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# A Complete Analytical Analysis and Modeling of Few Mode Non-Uniform Fiber Bragg Grating Assisted Sensing Waveguide Devices

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**Abstract**—In this paper, we develop and present a complete analytical method to analyze the spectral response of a non-uniform multimode fiber Bragg grating assisted devices supporting a few modes. We present the analytical solution while taking into account the two forward and two backward propagating even or odd normal modes of the grating using the matrix method of multimode coupled grating assisted coupler, for sensing application. Earlier, these types of numerical technique based analysis were presented by other researchers, but no one seems to present a complete analytical solution for the given case. The present analytical analysis can simulate a single mode to multimode coupled sensing waveguide devices based on non-uniform grating assisted operation in a coupled structure. The potential applications of our findings will be mostly in sensing devices.

## 1. INTRODUCTION

Nowadays, Fiber Bragg Grating (FBG) sensors have great advantages over other types of sensors in the field of structural health monitoring. The FBG sensors are immune to EMI (Electromagnetic Interference), high voltage and current, and harsh environments as stated in [1]. A Bragg grating has great advantages and a wide range of applications in optical communication and sensing systems as stated in [2–6]. The grating-assisted coupler also has different applications as stated in [7–10]. The FBG fabrication was possible due to the discovery of photosensitivity property of optical fibers. The mechanism of FBG is the reflection at the Bragg wavelength and coupling of power from the forward-propagating mode to the backward-propagating mode. This coupling phenomenon is very important in the case of optical fiber sensors or any other types of devices [11, 12]. The widely established approach for the simulation of waveguide devices and grating assisted coupled waveguide devices is the coupled mode theory (CMT), which combines a deep insight into the problem together with a much greater computational simplicity than other numerical methods such as Finite Difference (FD), Finite Element (FE), and Beam Propagation Methods (BPM). There are a number of different coupled mode formulations which have been developed from the early 1980s until today [13–21]. There are two main representative formulations, the orthogonal CMT and rigorous non-orthogonal CMT. The first one is a simple approach, which can model devices with weakly coupled waveguides. The later can model a variety of problems such as strongly coupled waveguides involving grating assisted couplers. A more rigorous coupled mode approach uses exact composite modes such as normal modes of the waveguide structure. This approach is more accurate and can be applied equally to taper waveguide structure too. These methods have been applied previously to the modeling of strongly coupled waveguide structures with a very good agreement with the experimental results [22–24]. For the modeling and simulation of the grating assisted devices, we use the local normal mode analysis in this paper as discussed before.

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It has been proved as a very effective method for strongly coupled case [25–28]. In our formulation, the coupling in a two-mode Bragg grating structures is analyzed. Our CMT deals with the coupling between two forward- and backward-propagating normal modes. This technique can be extended to more number of modes, by taking the self-coupling and cross-coupling phenomena into consideration. The present analysis is extremely useful for analyzing the performance parameters of active coupled sensing devices. In Section 2, the coupled mode equations will be derived by a well-known perturbation theory, followed by a complete analytical model for the analysis of Bragg grating assisted devices in Section 3. Finally, the results are simulated for a few mode cases in Section 4.

## 2. ANALYSIS OF NON-UNIFORM BRAGG GRATING USING PERTURBATION TECHNIQUE

In this section, we develop and present the perturbation theory for the calculation of spectral response of a non-uniform Bragg grating. Here we consider two (even and odd) forward and two backward propagating normal modes of the grating for the special case of uniform grating as shown in Fig. 1.

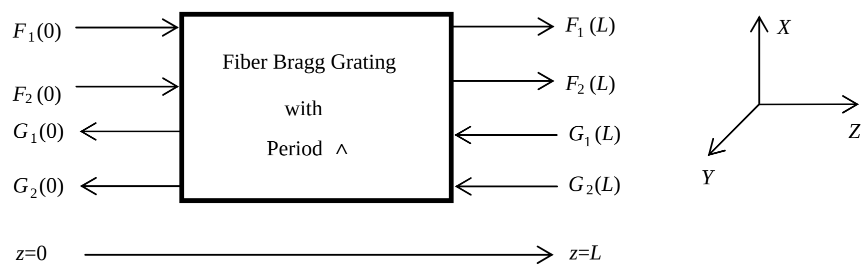


**Figure 1.** Model of FBG with two propagating modes along $z$.

In the grating region, $F_i(z)$ $(i = 1, 2)$ are the amplitudes of the two forward-propagating normal modes and $G_i(z)$ $(i = 1, 2)$ the amplitudes of the two backward-propagating normal modes.

The electric field in the grating can be expressed as:

$$E_g = \left[\left\{F_1(z)\,e^{-j\beta_{g1}z} + G_1(z)\,e^{j\beta_{g1}z}\right\}\Psi_{\beta_{g1}}(x,y) + \left\{F_2(z)\,e^{-j\beta_{g2}z} + G_2(z)\,e^{j\beta_{g2}z}\right\}\Psi_{\beta_{g2}}(x,y)\right]e^{j\omega t} \quad (1)$$

where $\Psi_{\beta_{gi}}$ are the normal modes of the refractive index-averaged waveguide, because of the grating perturbation, which correspond to the propagation constant $\beta_{gi}$ $(i = 1, 2)$. The normal modes in the above equation are normalized according to the following equation:

$$\iint_{-\infty}^{\infty} \Psi_{\beta_{gi}}(x,y)\,\Psi_{\beta_{gj}}(x,y)\,dxdy = \frac{2\omega\mu_0}{|\beta_{gi}|}\delta_{ij} \quad (\text{for } i,j = 1,2) \quad (2)$$

Here for $i = j$, $\delta_{ij} = 1$ & for $i \neq j$, $\delta_{ij} = 0$ and

$$\nabla_t^2\Psi_{\beta_{gi}} + \left[\omega^2\mu_0\varepsilon_0\varepsilon_r(x,y) - \beta_i^2\right]\Psi_{\beta_{gi}} = 0 \quad (3)$$

Here $\nabla_t^2 = \nabla^2 - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The wave equation for the electric field component is given by:

$$\nabla^2 E_g + \omega^2\mu_0\varepsilon_0\left[\varepsilon_r(x,y) + \Delta\varepsilon_r(x,y,z)\right]E_g = 0 \quad (4)$$

where $\varepsilon_r$ is the relative permittivity and $\Delta\varepsilon_r$ the perturbation of the dielectric permittivity because of the grating presence, which is periodic in the $z$-direction with zero average over a period. This perturbation can be expanded in Fourier series:

$$\Delta\varepsilon_r(x,y,z) = \sum_{m\neq 0}\Delta\varepsilon_m(x,y)e^{-jm2\frac{\pi}{\Lambda}z} \quad (5)$$

where $\Lambda$ is the period of the perturbation. Here only the fundamental harmonic $(m = 1)$ is retained.

From Eq. (1),

$$\frac{\partial^2 E_g}{\partial x^2} = \left(F_1(z)\,e^{-j\beta_{g1}z} + G_1(z)e^{j\beta_{g1}z}\right)\frac{\partial^2\Psi_{\beta_{g1}}}{\partial x^2}e^{j\omega t} + \left(F_2(z)\,e^{-j\beta_{g2}z} + G_2(z)e^{j\beta_{g2}z}\right)\frac{\partial^2\Psi_{\beta_{g2}}}{\partial x^2}e^{j\omega t} \quad (6)$$

$$\frac{\partial^2 E_g}{\partial y^2} = \left(F_1(z)\,e^{-j\beta_{g1}z} + G_1(z)e^{j\beta_{g1}z}\right)\frac{\partial^2\Psi_{\beta_{g1}}}{\partial y^2}e^{j\omega t} + \left(F_2(z)\,e^{-j\beta_{g2}z} + G_2(z)e^{j\beta_{g2}z}\right)\frac{\partial^2\Psi_{\beta_{g2}}}{\partial y^2}e^{j\omega t} \quad (7)$$

$$\begin{aligned}\frac{\partial^2 E_g}{\partial z^2} = &\left[\frac{\partial^2 F_1}{\partial z^2}e^{-j\beta_{g1}z} - 2j\beta_{g1}\frac{\partial F_1}{\partial z}e^{-j\beta_{g1}z} - \beta_{g1}^2 F_1 e^{-j\beta_{g1}z} + \frac{\partial^2 G_1}{\partial z^2}e^{j\beta_{g1}z} + 2j\beta_{g1}\frac{\partial G_1}{\partial z}e^{j\beta_{g1}z}\right.\\ &\left.-\beta_{g1}^2 G_1 e^{j\beta_{g1}z}\right]\Psi_{\beta_{g1}}e^{j\omega t} + \left[\frac{\partial^2 F_2}{\partial z^2}e^{-j\beta_{g2}z} - 2j\beta_{g2}\frac{\partial F_2}{\partial z}e^{-j\beta_{g2}z} - \beta_{g2}^2 F_2 e^{-j\beta_{g2}z} + \frac{\partial^2 G_2}{\partial z^2}e^{j\beta_{g2}z}\right.\\ &\left.+2j\beta_{g2}\frac{\partial G_2}{\partial z}e^{j\beta_{g2}z} - \beta_{g2}^2 G_2 e^{j\beta_{g2}z}\right]\Psi_{\beta_{g2}}e^{j\omega t} \quad (8)\end{aligned}$$

In the above equation, using Slowly Varying Envelope Approximation, the double derivatives of $F_i$ and $G_i$ (for $i, j = 1, 2$) is neglected. After this approximation, using the above three equations in Eq. (4) and taking Eq. (3) into consideration, we get:

$$\begin{aligned}&\left(2j\beta_{g1}\frac{\partial G_1}{\partial z}e^{j\beta_{g1}z} - 2j\beta_{g1}\frac{\partial F_1}{\partial z}e^{-j\beta_{g1}z}\right)\Psi_{\beta_{g1}} + \left(2j\beta_{g2}\frac{\partial G_2}{\partial z}e^{j\beta_{g2}z} - 2j\beta_{g2}\frac{\partial F_2}{\partial z}e^{-j\beta_{g2}z}\right)\Psi_{\beta_{g2}}\\ &+\omega^2\mu_0\varepsilon_0\Delta\varepsilon_r(x,y,z)\left[\left(F_1 e^{-j\beta_{g1}z} + G_1 e^{j\beta_{g1}z}\right)\Psi_{\beta_{g1}} + \left(F_2 e^{-j\beta_{g2}z} + G_2 e^{j\beta_{g2}z}\right)\Psi_{\beta_{g2}}\right] = 0 \quad (9)\end{aligned}$$

Multiplying $\Psi_{\beta g1}$ in Eq. (9), and integrating w.r.t. $x$, $y$:

$$\begin{aligned}&2j\beta_{g1}\left[\frac{\partial G_1}{\partial z}e^{j\beta_{g1}z} - \frac{\partial F_1}{\partial z}e^{-j\beta_{g1}z}\right]\iint \Psi_{\beta_{g1}}\Psi_{\beta_{g1}}dxdy\\ &= -\omega^2\mu_0\varepsilon_0\left[\left(F_1 e^{-j\beta_{g1}z} + G_1 e^{j\beta_{g1}z}\right)\iint \Psi_{\beta_{g1}}\Delta\varepsilon_r(x,y,z)\Psi_{\beta_{g1}}dxdy\right.\\ &\quad\left.+\left(F_2 e^{-j\beta_{g2}z} + G_2 e^{j\beta_{g2}z}\right)\iint \Psi_{\beta_{g2}}\Delta\varepsilon_r(x,y,z)\Psi_{\beta_{g1}}dxdy\right] \quad (10)\end{aligned}$$

Now using Eqs. (2) & (5), in the above equation,

$$\left[\frac{\partial G_1}{\partial z}e^{j\beta_{g1}z} - \frac{\partial F_1}{\partial z}e^{-j\beta_{g1}z}\right] = jk_{11}\left(F_1 e^{-j\left(\beta_{g1}+\frac{2\pi}{\Lambda}\right)z} + G_1 e^{j\left(\beta_{g1}-\frac{2\pi}{\Lambda}\right)z}\right) + jk_{21}\left(F_2 e^{-j\left(\beta_{g2}+\frac{2\pi}{\Lambda}\right)z} + G_2 e^{j\left(\beta_{g2}-\frac{2\pi}{\Lambda}\right)z}\right) \quad (11)$$

Similarly, multiplying $\Psi_{\beta g2}$ in Eq. (9), and integrating w.r.t. $x$, $y$:

$$\begin{aligned}&2j\beta_{g2}\left[\frac{\partial G_2}{\partial z}e^{j\beta_{g2}z} - \frac{\partial F_2}{\partial z}e^{-j\beta_{g2}z}\right]\iint \Psi_{\beta_{g2}}\Psi_{\beta_{g2}}dxdy\\ &= -\omega^2\mu_0\varepsilon_0\left[\left(F_1 e^{-j\beta_{g1}z} + G_1 e^{j\beta_{g1}z}\right)\iint \Psi_{\beta_{g1}}\Delta\varepsilon_r(x,y,z)\Psi_{\beta_{g2}}dxdy\right.\\ &\quad\left.+\left(F_2 e^{-j\beta_{g2}z} + G_2 e^{j\beta_{g2}z}\right)\iint \Psi_{\beta_{g2}}\Delta\varepsilon_r(x,y,z)\Psi_{\beta_{g2}}dxdy\right] \quad (12)\end{aligned}$$

Now using Eqs. (2) & (5), in the above equation,

$$\left[\frac{\partial G_2}{\partial z}e^{j\beta_{g2}z} - \frac{\partial F_2}{\partial z}e^{-j\beta_{g2}z}\right] = jk_{12}\left(F_1 e^{-j\left(\beta_{g1}+\frac{2\pi}{\Lambda}\right)z} + G_1 e^{j\left(\beta_{g1}-\frac{2\pi}{\Lambda}\right)z}\right) + jk_{22}\left(F_2 e^{-j\left(\beta_{g2}+\frac{2\pi}{\Lambda}\right)z} + G_2 e^{j\left(\beta_{g2}-\frac{2\pi}{\Lambda}\right)z}\right) \quad (13)$$

By multiplying $e^{j\beta_{g1}z}$, $e^{-j\beta_{g1}z}$ in Eq. (11), and $e^{j\beta_{g2}z}$, $e^{-j\beta_{g2}z}$ in Eq. (13), and substituting Eqs. (1) & (5) in Eq. (4), with some assumptions, we obtain the following set of differential equations:

$$\frac{dF_1}{dz} = -jk_{11}G_1(z)\,e^{j2\Delta\beta_1 z} - jk_{12}G_2(z)e^{j(\Delta\beta_1+\Delta\beta_2)z} \quad (13a)$$

$$\frac{dF_2}{dz} = -jk_{12}G_1(z)\,e^{j(\Delta\beta_1+\Delta\beta_2)z} - jk_{22}G_2(z)e^{j2\Delta\beta_2 z} \tag{13b}$$

$$\frac{dG_1}{dz} = jk_{11}^* F_1(z)\,e^{-j2\Delta\beta_1 z} + jk_{12}^* F_2(z)e^{-j(\Delta\beta_1+\Delta\beta_2)z} \tag{13c}$$

$$\frac{dG_2}{dz} = jk_{12}^* F_1(z)\,e^{-j(\Delta\beta_1+\Delta\beta_2)z} + jk_{22}^* F_2(z)e^{-j2\Delta\beta_2 z} \tag{13d}$$

where $\Delta\beta_i$ are the phase detuning from the Bragg condition:

$$\Delta\beta_i = \beta_{gi} - K_B = \beta_{gi} - \pi/\Lambda \tag{14}$$

And $k_{ij}$ are the coupling coefficients [16]:

$$k_{11} = \frac{\omega\varepsilon_0}{4}\iint \Delta\varepsilon_{+1}\Psi_1^2 dxdy \tag{15a}$$

$$k_{12} = k_{21} = \frac{\omega\varepsilon_0}{4}\iint \Delta\varepsilon_{+1}\Psi_1\Psi_2 dxdy \tag{15b}$$

$$k_{22} = \frac{\omega\varepsilon_0}{4}\iint \Delta\varepsilon_{+1}\Psi_2^2 dxdy \tag{15c}$$

The system of differential equations can be rewritten from Eqs. (13a)–(13d) as:

$$\frac{d}{dz}\begin{pmatrix} F \\ G \end{pmatrix} = S(z)\begin{pmatrix} F \\ G \end{pmatrix} \tag{16}$$

where $S(z)$ is the following complex $4 \times 4$ matrix

$$S(z) = \begin{pmatrix} 0 & 0 & -jk_{11}e^{j2\Delta\beta_1 z} & -jk_{12}e^{j(\Delta\beta_1+\Delta\beta_2)z} \\ 0 & 0 & -jk_{12}e^{j(\Delta\beta_1+\Delta\beta_2)z} & -jk_{22}e^{j2\Delta\beta_2 z} \\ jk_{11}^* e^{-j2\Delta\beta_1 z} & jk_{12}^* e^{-j(\Delta\beta_1+\Delta\beta_2)z} & 0 & 0 \\ jk_{12}^* e^{-j(\Delta\beta_1+\Delta\beta_2)z} & jk_{22}^* e^{-j2\Delta\beta_2 z} & 0 & 0 \end{pmatrix} \tag{17}$$

And $F$ & $G$ are vectors with components $[F_1(z),\, F_2(z)]$ and $[G_1(z),\, G_2(z)]$, respectively. i.e.,

$$F(z) = \begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix} \quad \text{and} \quad G(z) = \begin{pmatrix} G_1(z) \\ G_2(z) \end{pmatrix}$$

The solution of this system of equations which we are interested in is the matrix $P(0, L)$ such that:

$$\begin{pmatrix} F(L) \\ G(L) \end{pmatrix} = P(0,L)\begin{pmatrix} F(0) \\ G(0) \end{pmatrix} = \begin{pmatrix} P_{FF} & P_{FG} \\ P_{GF} & P_{GG} \end{pmatrix}\begin{pmatrix} F(0) \\ G(0) \end{pmatrix} \tag{18}$$

where $P_{ij}$ are $2 \times 2$ matrices [21]. The solution matrix can be found by numerical integration of the differential equation system (Eq. (16)), between 0 and $L$ for four different initial conditions such that each initial vector has one component equal to 1, and the others are zero (using Appendix A).

## 3. ANALYTICAL SOLUTION FOR THE COUPLED EQUATIONS

In the previous section, the coupled mode equations are derived by perturbation techniques. It is apparent from the nature of coupled equations that an analytical solution does not seem feasible. However, we would like to mention that the solution for single mode (even and odd) has been well established, but not found for many modes which have explicit applications in grating based sensing devices. Our main contribution in this paper is to present the above all four coupled equations in analytical form for the case of non-uniform fiber Bragg gratings. Let's proceed to the analytical solution which can be written in the form (using Appendix B as in [17])

$$P(0, L) = \exp(S_1 L)\exp(S_2 L) \tag{19}$$

and $S_1$, $S_2$ are $4 \times 4$ complex matrices, independent of $z$ (using Appendix B):

$$S_1 = \begin{pmatrix} j\Delta\beta_1 & 0 & 0 & 0 \\ 0 & j\Delta\beta_2 & 0 & 0 \\ 0 & 0 & -j\Delta\beta_1 & 0 \\ 0 & 0 & 0 & -j\Delta\beta_2 \end{pmatrix} \tag{20}$$

$$S_2 = S(0) - S_1 = \begin{pmatrix} -j\Delta\beta_1 & 0 & -jk_{11} & -jk_{12} \\ 0 & -j\Delta\beta_2 & -jk_{12} & -jk_{22} \\ jk_{11}^* & jk_{12}^* & j\Delta\beta_1 & 0 \\ jk_{12}^* & jk_{22}^* & 0 & j\Delta\beta_2 \end{pmatrix} \tag{21}$$

actually, Eq. (19) is of the form:

$$P(L_a, L_b) = \exp(S_1 \Delta L)\exp(S_2 \Delta L), \quad \Delta L = L_b - L_a \tag{22}$$

But here we have taken $P(0, L)$, that means $\Delta L = L - 0$. So the length varies from 0 to $L$. With the help of Appendix A, let $p_1(\lambda)$ be given by:

$$p_1(\lambda) = \det\{S_1 - \lambda I\} \tag{23}$$

$$\text{and} \quad p_2(\lambda) = \det\{S_2 - \lambda I\} \tag{24}$$

Then from Eq. (23),

$$p_1(\lambda) = \det \begin{bmatrix} (j\Delta\beta_1 - \lambda) & 0 & 0 & 0 \\ 0 & (j\Delta\beta_2 - \lambda) & 0 & 0 \\ 0 & 0 & (-j\Delta\beta_1 - \lambda) & 0 \\ 0 & 0 & 0 & (-j\Delta\beta_2 - \lambda) \end{bmatrix} \tag{25}$$

By evaluating the determinant defined above, the eigenvalues are given by:

$$p_1(\lambda) = \lambda^4 + \lambda^2[(\Delta\beta_1)^2 + (\Delta\beta_2)^2] + \left[(\Delta\beta_1)^2(\Delta\beta_2)^2\right] \tag{26}$$

Therefore, the polynomial coefficients are:

$$C_0 = \left[(\Delta\beta_1)^2(\Delta\beta_2)^2\right] \tag{27a}$$

$$C_1 = 0 \tag{27b}$$

$$C_2 = \left[(\Delta\beta_1)^2 + (\Delta\beta_2)^2\right] \tag{27c}$$

$$C_3 = 0 \tag{27d}$$

$$C_4 = 1 \tag{27e}$$

Now $\exp(S_1 L)$ can be written as:

$$\exp(S_1 L) = f_1(z)I + f_2(z)S_1 + f_3(z)S_1^2 + f_4(z)S_1^3 \tag{28}$$

where,

$$C_4\frac{d^4 f}{dz^4} + C_3\frac{d^3 f}{dz^3} + C_2\frac{d^2 f}{dz^2} + C_1\frac{df}{dz} + C_0 f = 0 \tag{29}$$

Here the above equation will be:

$$\begin{aligned} & \frac{d^4 f}{dz^4} + C_2\frac{d^2 f}{dz^2} + C_0 f = 0 \\ \Rightarrow\ & (D^4 + C_2 D^2 + C_0)f = 0 \\ \Rightarrow\ & D^2 = -(\Delta\beta_1)^2 \text{ or } -(\Delta\beta_2)^2 \\ \Rightarrow\ & D = \pm j\Delta\beta_1 \text{ or } \pm j\Delta\beta_2 \end{aligned} \tag{30}$$

Then:

$$f(z) = a_1\cos(\Delta\beta_1 z) + a_2\sin(\Delta\beta_1 z) + a_3\cos(\Delta\beta_2 z) + a_4\sin(\Delta\beta_2 z) \tag{31}$$

Using Appendix A, the conditions for $f_1(z)$ are:

$$a_1 + a_3 = 1, \quad a_2(\Delta\beta_1) + a_4(\Delta\beta_2) = 0, \quad -a_1(\Delta\beta_1)^2 - a_3(\Delta\beta_2)^2 = 0, \quad -a_2(\Delta\beta_1)^3 - a_4(\Delta\beta_2)^3 = 0$$

Solving these equations we get:

$$a_2 = a_4 = 0 \tag{32a}$$

$$a_3 = \frac{(\Delta\beta_1)^2}{(\Delta\beta_1)^2 - (\Delta\beta_2)^2} \tag{32b}$$

$$a_1 = \frac{-(\Delta\beta_2)^2}{(\Delta\beta_1)^2 - (\Delta\beta_2)^2} \tag{32c}$$

Hence,

$$f_1(z) = a_1 \cos(\Delta\beta_1 z) + a_3 \cos(\Delta\beta_2 z) \tag{33}$$

Conditions for $f_2(z)$ are:

$$a_1 + a_3 = 0, \quad a_2(\Delta\beta_1) + a_4(\Delta\beta_2) = 1, \quad -a_1(\Delta\beta_1)^2 - a_3(\Delta\beta_2)^2 = 0, \quad -a_2(\Delta\beta_1)^3 - a_4(\Delta\beta_2)^3 = 0$$

Solving these sets of equations we get:

$$a_1 = a_3 = 0 \tag{34a}$$

$$a_2 = \frac{-(\Delta\beta_2)^2}{(\Delta\beta_1)\left[(\Delta\beta_1)^2 - (\Delta\beta_2)^2\right]} \tag{34b}$$

$$a_4 = \frac{(\Delta\beta_1)^2}{\left[(\Delta\beta_1)^2(\Delta\beta_2) - (\Delta\beta_2)^3\right]} \tag{34c}$$

Hence,

$$f_2(z) = a_2 \sin(\Delta\beta_1 z) + a_4 \sin(\Delta\beta_2 z) \tag{35}$$

Conditions for $f_3(z)$ are:

$$a_1 + a_3 = 0, \quad a_2(\Delta\beta_1) + a_4(\Delta\beta_2) = 0, \quad -a_1(\Delta\beta_1)^2 - a_3(\Delta\beta_2)^2 = 1, \quad -a_2(\Delta\beta_1)^3 - a_4(\Delta\beta_2)^3 = 0$$

The solutions of these sets of equations are:

$$a_2 = a_4 = 0 \tag{36a}$$

$$a_3 = \frac{1}{(\Delta\beta_1)^2 - (\Delta\beta_2)^2} \tag{36b}$$

$$a_1 = \frac{-1}{(\Delta\beta_1)^2 - (\Delta\beta_2)^2} \tag{36c}$$

Hence,

$$f_3(z) = a_1 \cos(\Delta\beta_1 z) + a_3 \cos(\Delta\beta_2 z) \tag{37}$$

Similarly, conditions for $f_4(z)$ are:

$$a_1 + a_3 = 0, \quad a_2(\Delta\beta_1) + a_4(\Delta\beta_2) = 0, \quad -a_1(\Delta\beta_1)^2 - a_3(\Delta\beta_2)^2 = 0, \quad -a_2(\Delta\beta_1)^3 - a_4(\Delta\beta_2)^3 = 1$$

The solutions for these sets of equations are:

$$a_1 = a_3 = 0 \tag{38a}$$

$$a_2 = \frac{-1}{(\Delta\beta_1)\left[(\Delta\beta_1)^2 - (\Delta\beta_2)^2\right]} \tag{38b}$$

$$a_4 = \frac{1}{\left[(\Delta\beta_1)^2(\Delta\beta_2) - (\Delta\beta_2)^3\right]} \tag{38c}$$

Hence,

$$f_4(z) = a_2 \sin(\Delta\beta_1 z) + a_4 \sin(\Delta\beta_2 z) \tag{39}$$

Let

$$f_1(z) = A_1, \quad f_2(z) = B_1, \quad f_3(z) = C_1, \quad f_4(z) = D_1$$

Then from Eq. (28) we can get the solution of $\exp(S_1L)$ as:

$$\exp(S_1L) = \begin{pmatrix} \Upsilon_{11} & 0 & 0 & 0 \\ 0 & \Upsilon_{22} & 0 & 0 \\ 0 & 0 & \Upsilon_{33} & 0 \\ 0 & 0 & 0 & \Upsilon_{44} \end{pmatrix} \tag{40}$$

where

$$\Upsilon_{11} = A_1 + jB_1(\Delta\beta_1) - C_1(\Delta\beta_1)^2 - jD_1(\Delta\beta_1)^3 \tag{40a}$$
$$\Upsilon_{22} = A_1 + jB_1(\Delta\beta_2) - C_1(\Delta\beta_2)^2 - jD_1(\Delta\beta_2)^3 \tag{40b}$$
$$\Upsilon_{33} = A_1 - jB_1(\Delta\beta_1) - C_1(\Delta\beta_1)^2 + jD_1(\Delta\beta_1)^3 \tag{40c}$$
$$\Upsilon_{44} = A_1 - jB_1(\Delta\beta_2) - C_1(\Delta\beta_2)^2 + jD_1(\Delta\beta_2)^3 \tag{40d}$$

Again from Eq. (24),

$$p_2(\lambda) = \det \begin{bmatrix} (-j\Delta\beta_1 - \lambda) & 0 & -jk_{11} & -jk_{12} \\ 0 & (-j\Delta\beta_2 - \lambda) & -jk_{12} & -jk_{22} \\ jk_{11}^* & jk_{12}^* & j\Delta\beta_1 - \lambda & 0 \\ jk_{12}^* & jk_{22}^* & 0 & j\Delta\beta_2 - \lambda \end{bmatrix} \tag{41}$$

By evaluating the determinant defined above, the eigenvalues are given by:

$$p_2(\lambda) = C_4\lambda^4 + C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 \tag{42}$$

where

$$\begin{aligned} C_0 = & \Big[(\Delta\beta_1)^2 (\Delta\beta_2)^2 - 2|k_{12}|^2 \Delta\beta_1\Delta\beta_2 - |k_{22}|^2 (\Delta\beta_1)^2 - |k_{11}|^2 (\Delta\beta_2)^2 + |k_{11}|^2 |k_{22}|^2 \\ & -k_{11}k_{22}k_{12}^*k_{12}^* - k_{12}k_{12}k_{11}^*k_{22}^* + |k_{12}|^4\Big] \end{aligned} \tag{43a}$$
$$C_1 = 0 \tag{43b}$$
$$C_2 = (\Delta\beta_1)^2 + (\Delta\beta_2)^2 - 2|k_{12}|^2 - |k_{22}|^2 - |k_{11}|^2 \tag{43c}$$
$$C_3 = 0 \tag{43d}$$
$$C_4 = 1 \tag{43e}$$

Now $\exp(S_2L)$ can be written as:

$$\exp(S_2L) = f_1(z)I + f_2(z)S_2 + f_3(z)S_2^2 + f_4(z)S_2^3 \tag{44}$$

where,

$$C_4\frac{d^4f}{dz^4} + C_3\frac{d^3f}{dz^3} + C_2\frac{d^2f}{dz^2} + C_1\frac{df}{dz} + C_0f = 0 \tag{45}$$

Here the above equation can be written as:

$$\frac{d^4f}{dz^4} + C_2\frac{d^2f}{dz^2} + C_0f = 0 \quad \Rightarrow \quad (D^4 + C_2D^2 + C_0)f = 0 \quad \Rightarrow \quad D = K, L, M, N \tag{46}$$

where:

$$K = \sqrt{\frac{-C_2 + \sqrt{C_2^2 - 4C_0}}{2}}, \quad L = -\sqrt{\frac{-C_2 + \sqrt{C_2^2 - 4C_0}}{2}} = -K \tag{46a}$$

$$M = \sqrt{\frac{-C_2 - \sqrt{C_2^2 - 4C_0}}{2}}, \quad N = -\sqrt{\frac{-C_2 - \sqrt{C_2^2 - 4C_0}}{2}} = -M \tag{46b}$$

Then,

$$f(z) = a_1 \exp(Kz) + a_2 \exp(Lz) + a_3 \exp(Mz) + a_4 \exp(Nz) \tag{47}$$

Using Appendix A, the conditions for $f_1(z)$ are given by set of following equations

$$a_1 + a_2 + a_3 + a_4 = 1, \quad Ka_1 + La_2 + Ma_3 + Na_4 = 0,$$
$$K^2a_1 + L^2a_2 + M^2a_3 + N^2a_4 = 0, \quad \text{and} \quad K^3a_1 + L^3a_2 + M^3a_3 + N^3a_4 = 0$$

Again by simplifying

$$\Rightarrow \quad a_1 + a_2 + a_3 + a_4 = 1, K(a_1 - a_2) + M(a_3 - a_4) = 0,$$
$$K^2(a_1 + a_2) + M^2(a_3 + a_4) = 0, \quad \text{and} \quad K^3(a_1 - a_2) + M^3(a_3 - a_4) = 0$$

The solutions for the above set of equations are:

$$a_1 = \frac{-M^2}{2\left[K^2 - M^2\right]} = a_2 \tag{48a}$$
$$a_3 = \frac{K^2}{2\left[K^2 - M^2\right]} = a_4 \tag{48b}$$

Hence,

$$f_1(z) = a_1 \exp(Kz) + a_2 \exp(Lz) + a_3 \exp(Mz) + a_4 \exp(Nz) \tag{49}$$

Conditions for $f_2(z)$ are:

$$a_1 + a_2 + a_3 + a_4 = 0, K(a_1 - a_2) + M(a_3 - a_4) = 1,$$
$$K^2(a_1 + a_2) + M^2(a_3 + a_4) = 0, \quad \text{and} \quad K^3(a_1 - a_2) + M^3(a_3 - a_4) = 0$$

Then the solutions for the above set of equations are:

$$a_1 = \frac{-M^3}{2K\left[K^2M - M^3\right]} = -a_2 \tag{50a}$$
$$a_3 = \frac{K^2}{2\left[K^2M - M^3\right]} = -a_4 \tag{50b}$$

Hence,

$$f_2(z) = a_1 \exp(Kz) + a_2 \exp(Lz) + a_3 \exp(Mz) + a_4 \exp(Nz) \tag{51}$$

Conditions for $f_3(z)$ are:

$$a_1 + a_2 + a_3 + a_4 = 0, \quad K(a_1 - a_2) + M(a_3 - a_4) = 0$$
$$K^2(a_1 + a_2) + M^2(a_3 + a_4) = 1, \quad K^3(a_1 - a_2) + M^3(a_3 - a_4) = 0$$

The solutions are:

$$a_1 = \frac{-1}{2\left[M^2 - K^2\right]} = a_2 \tag{52a}$$
$$a_3 = \frac{1}{2\left[M^2 - K^2\right]} = a_4 \tag{52b}$$

Hence,

$$f_3(z) = a_1 \exp(Kz) + a_2 \exp(Lz) + a_3 \exp(Mz) + a_4 \exp(Nz) \tag{53}$$

Conditions for $f_4(z)$ are:

$$a_1 + a_2 + a_3 + a_4 = 0, K(a_1 - a_2) + M(a_3 - a_4) = 0,$$
$$K^2(a_1 + a_2) + M^2(a_3 + a_4) = 0 \quad \text{and} \quad K^3(a_1 - a_2) + M^3(a_3 - a_4) = 1.$$

The solutions for the above set of equations are:

$$a_1 = \frac{-M}{2K\left[M^3 - K^2M\right]} = -a_2 \tag{54a}$$

$$a_3 = \frac{1}{2\left[M^3 - K^2M\right]} = -a_4 \tag{54b}$$

Hence,

$$f_4(z) = a_1\exp(Kz) + a_2\exp(Lz) + a_3\exp(Mz) + a_4\exp(Nz) \tag{55}$$

Let

$$f_1(z) = A_2, \quad f_2(z) = B_2, \quad f_3(z) = C_2, \quad f_4(z) = D_2$$

Then from Eq. (44) we can get the solution of $\exp(S_2L)$ as:

$$\exp(S_2L) = A_2I + B_2S_2 + C_2S_2^2 + D_2S_2^3 \tag{56}$$

Now from Eq. (40) & (56), we can get the solution given in Eq. (16)

$$P(0,L) = \exp(S_1L)\exp(S_2L) = \begin{pmatrix} \Upsilon_{11} & 0 & 0 & 0 \\ 0 & \Upsilon_{22} & 0 & 0 \\ 0 & 0 & \Upsilon_{33} & 0 \\ 0 & 0 & 0 & \Upsilon_{44} \end{pmatrix} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} \end{pmatrix} \tag{57}$$

where $\Upsilon_{11}$, $\Upsilon_{22}$, $\Upsilon_{33}$, $\Upsilon_{44}$ are given by Eqs. (40a)–(40d) and

$$\Gamma_{11} = A_2 - jB_2(\Delta\beta_1) + C_2[|k_{11}|^2 + |k_{12}|^2 - (\Delta\beta_1)^2] + D_2\Big[j(\Delta\beta_1)^3 + j|k_{12}|^2(\Delta\beta_2 - 2\Delta\beta_1) - j\Delta\beta_1|k_{11}|^2\Big] \tag{57a}$$

$$\Gamma_{12} = C_2\left[k_{11}k_{12}^* + k_{12}k_{22}^*\right] + D_2\left[-jk_{12}k_{22}^*\Delta\beta_1 - jk_{11}k_{12}^*\Delta\beta_2\right] \tag{57b}$$

$$\Gamma_{13} = -jB_2k_{11} + D_2\left[jk_{11}\left\{(\Delta\beta_1)^2 - |k_{11}|^2 - 2|k_{12}|^2\right\} - jk_{22}^*k_{12}k_{12}\right] \tag{57c}$$

$$\begin{aligned}\Gamma_{14} = {} & -jB_2k_{12} + C_2\left[k_{12}\left(\Delta\beta_2 - \Delta\beta_1\right)\right] + D_2\Big[(-jk_{12})\Big\{|k_{11}|^2 + |k_{12}|^2 - (\Delta\beta_1)^2\Big\} \\ & -jk_{22}\{k_{11}k_{12}^* + k_{12}k_{22}^*\} + j\Delta\beta_2\left\{k_{12}(\Delta\beta_2 - \Delta\beta_1)\right\}\Big]\end{aligned} \tag{57d}$$

$$\Gamma_{21} = C_2\left[k_{12}k_{11}^* + k_{22}k_{12}^*\right] + D_2\left[-jk_{22}k_{12}^*\Delta\beta_1 - jk_{12}k_{11}^*\Delta\beta_2\right] \tag{57e}$$

$$\Gamma_{22} = A_2 - jB_2(\Delta\beta_2) + C_2\Big[|k_{22}|^2 + |k_{12}|^2 - (\Delta\beta_2)^2\Big] + D_2\Big[j(\Delta\beta_2)^3 + j|k_{12}|^2(\Delta\beta_1 - 2\Delta\beta_2) - j\Delta\beta_2|k_{22}|^2\Big] \tag{57f}$$

$$\begin{aligned}\Gamma_{23} = {} & -jB_2k_{12} + C_2[k_{12}(\Delta\beta_1 - \Delta\beta_2)] + D_2\Big[(-jk_{12})\Big\{|k_{22}|^2 + |k_{12}|^2 - (\Delta\beta_2)^2\Big\} \\ & -jk_{11}\left\{k_{12}k_{11}^* + k_{22}k_{12}^*\right\} + j\Delta\beta_1\left\{k_{12}\left(\Delta\beta_1 - \Delta\beta_2\right)\right\}\Big]\end{aligned} \tag{57g}$$

$$\Gamma_{24} = -jB_2k_{22} + D_2\left[jk_{22}\left\{(\Delta\beta_2)^2 - |k_{22}|^2 - 2|k_{12}|^2\right\} - jk_{11}^*k_{12}k_{12}\right] \tag{57h}$$

$$\Gamma_{31} = jB_2k_{11}^* + D_2\left[jk_{11}^*\{-(\Delta\beta_1)^2 + |k_{11}|^2 + 2|k_{12}|^2\} + jk_{12}^*k_{12}^*k_{22}\right] \tag{57i}$$

$$\begin{aligned}\Gamma_{32} = {} & jB_2k_{12}^* + C_2\left[k_{12}^*(\Delta\beta_2 - \Delta\beta_1)\right] + D_2\left[(jk_{12}^*)\left\{|k_{11}|^2 + |k_{12}|^2 - (\Delta\beta_1)^2\right\}\right. \\ & \left. +jk_{22}^*\{k_{12}k_{11}^* + k_{22}k_{12}^*\} - j\Delta\beta_2\left\{k_{12}^*(\Delta\beta_2 - \Delta\beta_1)\right\}\right]\end{aligned} \tag{57j}$$

$$\Gamma_{33} = A_2 + jB_2(\Delta\beta_1) + C_2\Big[|k_{11}|^2 + |k_{12}|^2 - (\Delta\beta_1)^2\Big] + D_2\left[-j(\Delta\beta_1)^3 + j|k_{12}|^2(2\Delta\beta_1 - \Delta\beta_2) + j\Delta\beta_1|k_{11}|^2\right] \tag{57k}$$

$$\Gamma_{34} = C_2\left[k_{12}k_{11}^* + k_{22}k_{12}^*\right] + D_2\left[jk_{22}k_{12}^*\Delta\beta_1 + jk_{12}k_{11}^*\Delta\beta_2\right] \tag{57l}$$

$$\begin{aligned}\Gamma_{41} = {} & jB_2k_{12}^* + C_2\left[k_{12}^*(\Delta\beta_1 - \Delta\beta_2)\right] + D_2\left[(jk_{12}^*)\left\{|k_{22}|^2 + |k_{12}|^2 - (\Delta\beta_2)^2\right\}\right. \\ & \left. +jk_{11}^*\left\{k_{11}k_{12}^* + k_{12}k_{22}^*\right\} - j\Delta\beta_1\left\{k_{12}^*\left(\Delta\beta_1 - \Delta\beta_2\right)\right\}\right]\end{aligned} \tag{57m}$$

$$\Gamma_{42} = jB_2k_{22}^* + D_2\left[jk_{22}^*\{-(\Delta\beta_2)^2 + |k_{22}|^2 + 2|k_{12}|^2\} + jk_{12}^*k_{12}^*k_{11}\right] \tag{57n}$$

$$\Gamma_{43} = C_2\left[k_{11}k_{12}^* + k_{12}k_{22}^*\right] + D_2\left[jk_{11}k_{12}^*\Delta\beta_2 + jk_{12}k_{22}^*\Delta\beta_1\right] \tag{57o}$$

$$\Gamma_{44} = A_2 + jB_2(\Delta\beta_2) + C_2\Big[|k_{22}|^2 + |k_{12}|^2 - (\Delta\beta_2)^2\Big] + D_2\left[-j(\Delta\beta_2)^3 + j|k_{12}|^2(2\Delta\beta_2 - \Delta\beta_1) + j\Delta\beta_2|k_{22}|^2\right] \tag{57p}$$

## 4. RESULTS OF SIMULATION

There are several types of numerical methods (Matrix method being the easiest one) for the analysis of non-uniform grating. The Matrix method has certain limitations, despite which it is widely applied to the analysis of single-mode co-directional and contra-directional coupling cases. However, we have tried to develop an analytical method based on a few theorems for similar constraints, but considering two-mode coupling effect which is a more realistic and practical model in various applications of grating assisted devices based on sensors application. To be specific about its benefit, our mathematical model gives an analytical solution for multimode grating assisted structure which has many applications in sensing devices such as the taper assisted coupler, tunable Bragg grating filters, optical add-drop filters to mention a few. After analytical formulation and finding each element of transfer matrix, we can

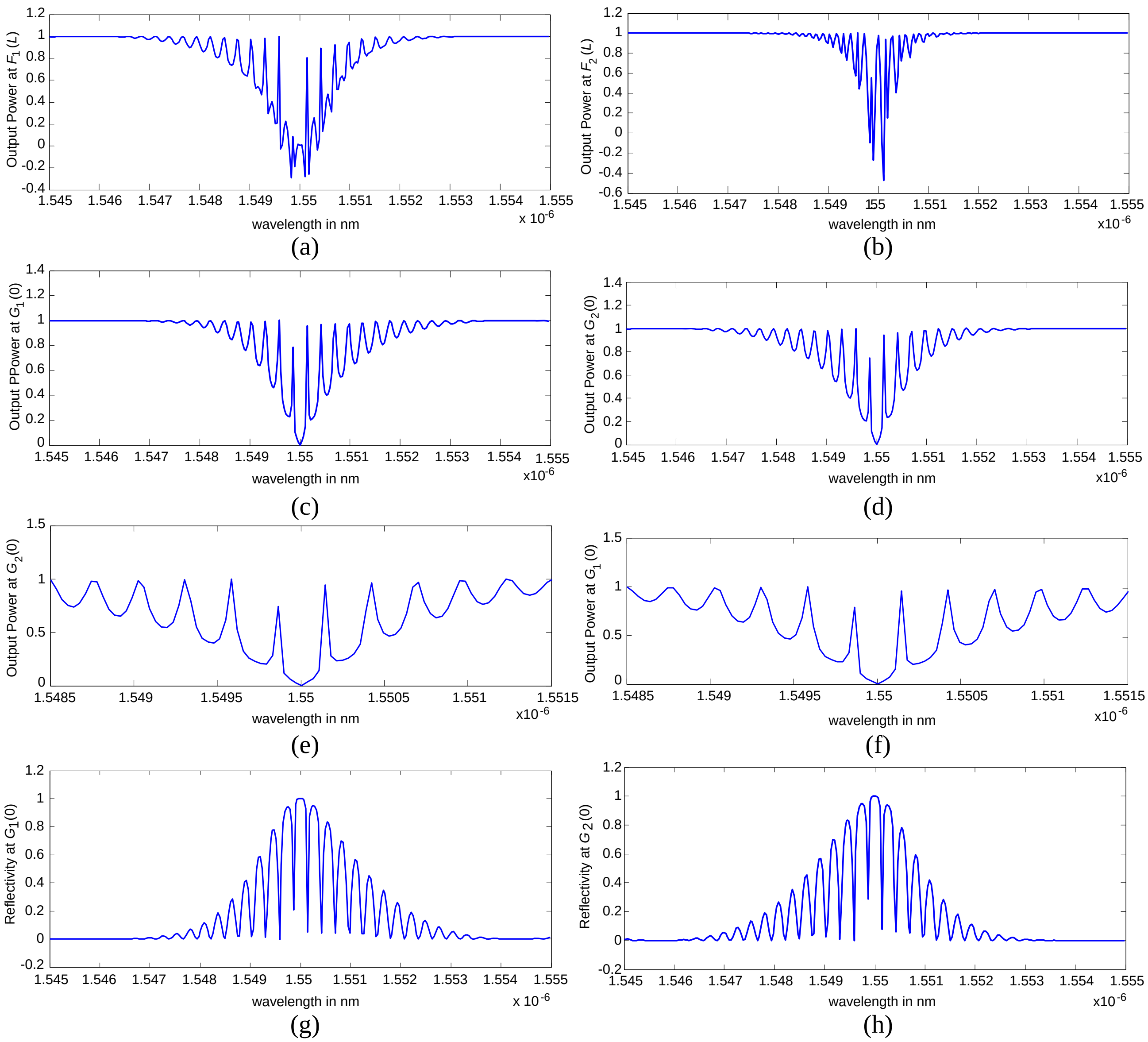


**Figure 2.** Output Power at the ports (a) $F_1(L)$, (b) $F_2(L)$, (c) $G_1(0)$, and (d) $G_2(0)$, respectively when inputs are taken as $F_1(0) = F_2(0) = G_1(L) = G_2(L) = 1$ in the CMT formulation shown in this paper. Expanded view of the output power at (e) $G_1(0)$, and (f) $G_2(0)$ is shown which is the output power due to the effect of both self-coupling and cross-coupling. Reflectivity at (g) $G_1(0)$, and (h) $G_2(0)$.

calculate the effect of cross coupling of the modes for the present case, provided that we have known coupling coefficients. For the above relation between the two forward- and backward-propagating modes, we can consider that the grating is excited from both the right and left side as in [16], and the spectral response is obtained as shown in Fig. 2. By some calculation as in [16], the relation will be as follows:

$$\begin{pmatrix} F_1(L) \\ F_2(L) \\ G_1(0) \\ G_2(0) \end{pmatrix} = \begin{pmatrix} \left(P_{FF} - P_{FG} \cdot P_{GG}^{-1} \cdot P_{GF}\right)_{2\times 2} & \left(P_{FG} \cdot P_{GG}^{-1}\right)_{2\times 2} \\ -\left(P_{GG}^{-1} \cdot P_{GF}\right)_{2\times 2} & \left(P_{GG}^{-1}\right)_{2\times 2} \end{pmatrix} \begin{pmatrix} F_1(0) \\ F_2(0) \\ G_1(L) \\ G_2(L) \end{pmatrix} \tag{58}$$

In the above simulation, $n_{eff}$ was taken as 1.455, the Bragg wavelength taken as 1550 nm, $k_{11} = k_{22} = 0.0012\,\mu\mathrm{m}^{-1}$, and $k_{12} = k_{21} = 0.002\,\mu\mathrm{m}^{-1}$. Using these parameter values, the output power was calculated for the two-mode FBG assumed in this paper. The proper derivation of self and cross-coupling coefficients can be obtained which will give perfect shape of the reflected and transmitted power. From the simulation result, it is also apparent that the spectral response improves in terms of the bandwidth reduction. As the response at the Bragg wavelength has a narrower bandwidth, it can be used as a filter with better tuning capacity. The quality factor is also increased as it has an inverse relation with the bandwidth. By varying the mode numbers, the response can be changed as per the application.

## 5. CONCLUSION

An analytical modeling method for analyzing non-uniform Bragg grating assisted devices based on coupled waveguide structures has been described. The presented analytical method is based on local normal mode analysis and is a powerful tool, which can be applied to a wide variety of problems and structures. It does not use weak coupling approximation as the traditional coupled mode theory does, thus this method is directly applicable to even strongly coupled waveguides. Also the shape of the structure is taken directly into consideration through two-mode formulation, which is important when the device is parameterized by its branch shape. In fact, our results are valid for more than one mode coupling cases. For these cases, there are many variables in coupled mode formulation. Hence for so many coupled variables, it is very tedious to derive an exact analytical expression for coupling coefficients and finally the power at each mode. Moreover, our analytical formulation can be converted into single-mode grating assisted coupler case as we remove the parameters associated with the second mode from the formulation process. Here in our analysis, the reflected and transmitted powers at the outputs differ from the ones in the single-mode FBG case, because here the effects of both self-coupling and cross-coupling coefficients are present.

## APPENDIX A.

### Theorem:

Let $A$ be a constant $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0;$$

then

$$e^{At} = x_1(t)I + x_2(t)A + x_3(t)A^2 + \ldots + x_n(t)A^{n-1},$$

where $x_k(t)$, $1 \le k \le n$, are the solutions to the $n$th order scalar differential equation

$$x^{(n)} + c_{n-1}x^{(n-1)} + \ldots + c_1x' + c_0x = 0,$$

satisfying the following initial conditions:

$$\left.\begin{array}{r} x_1(0) = 1 \\ x_1'(0) = 0 \\ \vdots \\ x_1^{(n-1)}(0) = 0 \end{array}\right\} \quad \left.\begin{array}{r} x_2(0) = 0 \\ x_2'(0) = 1 \\ \vdots \\ x_2^{(n-1)}(0) = 0 \end{array}\right\} \cdots \quad \left.\begin{array}{r} x_3(0) = 0 \\ x_3'(0) = 0 \\ \vdots \\ x_3^{(n-1)}(0) = 1 \end{array}\right\}$$

## APPENDIX B.

### Theorem:

If $S(z)$ is a matrix such that:

$$\frac{d}{dz}S(z) = S_1 S(z) - S(z) S_1 \tag{B1}$$

Then the fundamental matrix $\Upsilon(z)$ for the system of equations

$$\frac{d}{dz}x = S(z)\,x$$

is given by:

$$\Upsilon(z) = \exp(S_1 z)\exp(S_2 z) \tag{B2}$$

where

$$S_2 = S(0) - S_1$$

The fundamental matrix is such that if $x(0)$ is the vector of initial conditions at $z = 0$, the system's solution vector at $z_f$ is:

$$x(z_f) = \Upsilon(z_f)x(0) \tag{B3}$$

### Proof:

Given that differential equation is:

$$\begin{aligned} &\frac{d}{dz}x = S(z)\,x \\ \Rightarrow\quad & \ln(x) = \int S(z)dz + \ln C \text{ (where } C \text{ is a constant)} \\ \Rightarrow\quad & x(z) = C\exp\left(\int S(z)dz\right) \end{aligned} \tag{B4}$$

Then

$$x(0) = C\exp\left(\int S(z)dz|_{z=0}\right) \quad \Rightarrow \quad C = \frac{x(0)}{\exp\left(\int S(z)dz|_{z=0}\right)} \tag{B5}$$

Now Eq. (B4) will be:

$$x(z) = \frac{x(0)}{\exp\left(\int S(z)dz|_{z=0}\right)}\exp\left(\int S(z)dz\right) \tag{B6}$$

From Eqs. (B2), (B3) & (B6), we can conclude that:

$$\frac{\exp\left(\int S(z)dz\right)}{\exp\left(\int S(z)dz\Big|_{z=0}\right)} = \exp(S(0)z) \quad \Rightarrow \quad \int S(z)dz - \int S(z)dz\Big|_{z=0} = S(0)z \tag{B7}$$

In our case, $S(z)$ is a $4 \times 4$ matrix. If we take an element to prove Eq. (B7), then it can be confirmed about the theorem.

Let's take an element from our $S(z)$ matrix, i.e., $S_{13} = -jk_{11}e^{j2\Delta\beta_1 z}$.

Putting it into Eq. (B7), we will get:

$$\frac{-jk_{11}e^{j2\Delta\beta_1 z}}{j2\Delta\beta_1} + \frac{jk_{11}}{j2\Delta\beta_1} = -jk_{11}z \tag{B8}$$

**Proof:**

$$\begin{aligned}\text{LHS} &= \frac{-jk_{11}e^{j2\Delta\beta_1 z}}{j2\Delta\beta_1}+\frac{jk_{11}}{j2\Delta\beta_1}=\frac{jk_{11}(1-e^{j2\Delta\beta_1 z})}{j2\Delta\beta_1}\\ &= \frac{jk_{11}(1-1-j2\Delta\beta_1 z-\ldots)}{j2\Delta\beta_1}\quad \text{(By Series expansion of } e^x\text{)}\end{aligned}$$

Neglecting higher terms, we will get:

$$\text{LHS} = -jk_{11}z = \text{RHS (Proved)}$$

That is why here $P(0,L)$ is taken as $[\exp(S_1L)\exp(S_2L)]$ which is equal to $\exp[S(0)z]$.

In our case, matrix $S(z)$ also obeys the relation given in Eq. (B1)

**Proof:**

$$S(z)=\begin{pmatrix} 0 & 0 & -jk_{11}e^{j2\Delta\beta_1 z} & -jk_{12}e^{j(\Delta\beta_1+\Delta\beta_2)z}\\ 0 & 0 & -jk_{12}e^{j(\Delta\beta_1+\Delta\beta_2)z} & -jk_{22}e^{j2\Delta\beta_2 z}\\ jk_{11}^*e^{-j2\Delta\beta_1 z} & jk_{12}^*e^{-j(\Delta\beta_1+\Delta\beta_2)z} & 0 & 0\\ jk_{12}^*e^{-j(\Delta\beta_1+\Delta\beta_2)z} & jk_{22}^*e^{-j2\Delta\beta_2 z} & 0 & 0\end{pmatrix} \tag{B9}$$

$$S_1=\begin{pmatrix} j\Delta\beta_1 & 0 & 0 & 0\\ 0 & j\Delta\beta_2 & 0 & 0\\ 0 & 0 & -j\Delta\beta_1 & 0\\ 0 & 0 & 0 & -j\Delta\beta_2\end{pmatrix} \tag{B10}$$

$$dS(z)/dz=\begin{pmatrix} 0 & 0 & k_{11}2\Delta\beta_1 e^{j2\Delta\beta_1 z} & \begin{matrix}k_{12}\,(\Delta\beta_1{+}\Delta\beta_2)\\ e^{j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix}\\ 0 & 0 & \begin{matrix}k_{12}\,(\Delta\beta_1{+}\Delta\beta_2)\\ e^{j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & 2\Delta\beta_2 k_{22}e^{j2\Delta\beta_2 z}\\ 2\Delta\beta_1 k_{11}^*e^{-j2\Delta\beta_1 z} & \begin{matrix}k_{12}^*\,(\Delta\beta_1{+}\Delta\beta_2)\\ e^{-j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & 0 & 0\\ \begin{matrix}k_{12}^*\,(\Delta\beta_1{+}\Delta\beta_2)\\ e^{-j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & 2\Delta\beta_2 k_{22}^*e^{-j2\Delta\beta_2 z} & 0 & 0\end{pmatrix} \tag{LHS}$$

$$S_1S(z)=\begin{pmatrix} 0 & 0 & k_{11}\Delta\beta_1 e^{j2\Delta\beta_1 z} & \begin{matrix}k_{12}\,(\Delta\beta_1)\\ e^{j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix}\\ 0 & 0 & \begin{matrix}k_{12}\,(\Delta\beta_2)\\ e^{j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & \Delta\beta_2 k_{22}e^{j2\Delta\beta_2 z}\\ \Delta\beta_1 k_{11}^*e^{-j2\Delta\beta_1 z} & \begin{matrix}k_{12}^*\,(\Delta\beta_1)\\ e^{-j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & 0 & 0\\ \begin{matrix}k_{12}^*\,(\Delta\beta_2)\\ e^{-j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & \Delta\beta_2 k_{22}^*e^{-j2\Delta\beta_2 z} & 0 & 0\end{pmatrix}$$

$$S(z)S_1=\begin{pmatrix} 0 & 0 & -k_{11}\Delta\beta_1 e^{j2\Delta\beta_1 z} & \begin{matrix}-k_{12}\,(\Delta\beta_2)\\ e^{j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix}\\ 0 & 0 & \begin{matrix}-k_{12}\,(\Delta\beta_1)\\ e^{j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & -\Delta\beta_2 k_{22}e^{j2\Delta\beta_2 z}\\ \Delta\beta_1 k_{11}^*e^{-j2\Delta\beta_1 z} & \begin{matrix}-k_{12}^*\,(\Delta\beta_2)\\ e^{-j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & 0 & 0\\ \begin{matrix}-k_{12}^*\,(\Delta\beta_1)\\ e^{-j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & -\Delta\beta_2 k_{22}^*e^{-j2\Delta\beta_2 z} & 0 & 0\end{pmatrix}$$

$$\text{Now } S_1S(z)-S(z)S_1=\begin{pmatrix} 0 & 0 & k_{11}2\Delta\beta_1 e^{j2\Delta\beta_1 z} & \begin{matrix}k_{12}\,(\Delta\beta_1{+}\Delta\beta_2)\\ e^{j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix}\\ 0 & 0 & \begin{matrix}k_{12}\,(\Delta\beta_1{+}\Delta\beta_2)\\ e^{j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & 2\Delta\beta_2 k_{22}e^{j2\Delta\beta_2 z}\\ 2\Delta\beta_1 k_{11}^*e^{-j2\Delta\beta_1 z} & \begin{matrix}k_{12}^*\,(\Delta\beta_1{+}\Delta\beta_2)\\ e^{-j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & 0 & 0\\ \begin{matrix}k_{12}^*\,(\Delta\beta_1{+}\Delta\beta_2)\\ e^{-j(\Delta\beta_1+\Delta\beta_2)z}\end{matrix} & 2\Delta\beta_2 k_{22}^*e^{-j2\Delta\beta_2 z} & 0 & 0\end{pmatrix} \tag{RHS}$$

From the above matrix we can prove Eq. (B1), as LHS & RHS are equal.

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