symmetry. It is to be emphasized that this is no restriction on the antenna or environment (which may contain inhomogeneous anisotropic media). So reciprocally adjoint quantities can be identified with the respective original quantities if they refer to the waveguides, and the transverse field components of the two branches ± 1 of a waveguide mode m can be chosen in such a way that $\mathbf{e}^+_{Tm} = \mathbf{e}^-_{Tm}$ and $\mathbf{h}^+_{Tm} = -\mathbf{h}^-_{Tm}$ [12, Chapter 5]. Under these preconditions (19) and (20) result in

$$2\,\hat{\mathbf{z}} \cdot \int_{S_w} \mathbf{e}^{\sigma}_{Tm} \times \mathbf{h}^{\tau}_{Tq}\, da = \langle \mathbf{f}^{\tau}_q, \mathbf{f}^{\sigma}_m \rangle + \langle \mathbf{f}^{\tau}_q, \mathbf{f}^{-\sigma}_m \rangle = N^{(\tau)}_m \delta_{mq} = -N^{(-\tau)}_m \delta_{mq} \tag{116}$$

This formula is used to calculate the average complex power $P + jQ$ transferred to the antenna through the waveguides $w = 1 \ldots W$, with P being the real (average) power transferred to the antenna and Q the corresponding reactive power. For that purpose we further assume that the field vectors $\mathbf{e}^{\pm}_{Tm}$ and $\mathbf{h}^{\pm}_{Tm}$ are real. This freedom of choice is guaranteed, for instance, when the guide walls are perfect conductors [18]. On the basis of the expansion (9) $P + jQ$ is calculated (W is the number of waveguides and n_w the number of modes in the waveguide w; the dependence of the quantities on the waveguide number w is suppressed)

$$\begin{aligned} P + jQ &= \sum_{w=1}^{W} \frac{1}{2}\, \hat{\mathbf{z}} \cdot \int_{S_w} \mathbf{E}_T \times \mathbf{H}^*_T da = \frac{1}{2} \sum_{w=1}^{W} \hat{\mathbf{z}} \cdot \sum_{m,q=1}^{n_w} \sum_{\sigma,\tau}^{\pm 1} \alpha^{\sigma}_m \alpha^{\tau *}_q \int_{S_w} \mathbf{e}^{\sigma}_{Tm} \times \mathbf{h}^{\tau *}_{Tq}\, da \\ &= \frac{1}{4} \sum_{w=1}^{W} \sum_{m=1}^{n_w} \left[|\alpha^+_m|^2 - |\alpha^-_m|^2 - \alpha^+_m \alpha^{-*}_m + \alpha^{+*}_m \alpha^-_m\right] N^+_m = \frac{1}{4} \left(\mathbf{a}^+ - \mathbf{a}^-\right)^* \cdot \mathbf{N}_+ \cdot \left(\mathbf{a}^+ + \mathbf{a}^-\right) \end{aligned} \tag{117}$$

The term $-\alpha^+_m \alpha^{-*}_m + \alpha^{+*}_m \alpha^-_m$ in the square brackets is purely imaginary and so determines Q and does not contribute to the total average power P transferred to the antenna (as N^+_m is real). Thus,

$$P = \frac{1}{4} \sum_{w=1}^{W} \sum_{m=1}^{n_w} \left[|\alpha^+_m|^2 - |\alpha^-_m|^2\right] N^+_m = \frac{1}{4} \left(\mathbf{a}^{+*} \cdot \mathbf{N}_+ \cdot \mathbf{a}^+ - \mathbf{a}^{-*} \cdot \mathbf{N}_+ \cdot \mathbf{a}^-\right) \tag{118}$$

$$Q = \frac{1}{4j} \sum_{w=1}^{W} \sum_{m=1}^{n_w} \left[\alpha^{+*}_m \alpha^-_m - \alpha^+_m \alpha^{-*}_m\right] N^+_m = \frac{1}{4j} \left(\mathbf{a}^{+*} \cdot \mathbf{N}_+ \cdot \mathbf{a}^- - \mathbf{a}^{-*} \cdot \mathbf{N}_+ \cdot \mathbf{a}^+\right) = \frac{\Im(\mathbf{a}^{+*} \cdot \mathbf{N}_+ \cdot \mathbf{a}^-)}{2} \tag{119}$$

With the canonical normalization $\mathbf{N}_+ = 2\mathbf{1}_n$ we obtain the familiar power expressions [5, 16, 31]

$$P = \frac{1}{2}\left(|\mathbf{a}^+|^2 - |\mathbf{a}^-|^2\right), \quad Q = \Im\left(\mathbf{a}^{+*}\cdot\mathbf{a}^-\right) \tag{120}$$

It is only valid if all waveguide modes are normalized in the same way (i.e., all $N_m^+ = -N_m^-$ are equal).

The relations (36) between travelling wave coefficients and voltages/currents are quite general. In order that $\mathbf{V}$ and $\mathbf{I}$ have the properties of voltages and currents (e.g., actual voltages and currents of a coaxial cable) they have not only to adopt the right dimensions but also render the power relation

$$P + jQ = \frac{1}{2}\mathbf{I}^*\cdot\mathbf{V} \tag{121}$$

The conditions under which this relation is valid are obtained by substituting (36) in (117), producing

$$\begin{aligned}&\frac{1}{16}\left[(\xi^+ - \xi^-)\cdot\mathbf{V} + (\xi^+\cdot\zeta^+ + \xi^-\cdot\zeta^-)\cdot\mathbf{I}\right]^*\cdot\mathbf{N}_+\cdot\left[(\xi^+ + \xi^-)\cdot\mathbf{V} + (\xi^+\cdot\zeta^+ - \xi^-\cdot\zeta^-)\cdot\mathbf{I}\right]\\ &= \frac{1}{16}\begin{pmatrix}\mathbf{V}\\ \mathbf{I}\end{pmatrix}^*\cdot\begin{pmatrix}\Xi_{11} & \Xi_{12}\\ \Xi_{21} & \Xi_{22}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{V}\\ \mathbf{I}\end{pmatrix}\end{aligned} \tag{122}$$

with the $n\times n$ matrices

$$\Xi_{11} = (\xi^+ - \xi^-)^*\cdot\mathbf{N}_+\cdot(\xi^+ + \xi^-) \tag{123}$$

$$\Xi_{12} = (\xi^+ - \xi^-)^*\cdot\mathbf{N}_+\cdot(\xi^+\cdot\zeta^+ - \xi^-\cdot\zeta^-) \tag{124}$$

$$\Xi_{21} = (\xi^+\cdot\zeta^+ + \xi^-\cdot\zeta^-)^*\cdot\mathbf{N}_+\cdot(\xi^+ + \xi^-) \tag{125}$$

$$\Xi_{22} = (\xi^+\cdot\zeta^+ + \xi^-\cdot\zeta^-)^*\cdot\mathbf{N}_+\cdot(\xi^+\cdot\zeta^+ - \xi^-\cdot\zeta^-) \tag{126}$$

The power expressions (121) and (122) are equal if and only if $\Xi_{11} = \Xi_{22} = \Xi_{12} = \mathbf{0}_n$ and $\Xi_{21} = 8\mathbf{1}_n$. These matrix equations are rewritten expediently in the form

$$\Xi_{11} \pm \Xi_{11}^* = \mathbf{0}_n, \quad \Xi_{22} \pm \Xi_{22}^* = \mathbf{0}_n, \quad \Xi_{21} \pm \Xi_{12}^* = 8\mathbf{1}_n \tag{127}$$

The equations obtained on the basis of the respective upper signs are a result of the identification of the real power (118) with the real part of (121), i.e., with $(\mathbf{V}^*\cdot\mathbf{I} + \mathbf{I}^*\cdot\mathbf{V})/4$. The corresponding lower-sign equations are the identification of the reactive power (119) with the imaginary part of (121), i.e., with $(\mathbf{I}^*\cdot\mathbf{V} - \mathbf{V}^*\cdot\mathbf{I})/4j$. Thus, the real power identity leads to

$$0 = \xi^{+*}\cdot\mathbf{N}_+\cdot\xi^+ - \xi^{-*}\cdot\mathbf{N}_+\cdot\xi^- \tag{128}$$

$$0 = (\xi^+\cdot\zeta^+)^*\cdot\mathbf{N}_+\cdot\xi^+\cdot\zeta^+ - (\xi^-\cdot\zeta^-)^*\cdot\mathbf{N}_+\cdot\xi^-\cdot\zeta^- \tag{129}$$

$$4\mathbf{1}_n = \xi^{+*}\cdot\mathbf{N}_+\cdot\xi^+\cdot\zeta^+ + \xi^{-*}\cdot\mathbf{N}_+\cdot\xi^-\cdot\zeta^- \tag{130}$$

and the reactive power identity to

$$0 = \xi^{+*}\cdot\mathbf{N}_+\cdot\xi^- - \xi^{-*}\cdot\mathbf{N}_+\cdot\xi^+ \tag{131}$$

$$0 = (\xi^+\cdot\zeta^+)^*\cdot\mathbf{N}_+\cdot\xi^-\cdot\zeta^- - (\xi^-\cdot\zeta^-)^*\cdot\mathbf{N}_+\cdot\xi^+\cdot\zeta^+ \tag{132}$$

$$4\mathbf{1}_n = \xi^{+*}\cdot\mathbf{N}_+\cdot\xi^-\cdot\zeta^- + \xi^{-*}\cdot\mathbf{N}_+\cdot\xi^+\cdot\zeta^+ \tag{133}$$

First the Equations (128)–(130) are analysed. By virtue of (128) the matrices $\xi^{-*}\cdot\mathbf{N}_+\cdot\xi^-$ and $\xi^{+*}\cdot\mathbf{N}_+\cdot\xi^+$ are equal. So (130) implies $\xi^{+*}\cdot\mathbf{N}_+\cdot\xi^+ = 2\underline{\zeta}^{-1}$, which also shows that $\underline{\zeta} = \underline{\zeta}^*$. With these results we can rearrange (129) into $\zeta^{+*}\cdot\underline{\zeta}^{-1}\cdot\zeta^+ = \zeta^{-*}\cdot\underline{\zeta}^{-1}\cdot\zeta^-$ and further into $\zeta^- = \zeta^{+*}$. This means that ζ^+ and ζ^- are hermitian adjoints of each other. Since the path of derivation can be reversed, the requirement $P = \Re(\mathbf{I}^*\cdot\mathbf{V})/2$ appears to be equivalent to

$$\zeta^{\pm} = \zeta^{\mp*} \tag{134}$$

$$\xi^{\pm*}\cdot\mathbf{N}_+\cdot\xi^{\pm} = 2\underline{\zeta}^{-1} \tag{135}$$

From the first equation follows that $\underline{\zeta}$ is a hermitian matrix, i.e., $\underline{\zeta} = \underline{\zeta}^*$.

In a similar way it is shown by means of (131)–(133) that the requirement $Q = \Im(\mathbf{I}^*\cdot\mathbf{V})/2$ is identical with the equations

$$\zeta^{\pm} = \zeta^{\pm*} \tag{136}$$

$$\xi^{\pm*}\cdot\mathbf{N}_+\cdot\xi^{\mp} = 2\underline{\zeta}^{-1} \tag{137}$$

The first equation states that ζ^+ and ζ^- (and so also $\underline{\zeta}$) are hermitian matrices.

Finally, if we claim the validity of (121), in other words that we can represent both the real as well as the reactive power on the basis of voltage and current values in the usual way, all Equations (134)–(137) must hold. They imply $\zeta^+ = \zeta^- = \underline{\zeta}$ and $\xi^+ = \xi^-$. Since the superscript signs of $\xi^\pm$ and $\zeta^\pm$ can be omitted in this case, (135) and (137) read

$$\mathbf{N}_+ = 2\,(\xi \cdot \zeta \cdot \xi^*)^{-1} \tag{138}$$

So far we have not assumed that the inter-reciprocity principle is expressible in the form (45) as is usually done in abstract network theory. If we add this condition, the Equations (54) and (56) must be taken into account, which are simplified in the present case (with $\zeta^+ = \zeta^-$ and $\xi^+ = \xi^-$, and because of the reciprocal nature of the waveguides):

$$\zeta = \zeta^t \tag{139}$$

$$\pm\mathbf{N}_\pm = \lambda\,\left[\xi \cdot \zeta \cdot \xi^t\right]^{-1} \tag{140}$$

The Equations (136) and (139) reveal ζ to be a real symmetric matrix. Eventually, (138) and (140) imply $\lambda\xi^* = 2\xi^t$, i.e.,

$$|\lambda| = 2, \quad \arg\lambda = 2\arg\xi_{ik} + 2\pi m \quad (m \in \mathbb{Z}) \tag{141}$$

It requires all non-zero elements of ξ to have the same argument (modulo pi), for instance $\lambda = 2$ and $\Im\xi = \mathbf{0}_n$.

Exactly this situation ($\lambda = 2$) occurs most frequently in practice where ζ and ξ are diagonal matrices, with $\zeta_{mm} > 0$ ($m = 1, \ldots, n$) being characteristic mode impedances and $\xi_{mm} = 1/\sqrt{\zeta_{mm}}$ (which entails $\mathbf{N}_+ = 2\mathbf{1}_n$ and so (120)). Because of the practical importance of this special case, the respective antenna equations and the corresponding inter-reciprocity properties are summarized:

$$\mathbf{Q}_+ \cdot \mathbf{a}_+ + \mathbf{Q}_- \cdot \mathbf{a}_- + \tilde{\mathsf{T}}_{\mathsf{f}}\left[\tilde{\mathcal{B}}_{+-}\right]\mathbf{f}^i = 0, \qquad \tilde{\mathcal{B}}_{+-} = \frac{1}{2}\begin{pmatrix}\mathbf{Q}_-^t\\ -\mathbf{Q}_+^t,\end{pmatrix}, \qquad \tilde{\mathbf{Q}}_+ \cdot \mathbf{Q}_-^t = \tilde{\mathbf{Q}}_- \cdot \mathbf{Q}_+^t \tag{142}$$

$$\mathbf{a}^- = \mathbf{S} \cdot \mathbf{a}^+ - \tilde{\mathsf{T}}_{\mathsf{f}}\left[\tilde{\mathcal{B}}_+\right]\mathbf{f}^i, \qquad \tilde{\mathcal{B}}_+ = \frac{1}{2}\begin{pmatrix}\mathbf{1}_n\\ \tilde{\mathbf{S}}\end{pmatrix}, \qquad \tilde{\mathbf{S}} = \mathbf{S}^t \tag{143}$$

$$\mathbf{Q}_{\mathbf{V}} \cdot \mathbf{V} + \mathbf{Q}_{\mathbf{I}} \cdot \mathbf{I} + \tilde{\mathsf{T}}_{\mathsf{f}}\left[\tilde{\mathcal{B}}_{\mathbf{VI}}\right]\mathbf{f}^i = 0, \qquad \tilde{\mathcal{B}}_{\mathbf{VI}} = \begin{pmatrix}-\mathbf{Q}_{\mathbf{I}}^t\\ \mathbf{Q}_{\mathbf{V}}^t\end{pmatrix}, \qquad \tilde{\mathbf{Q}}_{\mathbf{V}} \cdot \mathbf{Q}_{\mathbf{I}}^t = \tilde{\mathbf{Q}}_{\mathbf{I}} \cdot \mathbf{Q}_{\mathbf{V}}^t \tag{144}$$

$$\mathbf{V} = \mathbf{Z} \cdot \mathbf{I} - \tilde{\mathsf{T}}_{\mathsf{f}}\left[\tilde{\mathcal{B}}_{\mathbf{I}}\right]\mathbf{f}^i, \qquad \tilde{\mathcal{B}}_{\mathbf{I}} = \begin{pmatrix}\tilde{\mathbf{Z}}\\ \mathbf{1}_n\end{pmatrix}, \qquad \tilde{\mathbf{Z}} = \mathbf{Z}^t \tag{145}$$

$$\mathbf{I} = \mathbf{Y} \cdot \mathbf{V} - \tilde{\mathsf{T}}_{\mathsf{f}}\left[\tilde{\mathcal{B}}_{\mathbf{V}}\right]\mathbf{f}^i, \qquad \tilde{\mathcal{B}}_{\mathbf{V}} = -\begin{pmatrix}\mathbf{1}_n\\ \tilde{\mathbf{Y}}\end{pmatrix}, \qquad \tilde{\mathbf{Y}} = \mathbf{Y}^t \tag{146}$$

The relations (143), (145) and (146) are valid if the respective scattering, impedance and admittance matrices exist.

6.5. On the Application to Radio Astronomy Antennas aboard Spacecraft

As an example some consequences of the theory are discussed for antenna systems which consist of several monopoles mounted on a carrier. Such antennas have been used extensively as part of radio astronomy instruments aboard spacecraft [28], in particular when goniopolarimetry (direction finding and polarization analysis of radio waves) is among the main objectives [8, 22, 34]. Typical examples are the Cassini/RPWS [14, 40, 45] and STEREO/WAVES experiments [2, 8, 35]; the given references contain detailed information on the installed antennas. We consider the whole configuration (n monopoles and the spacecraft) as an antenna system with n ports, each monopole defining one terminal of a port, the other terminal of the port being given by the spacecraft ground. The coaxial cables connecting the antennas to the on-board receivers or pre-amplifiers contain no non-reciprocal materials, so the Formulas (142)–(146) apply. Due to the high conductivity of the monopoles and the spacecraft surface (many parts covered by gold foils), the whole antenna system can be regarded as reciprocal. However, if the spacecraft moves through a magnetized plasma (e.g., in regions of a planetary magnetosphere where the plasma is sufficiently dense) the environment is non-reciprocal and we have to consider the adjoint antenna system when determining the reception properties via any of the Equations (142)–(146).

Usually antennas are represented by effective length vectors in this context, which agree with the rows of the respective transfer matrix (explained in Section 5.4). As described after (82) the surrogate currents determining the transmission properties and the transfer operators of the adjoint antenna system can be chosen to be purely of electric type, which is taken for granted in the following: $\tilde{\mathbf{T}}_{\mathsf{H}} = \mathbf{0}_n$. Let $\tilde{\mathbf{h}}^{\circ}_m$ be the open port effective length vector (of the electric kind) associated with the m-th monopole of the adjoint system, i.e., $\tilde{\mathbf{T}}_{\mathsf{E}}[\tilde{\mathcal{B}}_{\mathbf{I}}] = (\tilde{\mathbf{h}}^{\circ}_1, \dots, \tilde{\mathbf{h}}^{\circ}_n)^t$. Hence, (145) shows that $-\tilde{\mathbf{h}}^{\circ}_p \cdot \mathbf{E}_0$ is the voltage induced at port p when the electromagnetic plane wave (77) is incident to the antenna system with all open ports. If the antennas are connected to receivers or any other load which can be represented by a load impedance matrix $\mathbf{Z}_L$, the port voltages change to

$$V_p = \sum_{m=1}^{n} Z_{pm} I_m - \tilde{\mathbf{h}}^{\circ}_p(-\mathbf{k}) \cdot \mathbf{E}_0 = -\sum_{m=1}^{n} Z_{\mathrm{L},pm} I_m \quad \text{with} \quad \mathbf{I} = (\mathbf{Z} + \mathbf{Z}_L)^{-1} \cdot \tilde{\mathbf{T}}_{\mathsf{E}} \left[\tilde{\mathcal{B}}_{\mathbf{I}}\right] (-\mathbf{k}) \cdot \mathbf{E}_0 \tag{147}$$

It proves that we can define a loaded port reception transfer matrix $\tilde{\mathbf{T}}_{\mathsf{E}}[\mathbf{Z}_L]$ which maps the incident plane wave amplitude to the voltage drop values appearing at the ports of the '$\mathbf{Z}_L$-loaded' antenna:

$$\mathbf{V} = -\tilde{\mathbf{T}}_{\mathsf{E}}[\mathbf{Z}_L](-\mathbf{k}) \cdot \mathbf{E}_0 \quad \text{with} \quad \tilde{\mathbf{T}}_{\mathsf{E}}[\mathbf{Z}_L] = \mathbf{Z}_L \cdot (\mathbf{Z} + \mathbf{Z}_L)^{-1} \cdot \tilde{\mathbf{T}}_{\mathsf{E}} \left[\tilde{\mathcal{B}}_{\mathbf{I}}\right] \tag{148}$$

In analogy to the open port case, we define the effective length vectors for the loaded antenna as the rows of $\tilde{\mathbf{T}}_{\mathsf{E}}[\mathbf{Z}_L]$, so they map the incident electric field amplitude on the voltage received at the respective port,

$$V_p = -\tilde{\mathbf{h}}_p \left[\mathbf{Z}_L\right](-\mathbf{k}) \cdot \mathbf{E}_0 \quad \text{where} \quad \left\{\tilde{\mathbf{h}}_p[\mathbf{Z}_L]\right\}_m = \left\{\tilde{\mathbf{T}}_{\mathsf{E}}\left[\mathbf{Z}_L\right]\right\}_{pm} \tag{149}$$

In order to describe the behaviour of the irradiated antenna as part of an electronic circuit, one has to determine $\mathbf{Z}$ for the antenna system and $\tilde{\mathbf{T}}_{\mathsf{E}}[\tilde{\mathcal{B}}_{\mathbf{I}}] = (\tilde{\mathbf{h}}^{\circ}_1, \dots, \tilde{\mathbf{h}}^{\circ}_n)^t$ for the adjoint counterpart. In practice, this can be accomplished by measurements with a scale model or by computer simulations. All other mentioned quantities can be calculated by means of the above formulas. If the external medium is a magnetized cold plasma we have the material tensors $\mu = \tilde{\mu} = \mu_0 \mathbf{1}_3$, $\tau = \tilde{\tau} = \nu = \tilde{\nu} = \mathbf{0}_3$, and the permittivity tensor $\epsilon[\mathbf{B}_0]$ depends on the static magnetic field $\mathbf{B}_0$, with $\epsilon[-\mathbf{B}_0] = (\epsilon[\mathbf{B}_0])^t$. Hence, the permittivity tensor of the adjoint magnetized plasma is obtained from the original by inverting the orientation of the static magnetic field. This fact can be utilized in the computation of the open port effective length vectors of the adjoint system. In virtue of (145), $\tilde{\mathbf{h}}^{\circ}_m$ is determined by driving unit current through the m-th port (towards monopole m, with spacecraft ground being charged oppositely) and keeping the other monopoles insulated from the spacecraft ground (no currents through ports $m' \neq m$). After having calculated a corresponding surrogate current density $\mathbf{J}^{\mathbf{o}}_m{}'(\mathbf{r})$ which radiates the same fields as the adjoint antenna under these driving conditions, the sought $\tilde{\mathbf{h}}^{\circ}_m(\mathbf{k})$ is calculated by applying (80):

$$\tilde{\mathbf{h}}^{\circ}_m(\mathbf{k}) = \int \tilde{\mathbf{J}}^{\circ}_m{}'(\mathbf{r})\, e^{j\mathbf{k}\cdot\mathbf{r}}\, dv \tag{150}$$

Here the integration volume contains the whole spacecraft (including receivers) and all monopoles.

The above procedure does not take plasma sheath effects around the antenna and spacecraft into account, which are often approximated by suitable artificial capacitances and resistances (see e.g., [2] and further references therein). For isotropic environment and quasi-static frequencies (147)–(150) are simplified to the analogous $\mathbf{k}$-independent formulas given in [24]. The anisotropy introduces an additional complication: At a given frequency there are two kinds of waves which may propagate in the same direction. They have, in general, different polarizations and phase velocities $\omega/|\mathbf{k}|$. From (150) we therefore obtain two effective length vectors $\tilde{\mathbf{h}}^{oF}_m$ and $\tilde{\mathbf{h}}^{oL}_m$ as functions of frequency ω and direction of incidence $\hat{\mathbf{k}}$, for the waves with fast and slow (superscript F and S) phase velocity, respectively.

7. CONCLUSION

The transfer operator (transfer matrix) formalism can be regarded as a comprehensive extension of the effective length vector description of antennas. Together with generalized scattering matrices, or generalized admittance and impedance matrices, it facilitates the n-port representation of non-reciprocal multi-port antennas in a non-homogeneous anisotropic environment, taking into account the impact of arbitrary external electromagnetic fields. Since the focus is on the characterization of the antenna as an n-port, with reception properties being accounted for by corresponding transmission properties of the adjoint system, no reference to the total field occurring in and around the receiving antenna is required. Hence, this approach avoids the scattering problem in 3d-space, i.e., the determination of the field change and the currents induced in the antenna when inserted in the external field. It is shown that the formalism can be kept so general that only the following preconditions need be fulfilled: the waveguides and their adjoint ones are mutually bidirectional, the antenna and the environment are composed of linear media, and the antenna system acts as an n-port of dimensionality n at transmission. Further advantages of this generality are as follows.

1. In comparison with the standard scattering, impedance and admittance matrices, their generalized versions cover a greater class of antenna structures. They allow the inclusion of circuit elements like negative resistances which may preclude the use of the respective standard matrices (sometimes only at certain frequencies [27]).

2. The use of a transfer operator for the whole system instead of one effective length vector for each individual antenna, emphasises a one-entity perspective: the antenna(s), the transmission lines and the environment are treated as part of one system. It is beneficial in the study of radiation coupling effects and cross-talk between the ports under the influence of incident electromagnetic waves [25]. The effect of the termination of the ports by loads (base capacitances at antenna feeds, receiver input impedances) has been investigated in this way for radio astronomy antennas aboard spacecraft [24].

3. Since the polarization of received waves is accounted for in the theory, it is an optimum prerequisite for the study of polarization effects in radar and radio astronomy instruments. In particular, the mathematical treatment of multi-antenna detectors on spacecraft for the observation of Stokes parameters of planetary or solar radio emissions profit.

4. The consideration of non-reciprocal configurations makes the inclusion of anisotropic media in the antenna, waveguides and environment possible. It is of special importance for magnetized plasma environments, which often occurs with antennas aboard spacecraft. The presence of a magnetic field in an ambient plasma (e.g., in planetary magnetospheres) is responsible for the anisotropy of the permittivity tensor. So a surrounding magnetized plasma is a non-reciprocal environment. The presented formalism provides the prerequisites for future applications to antennas in plasmas (it does not account for sheath effects, which require different means [2, 43]).

5. The time-consuming part in the computation of antenna reception properties by means of transfer matrices is the calculation of currents induced in the adjoint system at transmission. Hence, many different operation scenarios (variation of loads and/or external excitations by incident waves) can be studied without significant increase of computation time.

Since conventional waveguides are reflection-symmetric and reciprocal, the consequences for these specific guides have been examined in detail. It has been shown that the governing equations can be simplified considerably if voltages and currents are defined in such a way that the abstract network reciprocity $\tilde{\mathbf{I}} \cdot \mathbf{V} = \tilde{\mathbf{V}} \cdot \mathbf{I}$ maintains and the complex power exchanged via the ports is given by $\mathbf{I}^* \cdot \mathbf{V}/2$, which are common demands on voltage and current variables.

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