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The Equivalent Self-Inductance of N Coupled Parallel Coils

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Abstract—Based on Faraday's law of electromagnetic induction and the existence condition of non-trivial solution to a homogeneous and linear differential system of equations, the equivalent self-inductance of N coupled parallel coils has been derived by using some algebraic techniques. It can be expressed as the ratio of the determinants of two matrices, with ranks of N and $N-1$, respectively, and constructed with the self and mutual inductance of those coils. In addition, special conclusions are deduced and/or discussed in detail for three particular cases: 1. the completely uncoupled case, 2. the identical and symmetrical case, and 3. the completely coupled case, which are coincident with the existing results in the references.

1. INTRODUCTION

The problem of coupled coils and their application, in parallel or in series, is one of the most important subjects and permanent topics in the fields of electromagnetism and electro-technology [1–5]. In recent years, renewed attention has been aroused [4, 5]. The equivalent self-inductance of N uncoupled coils, whether for the series or the parallel case, can be calculated by using the corresponding formula for the complex impedance of alternating current. For N coupled series coils, the equivalent self-inductance can be easily calculated by the method of equivalent self-induction energy of magnetic field [6–10]. The conclusion for the $N = 2$ case can be found in some references [6–8] and can be easily generalized into $N > 2$ case. In contrast, the parallel coupled case has been paid little attention. References [6–8] have deduced or given following formula for two coupled parallel coils with no internal DC resistance:

$$L_e = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} \tag{1}$$

where L_1 and L_2 denote the self-inductance of the two coils respectively, and M denotes the mutual-inductance between the two coils. Reference [7] has also derived the equivalent decoupled circuit of two coupled parallel coils with internal DC resistance. However, it is not easy to generalize them into the $N > 2$ case by repeatedly using Formula (1) since self and mutual inductances coexist among these coils simultaneously. Starting from the fundamental Faraday's law of electromagnetic induction, using some techniques of high-order determinants of matrices in linear algebra and considering the existence condition of non-trivial solution to a homogeneous and linear differential system of equations, the equivalent self-inductance of N coupled parallel coils with no internal DC resistance is derived. It can be expressed as the ratio of determinants of two matrices of ranks N and $N-1$, respectively, which are constructed with the self-inductance and mutual inductance of these coils. The rule of signs has also been enacted to deal with the reversely-coupled cases. The concrete expressions of several particular cases, especially for completely uncoupled N coils, are deduced, which agree with the existing results in textbooks. In addition, the completely coupled parallel case and other cases have also been analyzed.

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2. A THEOREM ABOUT THE SELF-INDUCTANCE OF N COUPLED PARALLEL COILS

Since the mutual inductance between the ith and jth coils obeys the so-called Neumann relation [9,10], i.e., $M_{ij} = M_{ji}$, considering the demand of brevity of theoretical derivation and symmetry of expression, we re-label the self-inductance of the ith coil by $L_i \equiv M_{ii}$, $i = 1, 2, \ldots, N$, which means the self-inductance of a coil can be interpreted as the mutual inductance of itself. Thus we have the following theorem.

Theorem: The equivalent self-inductance of N coupled parallel coils with no internal DC resistance is the ratio of determinants of two matrices, $\tilde{\mathbf{M}}_1$ and $\tilde{\mathbf{M}}_2$, with ranks of N and $N-1$, respectively. These two matrices are constructed with their self and mutual inductance:

$$L_e = \det \tilde{\mathbf{M}}_1 / \det \tilde{\mathbf{M}}_2 \tag{2}$$

The arbitrary elements of $\tilde{\mathbf{M}}_1$ and $\tilde{\mathbf{M}}_2$ are respectively defined by

$$(\tilde{\mathbf{M}}_1)_{ij} = M_{ij}, \quad L_i \equiv M_{ii}, \quad (i, j = 1, 2, \ldots, N) \tag{3}$$

and

$$(\tilde{\mathbf{M}}_2)_{ij} = M_{11} + M_{i+1,j+1} - M_{i+1,1} - M_{1,j+1}, \quad (i, j = 1, 2, \ldots, N-1) \tag{4}$$

Proof: The $N = 3$ case is shown in Figure 1 as a reference. In the light of the properties of a parallel circuit, the total electromotive force, $\varepsilon(t)$, equals each branch electromotive force, $\varepsilon_i(t)$; and the total instantaneous current, $I(t)$, equals the sum of each branch instantaneous current, $I_i(t)$. Meanwhile, we should also note the simultaneous coexistence of self and mutual induction in the parallel coupled circuit, neglect the internal DC resistance of each coil. Finally, we define L_e to be the equivalent self-inductance of N coupled parallel coils, apply Faraday's law of electromagnetic induction and get the following equations:

$$\varepsilon(t) = \varepsilon_1(t) = \varepsilon_2(t) = \ldots = \varepsilon_N(t) \equiv -L_e \frac{dI}{dt} \tag{5}$$

$$I(t) = I_1(t) + I_2(t) + \ldots I_N(t) \tag{6}$$

$$\varepsilon_1 = -M_{11}\frac{dI_1}{dt} - M_{12}\frac{dI_2}{dt} - \ldots - M_{1N}\frac{dI_N}{dt} = -L_e\frac{dI}{dt} \tag{7a}$$

$$\varepsilon_2 = -M_{21}\frac{dI_1}{dt} - M_{22}\frac{dI_2}{dt} - \ldots - M_{2N}\frac{dI_N}{dt} = -L_e\frac{dI}{dt} \tag{7b}$$

$$\cdots$$

$$\varepsilon_N = -M_{N1}\frac{dI_1}{dt} - M_{N2}\frac{dI_2}{dt} - \ldots - M_{NN}\frac{dI_N}{dt} = -L_e\frac{dI}{dt} \tag{7c}$$

According to Equations (5) and (6), the system of Equations (7a)–(7c) has the following matrix form:

$$\begin{pmatrix} M_{11} & M_{12} & \ldots & M_{1N} \\ M_{21} & M_{22} & \ldots & M_{2N} \\ \vdots & \vdots & \ldots & \vdots \\ M_{N1} & M_{N2} & \ldots & M_{NN} \end{pmatrix} \begin{pmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \\ \vdots \\ \frac{dI_N}{dt} \end{pmatrix} = \begin{pmatrix} L_e & L_e & \ldots & L_e \\ L_e & L_e & \ldots & L_e \\ \vdots & \vdots & \ldots & \vdots \\ L_e & L_e & \ldots & L_e \end{pmatrix} \begin{pmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \\ \vdots \\ \frac{dI_N}{dt} \end{pmatrix} \tag{8a}$$

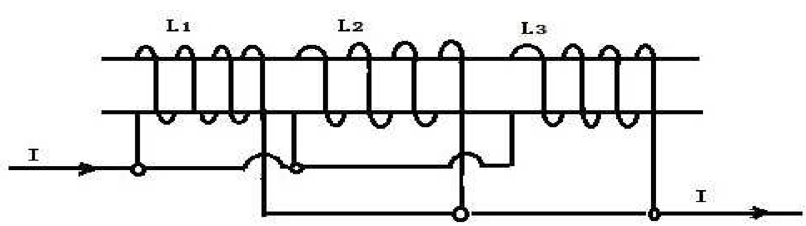

Figure 1. Three coupled parallel coils.

where the left hand matrix of (8) is the coefficient matrix $\tilde{\mathbf{M}}_1$ defined in (3). By translation, Equations (7a)–(7c) or (8a) can be rewritten as a homogeneous and linear first-order differential system of equations with N unknowns:

$$\begin{cases} (M_{11}-L_e)\frac{dI_1}{dt}+(M_{12}-L_e)\frac{dI_2}{dt}\ldots+(M_{1N}-L_e)\frac{dI_N}{dt}=0 \\ (M_{21}-L_e)\frac{dI_1}{dt}+(M_{22}-L_e)\frac{dI_2}{dt}\ldots+(M_{2N}-L_e)\frac{dI_N}{dt}=0 \\ \ldots \\ (M_{N1}-L_e)\frac{dI_1}{dt}+(M_{N2}-L_e)\frac{dI_2}{dt}\ldots+(M_{NN}-L_e)\frac{dI_N}{dt}=0 \end{cases} \tag{8b}$$

This differential system of equations has the following matrix form:

$$\begin{pmatrix} M_{11}-L_e & M_{12}-L_e & \ldots & M_{1N}-L_e \\ M_{21}-L_e & M_{22}-L_e & \ldots & M_{2N}-L_e \\ \vdots & \vdots & \ldots & \vdots \\ M_{N1}-L_e & M_{N2}-L_e & \ldots & M_{NN}-L_e \end{pmatrix} \begin{pmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \\ \vdots \\ \frac{dI_N}{dt} \end{pmatrix} = 0 \tag{8c}$$

Notice that the coefficient matrix in the system of Equation (8c) is a real and symmetrical matrix, due to Neumann's relation $M_{ij}=M_{ji}$, $i,j=1,2,\ldots,N$. If the system of Equations (8a)–(8c) have non-trivial solutions, (trivial solutions correspond to the steady DC case), for arbitrary time-varying currents such as alternating currents at any frequency, the determinant of coefficient matrix in (8c) must vanish, i.e.,

$$\begin{vmatrix} M_{11}-L_e & M_{12}-L_e & \ldots & M_{1N}-L_e \\ M_{21}-L_e & M_{22}-L_e & \ldots & M_{2N}-L_e \\ \vdots & \vdots & & \vdots \\ M_{N1}-L_e & M_{N2}-L_e & \ldots & M_{NN}-L_e \end{vmatrix} = 0 \tag{9}$$

At a first glimpse of Equation (9), it is an N-degree equation with only one unknown L_e, which seems difficult to solve. However, it is actually a first-order linear equation with only one unknown L_e. Using some manipulation techniques of a determinant, it is actually completely solvable. We make use of the properties of a determinant, that subtracting the first row from every other row, does not change the value of the determinant, i.e.,

$$\begin{vmatrix} M_{11}-L_e & M_{12}-L_e & \ldots & M_{1N}-L_e \\ M_{21}-M_{11} & M_{22}-M_{12} & \ldots & M_{2N}-M_{1N} \\ \vdots & \vdots & \ldots & \vdots \\ M_{N1}-M_{11} & M_{N2}-M_{12} & \ldots & M_{NN}-M_{1N} \end{vmatrix} = 0 \tag{10}$$

Then, using the addition property of a determinant, we can decompose the above determinant into two terms. By translating one term to the right hand side of the above equation and picking up the common factor L_e from its first row, we have:

$$\begin{vmatrix} M_{11} & M_{12} & \ldots & M_{1N} \\ M_{21}-M_{11} & M_{22}-M_{12} & \ldots & M_{2N}-M_{1N} \\ \vdots & \vdots & \ldots & \vdots \\ M_{N1}-M_{11} & M_{N2}-M_{12} & \ldots & M_{NN}-M_{1N} \end{vmatrix} = L_e \begin{vmatrix} 1 & 1 & \ldots & 1 \\ M_{21}-M_{11} & M_{22}-M_{12} & \ldots & M_{2N}-M_{1N} \\ \vdots & \vdots & \ldots & \vdots \\ M_{N1}-M_{11} & M_{N2}-M_{12} & \ldots & M_{NN}-M_{1N} \end{vmatrix} \tag{11}$$

Therefore, using the property of a determinant once more and adding the first row to every other row at the left hand of the above equation, the value of the left determinant does not change but the form changes into the $\det\tilde{\mathbf{M}}_1$ defined in Expressions (2)–(3):

$$\det\tilde{\mathbf{M}}_1 \equiv \begin{vmatrix} M_{11} & M_{12} & \ldots & M_{1N} \\ M_{21} & M_{22} & \ldots & M_{2N} \\ \vdots & \vdots & \ldots & \vdots \\ M_{N1} & M_{N2} & \ldots & M_{NN} \end{vmatrix} \tag{12}$$

For the determinant at the right hand of Equation (11), subtracting the first column from every other column, the value of the determinant does not change. Then we expand the right-hand determinant to the summation of cofactors of the first row, and get a determinant of rank $N-1$ in the following form:

$$\det\tilde{\mathbf{M}}_2=\begin{vmatrix} M_{11}+M_{22}-M_{12}-M_{21} & M_{11}+M_{23}-M_{13}-M_{21} & \dots & M_{11}+M_{2N}-M_{1N}-M_{21} \\ M_{11}+M_{32}-M_{12}-M_{31} & M_{11}+M_{33}-M_{13}-M_{31} & \dots & M_{11}+M_{3N}-M_{1N}-M_{31} \\ \vdots & \vdots & \dots & \vdots \\ M_{11}+M_{N2}-M_{12}-M_{N1} & M_{11}+M_{N3}-M_{13}-M_{N1} & \dots & M_{11}+M_{NN}-M_{1N}-M_{N1} \end{vmatrix} \quad (13)$$

The matrix $\tilde{\mathbf{M}}_2$ in the above determinant is as defined in Expression (4). Clearly, Equation (11) can be rewritten in the following form

$$\det\tilde{\mathbf{M}}_1=L_e\det\tilde{\mathbf{M}}_2 \quad (14)$$

which leads to Equation (2) when $\det\tilde{\mathbf{M}}_2\neq 0$. Our proof is thus completed. The physical meaning of the special case, $\det\tilde{\mathbf{M}}_2=0$, will be discussed in the seventh section.

3. THE RULE OF SIGNS WHEN REVERSE COUPLINGS EXIST

When Kirchhoff's laws are applied to deal with AC circuits with self and mutual induction, the rule of dot convention should be obeyed [6–8]. As shown in Figure 2, we are facing the same problem to deal with the equivalent self-inductance of the N coupled parallel coils when reverse couplings exist.

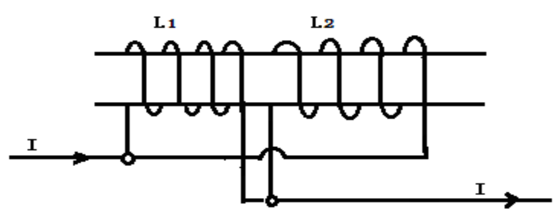

Figure 2. Reversely coupled parallel coils.

Notice that the magnetic energy of mutual induction for two parallel coils which are co-directionally coupled (reversely-coupled) is positive (negative), while the magnetic energy of self-induction for any coil is always positive, whether for co-directional coupling or for reverse coupling. Considering the above fact, we can multiply a sign factor ε_{ij} before each mutual inductance in matrices (3)–(4). The value of ε_{ij} is dependent on the relative signs of the magnetic energy of mutual induction, in contrast to those positive-definite magnetic energy of self-induction, just like in following formula of magnetic energy for coupled coils:

$$W_m=\sum_{i=1}^{n}\frac{1}{2}L_iI_i^2+\sum_{i=1,j>i}^{n}(\pm)M_{ij}I_iI_jW_m=\sum_{i=1}^{n}\frac{1}{2}\varepsilon_{ii}L_iI_i^2+\sum_{i=1,j>i}^{n}\varepsilon_{ij}M_{ij}I_iI_j \quad (15)$$

where W_m is the total magnetic energy excited by the whole current system, and I_i is the electric current of the ith coil, and $\varepsilon_{ij}=\pm1$, dependent on the sign of the magnetic energy of mutual induction. Hence we can rewrite the elements of two matrices in (3)–(4) as

$$\left(\tilde{\mathbf{M}}_1\right)_{ii}=L_i,\quad \text{or}\quad \varepsilon_{ii}=1 \quad (16)$$

$$\left(\tilde{\mathbf{M}}_1\right)_{ij}=\varepsilon_{ij}M_{ij} \quad (17)$$

$$\left(\tilde{\mathbf{M}}_2\right)_{ij}=M_{11}+\varepsilon_{i+i,j+1}M_{i+1,j+1}-\varepsilon_{i+1,1}M_{i+1,1}-\varepsilon_{1,j+1}M_{1,j+1} \quad (18)$$

for $i,j=1,2,\dots,N$. Due to Neumann's relation $M_{ij}=M_{ji}$, we always have $\varepsilon_{ij}=\varepsilon_{ji}$. Our rule of signs is actually equivalent to but slightly different from the rule of dot convention.

4. SOME CONCLUSIONS FOR THE SPECIAL CASES OF $N = 2, 3$

When $\det\tilde{\mathbf{M}}_2 \neq 0$, Equation (2) can be applied to arbitrary N coupled parallel coils. To check the equation, we can derive the equivalent self-inductances for some special cases.

When $N = 2$, corresponding to the case for two coupled parallel coils, $M_{12} = M_{21} = M$, $\det\tilde{\mathbf{M}}_1$ and $\det\tilde{\mathbf{M}}_2$ are determinants of order 2 and order 1, respectively:

$$\det\tilde{\mathbf{M}}_1 = \begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix} = M_{11}M_{22} - M_{12}M_{21} = L_1L_2 - M^2 \tag{19}$$

$$\det\tilde{\mathbf{M}}_2 = M_{11} + M_{22} - M_{12} - M_{21} = L_1 + L_2 - 2M \tag{20}$$

When $\det\tilde{\mathbf{M}}_2 \neq 0$, substituting Expressions (19)–(20) into Formula (2), we immediately get Formula (1), which is exactly the same result as found in literature [6–8]. When reverse coupling exists, as shown in Figure 2, following the rule of signs in Section 2, we find the sign factor before M should be negative, i.e., $\varepsilon_{12} = \varepsilon_{21} = -1$, thus $\det\tilde{\mathbf{M}}_1$ is invariant, but

$$\det\tilde{\mathbf{M}}_2 = L_1 + L_2 + 2M \tag{21}$$

Then we have derived the formula of the equivalent self-inductance for two reversely coupled parallel coils [8]:

$$L_e = \frac{L_1L_2 - M^2}{L_1 + L_2 + 2M} \tag{22}$$

When $N = 3$, corresponding to the case for three coupled parallel coils, we have

$$\begin{aligned} \det\tilde{\mathbf{M}}_1 &= \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix} \\ &= M_{11}M_{22}M_{33} + M_{12}M_{23}M_{31} + M_{21}M_{32}M_{13} - M_{31}M_{22}M_{13} - M_{21}M_{12}M_{33} - M_{32}M_{23}M_{11} \\ &= L_1L_2L_3 + 2M_{12}M_{23}M_{31} - L_1M_{23}^2 - L_2M_{31}^2 - L_3M_{12}^2 \end{aligned} \tag{23}$$

$$\begin{aligned} \det\tilde{\mathbf{M}}_2 &= (L_1 + L_2 - 2M_{12})(L_1 + L_3 - 2M_{13}) - (L_1 + M_{32} - M_{21} - M_{31})(L_1 + M_{23} - M_{12} - M_{13}) \\ &= (L_1 + L_2 - 2M_{12})(L_1 + L_3 - 2M_{13}) - (L_1 + M_{23} - M_{12} - M_{13})^2 \end{aligned} \tag{24}$$

Substituting Equations (23)–(24) into Formula (2), we get the formula of the equivalent self-inductance for three co-directionally coupled parallel coils, which has not been found in existing references.

5. THE EQUIVALENT SELF-INDUCTANCE OF N COMPLETELY UNCOUPLED PARALLEL COILS

When N coils are in parallel but completely uncoupled, note that every mutual inductance is zero, i.e., $M_{ij} = M_{ji} = 0$, $(i, j = 1, 2, \ldots, N$, and $i \neq j)$, $\det\tilde{\mathbf{M}}_1$ is given by

$$\det\tilde{\mathbf{M}}_1 = M_{11}M_{22}\ldots M_{NN} = L_1L_2\ldots L_N \tag{25}$$

while $\det\tilde{\mathbf{M}}_2$ satisfies following recursive relation in form

$$\begin{aligned} \det\tilde{\mathbf{M}}_2 &= \begin{vmatrix} L_1 + L_2 & L_1 & \ldots & L_1 \\ L_1 & L_1 + L_3 & \ldots & L_1 \\ \ldots & \ldots & \ldots & \ldots \\ L_1 & L_1 & \ldots & L_1 + L_N \end{vmatrix} \equiv D_{N-1}(L_1; L_2, L_3, \ldots, L_N) \\ &= L_1 \cdot L_3L_4\ldots L_N + L_2 \cdot D_{N-2}(L_1; L_3, L_4, \ldots, L_N), \end{aligned} \tag{26}$$

with $D_1(L_1; L_N) = L_1 + L_N$. Finally it gives

$$\begin{aligned} \det\tilde{\mathbf{M}}_2 &= L_2L_3\ldots L_N + L_1L_3\ldots L_N + \ldots + L_1L_2\ldots L_{N-1} \\ &= \sum_{i=1}^{N} \frac{L_1L_2L_3\ldots L_N}{L_i} = (L_1L_2L_3\ldots L_N)\sum_{i=1}^{N}\frac{1}{L_i} \end{aligned} \tag{27}$$

$$L_e = \det \tilde{\mathbf{M}}_1 / \det \tilde{\mathbf{M}}_2 = \left(\sum_{i=1}^{N} \frac{1}{L_i} \right)^{-1} \tag{28a}$$

Namely,

$$\frac{1}{L_e} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N} \tag{28b}$$

This obviously agrees with the formula of the equivalent complex impedance for N decoupled parallel coils in an alternating current circuit. Once again Equation (2) holds true.

6. THE EQUIVALENT SELF-INDUCTANCE OF *N* IDENTICAL AND SYMMETRICAL COUPLED PARALLEL COILS

In the case about N identical and symmetrical coupled parallel coils, i.e., $L_1 = L_2 = \ldots = L_N \equiv L$, $M_{ij} = M_{ji} \equiv M$, $(i, j = 1, 2, \ldots, N$, and $i \neq j)$, but $L \neq M$, some mathematical tricks are needed to get the results of the two determinants $\det \tilde{\mathbf{M}}_1$ and $\det \tilde{\mathbf{M}}_2$. They can be converted into problems of finding the general terms of two recursive sequences.

The $N \times N$ determinant, $\det \tilde{\mathbf{M}}_1 (\equiv J_N)$, satisfies following recursive relation:

$$J_n \equiv \begin{vmatrix} L & M & \ldots & M \\ M & L & \ldots & M \\ \ldots & \ldots & & \ldots \\ M & M & \ldots & L \end{vmatrix} = M(L-M)^{n-1} + (L-M) J_{n-1} \tag{29}$$

where $n = 2, 3, \ldots, N$; $J_1 = L$, $J_2 = L^2 - M^2$, which leads to

$$\det \tilde{\mathbf{M}}_1 = [L + (N-1)M](L-M)^{N-1} \tag{30}$$

The $(N-1) \times (N-1)$ determinant, $\det \tilde{\mathbf{M}}_2$, can be reduced to

$$\det \tilde{\mathbf{M}}_2 = \begin{vmatrix} 2(L-M) & L-M & \ldots & L-M \\ L-M & 2(L-M) & \ldots & L-M \\ \ldots & \ldots & & \ldots \\ L-M & L-M & \ldots & 2(L-M) \end{vmatrix} = (L-M)^{N-1} \begin{vmatrix} 2 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & 1 \\ \ldots & \ldots & & \ldots \\ 1 & 1 & \ldots & 2 \end{vmatrix} \equiv (L-M)^{N-1} K_{N-1} \tag{31}$$

where the recursion equation for K_n can be easily derived from Equation (29) with the substitution $L = 2$, $M = 1$ and $J_n = K_n$, which leads to $K_n = n + 1$. Thus $K_{N-1} = N$, and

$$\det \tilde{\mathbf{M}}_2 = N(L-M)^{N-1} \tag{32}$$

$$L_e = \det \tilde{\mathbf{M}}_1 / \det \tilde{\mathbf{M}}_2 = [L + (N-1)M]/N \tag{33}$$

For $N = 2$ case, according to Formula (33),

$$L_e = \det \tilde{M}_1 / \det \tilde{\mathbf{M}}_2 = [L + (N-1)M]/N = (L+M)/2 \tag{34}$$

This is obviously in agreement with the result calculated by using Formula (1) in an identical and symmetrical case. For instance $N = 3$, we can directly calculate the following two determinants:

$$\det \tilde{\mathbf{M}}_1 = \begin{vmatrix} L & M & M \\ M & L & M \\ M & M & L \end{vmatrix} = (L+2M)(L-M)^2 \tag{35}$$

$$\det \tilde{\mathbf{M}}_2 = 3(L-M)^2 \tag{36}$$

then using Formula (2), we have

$$L_e = \det \tilde{\mathbf{M}}_1 / \det \tilde{\mathbf{M}}_2 = (L+2M)/3 \tag{37}$$

This directly computed result agrees with the result from (33) for $N = 3$. Once again we are convinced of the validity of Equations (2) and (33).

7. DISCUSSION ON THE EQUIVALENT SELF-INDUCTANCE OF N COMPLETELY COUPLED PARALLEL COILS

References [6–8] point out that if two coils are completely coupled, their mutual inductance must be the geometric mean of their coefficients of self-inductance, i.e., $M_{12} = M_{21} = \sqrt{L_1 L_2}$; but the converse proposition is not always true. In other words, $M_{21} = M_{12} = \sqrt{L_1 L_2}$ is a necessary but not sufficient condition for two completely coupled coils [6–8]. It can be proved that, if any two coils of equal turns are completely coupled, they must have an identical self-inductance. For two coils of different turns but with the same shape, no matter whether the complete coupling between them can be experimentally realized, at least it should be recognized that theoretically the two coils of different turns can be completely coupled, although this has been the subject of much argument among scholars [10]. But once N coils are completely coupled, we can show the determinant $\det \tilde{\mathbf{M}}_1$ in the numerator of Formula (2) vanishes because any two different rows of this determinant are linearly correlated under condition $M_{ij} = M_{ji} = \sqrt{L_i L_j}$, $i, j = 1, 2, \ldots, N$, whereas the determinant $\det \tilde{\mathbf{M}}_2$ in the denominator of Formula (2) is zero for N identical and completely coupled parallel coils, and generally non-zero for other instances. This causes the equivalent self-inductance to have two different results for the completely coupled case. Concretely speaking, for the former instance, i.e., $M_{ij} = M_{ji} = \sqrt{L_i L_j}$, and $L_1 = L_2 = \ldots = L_N$, Formula (2) has an indefinite form of 0/0, but starting from the conclusion about the equivalent self-inductance of N identical and symmetrical partly-coupled parallel coils in the sixth section, we can compute the equivalent self-inductance of N completely coupled coils. Setting $L_1 = L_2 = \ldots = L_N \equiv L$, $M_{ij} = M_{ji} \equiv M$, $(i, j = 1, 2, \ldots, N)$, using Formulas (31) and (34)–(35) with $L \neq M$, removing the same factors $(L - M)^{N-1}$ from the numerator and denominator in Formula (2), and finally letting $M \to \sqrt{L_i L_j} = L$ in Formula (35), we can get the equivalent self-inductance of N completely coupled parallel coils as follows

$$L_e = L = L_1 = L_2 = \ldots = L_N \tag{38}$$

The latter instance, i.e., $\det \tilde{\mathbf{M}}_1 = 0$, but $\det \tilde{\mathbf{M}}_2 \neq 0$, due to possibility of $L_i \neq L_j$ or some other circumstances, will cause Formula (2) to give a vanishing equivalent self-inductance.

8. CONCLUSION AND PERSPECTIVE

In general circumstances, the coupled coils might be in a mixed parallel-serial circuit. For N coupled series coils, the equivalent self-inductance can be easily computed by the method of equivalent magnetic energy of self-induction. The theorem (Equation (2)) solves the problem of the equivalent self-inductance for N coupled parallel coils. Concrete expressions for several particular cases are also derived and discussed in detail, which are in accordance with the existing proven results in the references and verify the validity of Equation (2). Theorem given by this paper has laid the theoretical foundation for further dealing with coupling problems involving mixed parallel-serial N coils as well as problems of coupled coils with internal DC resistance, and it might benefit teaching and researching of electromagnetism, circuit analysis and electro-technology.

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