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Design of Dual- and Quad-Band E-CRLH-TLs with Arbitrary Phase Characteristics

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Abstract—In this paper, extended composite right/left-handed (E-CRLH) transmission line (TL) metamaterial structures, with two left-handed (backward) and two right-handed (forward) pass bands, are investigated. Also, design procedures in order to design dual- and quad-band E-CRLH-TLs are presented in detail and the parameters of these structures are extracted by clean formulas, while satisfying arbitrary phase shifts at the operating frequencies. Finally, the dispersion and characteristic impedances of these transmission lines are derived and plotted. The results of this paper can be applied to any type of TL-based dual- and quad-band microwave component.

1. INTRODUCTION

Extended Composite Right-Left Handed (E-CRLH) structures were first introduced in 2006 [1]. These structures are also known as Generalized Negative Refraction Index Transmission Line (G-NRL-TL) [2, 3]. E-CRLH-TLs are metamaterial-based transmission lines, which are, in fact, a hybrid of well-known Convenient Composite Right-Left Handed (C-CRLH) and Dual Composite Right-Left Handed (D-CRLH) structures [1]. These structures have recently found numerous applications in designing dual- and quad-band devices [4–11]. In [4], a quad-band Wilkinson power divider is proposed using E-CRLH structures, having narrow (low) bandwidth at each of its (four) frequency bands. The same structure was also used in designing a dual-band Wilkinson power divider [5] while the results in both frequency bands were satisfactory. Moreover, up to now, diverse applications and topologies of the E-CRLHs have been reported. In [6] the proposed E-CRLH structures without any lumped-element components have been used to design a quad-band filter. Also in [7], the same topology has been used to design a Leaky Wave Antenna (LWA) antenna capable with two frequency bands. In addition to these, another type of E-CRLH structure has been used to design a dual-band coupled-line coupler [8]. In [9, 10], diverse dual-band *passive* devices such as power dividers, hybrid couplers and bandpass filters, are designed without using any lumped elements and in a complete planar topology. Recently, application of these structures to active devices are also reported. For example, the power amplifier in [11] is an *E-CRLH-based* dual-band Class-E power amplifier.

Even though, in recent years, the application of the E-CRLH structures has been reported in the literature, a proper method for deriving the parameters of these structures, specifically to have zero phase shifts at four arbitrary frequencies, is still missing. In this paper, we investigate the E-CRLH-TL structures and present a novel procedure for designing dual- and quad-band transmission lines. Also, the parameters of these structures are extracted in detail. In addition, the dispersion and characteristic impedance of the E-CRLH-TLs are derived and sketched.

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2. E-CRLH STRUCTURES

2.1. Theory of E-CRLH Structures

Figure 1(a) depicts the circuit model of the unit cell of the E-CRLH structure [1] where the unit cell consists of four inductors L_R^c, L_R^d, L_L^c and L_L^d and four capacitors C_R^c, C_R^d, C_L^c and C_L^d. In this figure, inductors and capacitors related to C-CRLH structure and D-CRLH structure are denoted with superscript c and d, respectively. Also, Figure 1(b) displays the T-shaped format of the E-CRLH unit cell. It should be emphasized that the E-CRLH transmission lines can be implemented in different configurations. For instance, as discussed in [6, 7], these structures can be designed in fully-printed configurations. Furthermore, as shown in Figure 1(c), the T-shaped E-CRLH-TLs can be constructed with lumped-element components and microstrip technology [12, 13]. In this figure, some of the inductors and capacitors of the E-CRLH structure are implemented as lumped elements while others as distributed ones. For example, the capacitor C_R^d is modeled with a rectangular shaped planar surface (with dimension of $L_{cap} \times W_{cap}$) which together with the ground plane exhibits some capacitive properties. In addition, the inductor L_L^d is implemented using a stripline (with dimension of $L_{stub1} \times W_{stub1}$) connected to the rectangular shaped plane. In addition, a stripline (with dimension of $L_{stub2} \times W_{stub2}$), which is connected to the ground plane using a *Via*, models the inductor L_L^c.

Furthermore, as explained in [14], the propagation constant, β_{TL}, and the characteristic impedance, $Z_{0,TL}$, of a lossless transmission line considered here, can be written as

$$\beta_{TL} = -j\sqrt{Z_e Y_e} \quad Z_{0,TL} = \sqrt{\frac{Z_e}{Y_e}} \tag{1}$$

where $Z_e = Z_c + Z_d$ and $Y_e = Y_c + Y_d$ are the series impedance and parallel admittance of the transmission line's unit cell, respectively.

According to Figure 1, Z_c, Z_d, Y_c and Y_d are given by

$$Z_c = j\omega L_R^c \left[1 - \left(\frac{\omega_{se}^c}{\omega}\right)^2\right], \quad \omega_{se}^c \triangleq \frac{1}{\sqrt{L_R^c C_L^c}} \tag{2a}$$

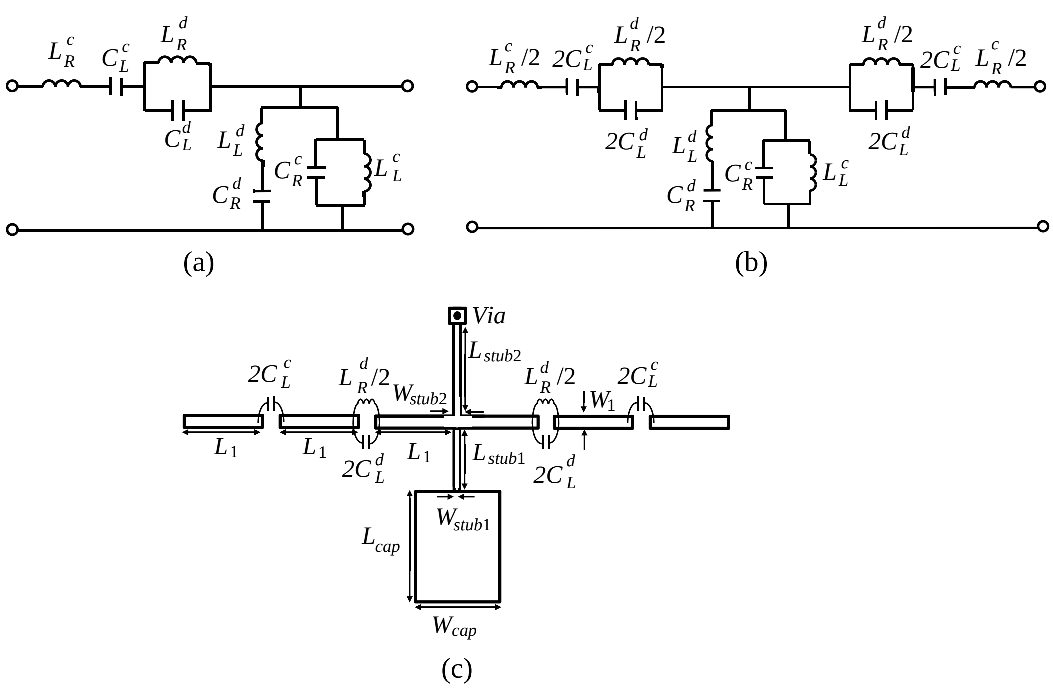


Figure 1. (a) Circuit model of the unit cell of the E-CRLH structure [1]. (b) T-shaped format of the E-CRLH-TLs. (c) Implementation of the E-CRLH-TLs [12].

$$Z_d = \frac{j\omega L_R^d}{1-\left(\dfrac{\omega}{\omega_{se}^d}\right)^2}, \quad \omega_{se}^d \triangleq \frac{1}{\sqrt{L_R^d C_L^d}} \tag{2b}$$

$$Y_c = j\omega C_R^c\left[1-\left(\frac{\omega_{sh}^c}{\omega}\right)^2\right], \quad \omega_{sh}^c \triangleq \frac{1}{\sqrt{L_L^c C_R^c}} \tag{2c}$$

$$Y_d = \frac{j\omega C_R^d}{1-\left(\dfrac{\omega}{\omega_{sh}^d}\right)^2}, \quad \omega_{sh}^d \triangleq \frac{1}{\sqrt{L_L^d C_R^d}} \tag{2d}$$

where ω is the operating angular frequency. Hence, for the unit cell of Figure 1(a), the propagation constant can be re-written as

$$\beta_{\text{E-CRLH}} = -j\sqrt{Z_e Y_e} = -j\sqrt{Z_c Y_c + Z_c Y_d + Z_d Y_c + Z_d Y_d}. \tag{3}$$

After mathematical manipulation, we obtain the propagation constant and characteristic impedance of the E-CRLH structure as Eq. (4) and Eq. (5), respectively [1].

$$\beta_{\text{E-CRLH}} = \frac{\sqrt{L_R^c C_R^c}}{\omega}\sqrt{\frac{\left(\omega^2-\omega_{se}^{c2}\right)\left(\omega^2-\omega_{se}^{d2}\right)-\dfrac{1}{L_R^c C_L^d}\omega^2}{\left(\omega^2-\omega_{se}^{d2}\right)}\cdot\frac{\left(\omega^2-\omega_{sh}^{c2}\right)\left(\omega^2-\omega_{sh}^{d2}\right)-\dfrac{1}{L_L^d C_R^c}\omega^2}{\left(\omega^2-\omega_{sh}^{d2}\right)}} \tag{4}$$

$$Z_{0,\text{E-CRLH}} = \sqrt{\frac{L_e}{C_e}} = Z_0\sqrt{\frac{1-\left(\omega_{se}^c/\omega\right)^2+\dfrac{L_R^d/L_R^c}{1-\left(\omega/\omega_{se}^d\right)^2}}{1-\left(\omega_{sh}^c/\omega\right)^2+\dfrac{C_R^d/C_R^c}{1-\left(\omega/\omega_{sh}^d\right)^2}}}, \quad Z_0 \triangleq \sqrt{\frac{L_R^c}{C_R^c}} \tag{5}$$

In addition, Eq. (4) suggests that, at four arbitrary frequencies, the phase shift of the E-CRLH-TL can be equal to 0°. The following lemma proves this intuition.

Lemma 1. *Let $P_2(x)$ be a quadratic polynomial. Any $P_2(x)$ of the form*

$$P_2(x) = x^2 - (a+b+c)x + ab$$

where a, b and c are three given positive constants, has two distinct positive (hence real) roots.

Proof. Refer to the appendix. □

In fact, the numerator of the square-rooted term, is the product of two quadratic polynomials in terms of ω^2. So, according to lemma 1, these two expressions can be written as

$$\left(\omega^2\right)^2-\left(\omega_{se}^{c2}+\omega_{se}^{d2}+\frac{1}{L_R^c C_L^d}\right)\omega^2+\omega_{se}^{c2}\omega_{se}^{d2} = \left(\omega^2-\omega_{1,0}^2\right)\left(\omega^2-\omega_{3,0}^2\right) \tag{6a}$$

$$\left(\omega^2\right)^2-\left(\omega_{sh}^{c2}+\omega_{sh}^{d2}+\frac{1}{L_L^d C_R^c}\right)\omega^2+\omega_{sh}^{c2}\omega_{sh}^{d2} = \left(\omega^2-\omega_{2,0}^2\right)\left(\omega^2-\omega_{4,0}^2\right) \tag{6b}$$

where $\omega_{1,0}$, $\omega_{2,0}$, $\omega_{3,0}$ and $\omega_{4,0}$ are the corresponding angular frequencies of zero phase shift in Figure 2(a). The subscript 0, in the $\omega_{i,0}$'s, emphasizes the fact that the phase shift, exhibited by the E-CRLH structure at these frequencies, is zero. According to these equations, the calculated propagation constant in Eq. (4) can be simplified to

$$\beta_{\text{E-CRLH}} = \frac{\sqrt{L_R^c C_R^c}}{\omega}\sqrt{\frac{\left(\omega^2-\omega_{1,0}^2\right)\left(\omega^2-\omega_{2,0}^2\right)\left(\omega^2-\omega_{3,0}^2\right)\left(\omega^2-\omega_{4,0}^2\right)}{\left(\omega^2-\omega_{se}^{d2}\right)\left(\omega^2-\omega_{sh}^{d2}\right)}}. \tag{7}$$

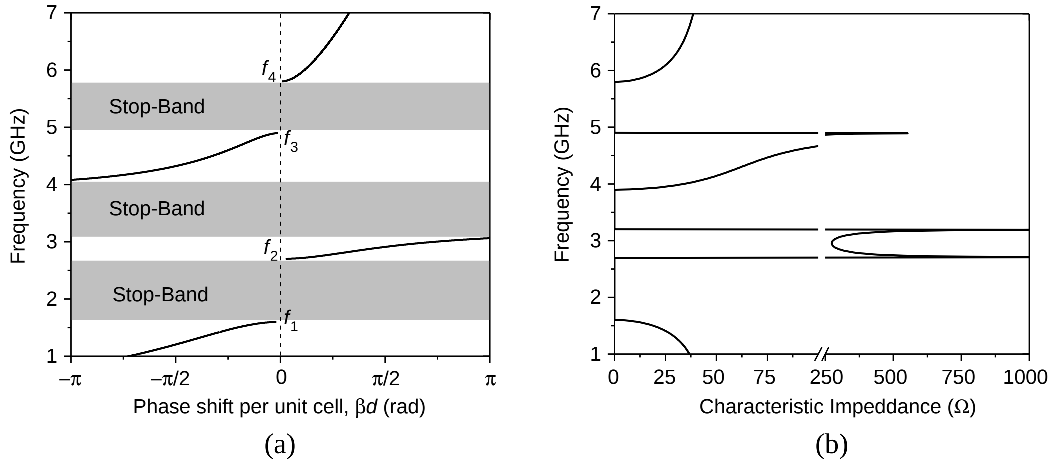


Figure 2. (a) Dispersion and (b) Characteristic impedance of the E-CRLH structure with $f_{1,0} = 1.60\,\text{GHz}$, $f_{2,0} = 2.70\,\text{GHz}$, $f_{3,0} = 4.90\,\text{GHz}$, $f_{4,0} = 5.80\,\text{GHz}$, $f_R^c = 3.3\,\text{GHz}$, $f_{se}^d = 3.2\,\text{GHz}$, $f_{sh}^d = 3.9\,\text{GHz}$, $Z_0 = 50\,\Omega$, $L_R^c = 2.41\,\text{nH}$, $C_L^c = 1.25\,\text{pF}$, $L_R^d = 4.13\,\text{nH}$, $C_L^d = 0.60\,\text{pF}$, $L_L^c = 2.28\,\text{nH}$, $C_R^c = 0.96\,\text{pF}$, $L_L^d = 5.73\,\text{nH}$, $C_R^d = 0.30\,\text{pF}$.

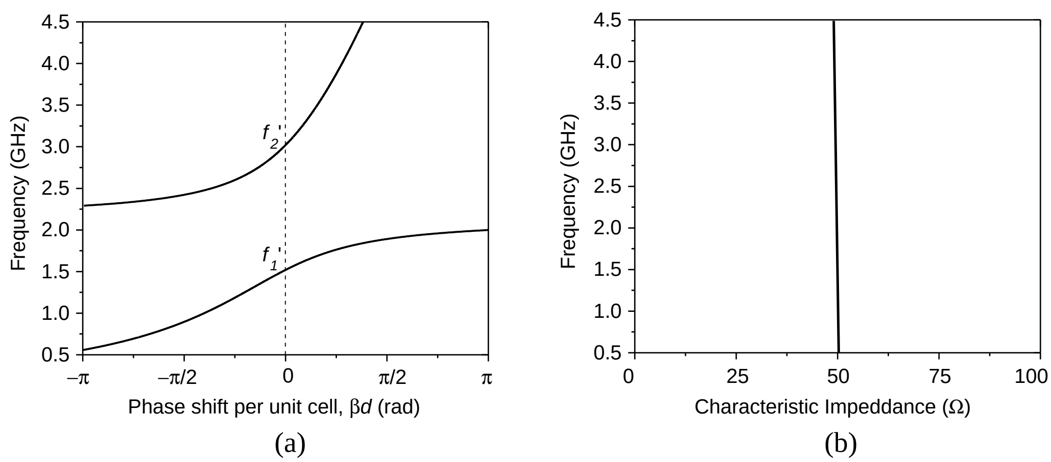


Figure 3. (a) Dispersion and (b) Characteristic impedance of the balanced E-CRLH structure with $f_{1,0} = f_{2,0} = 1.52\,\text{GHz}$, $f_{3,0} = f_{4,0} = 3.02\,\text{GHz}$, $f_R^c = 2.36\,\text{GHz}$, $f_{se}^d = 2.14\,\text{GHz}$, $f_{sh}^d = 2.14\,\text{GHz}$, $Z_0 = 50\,\Omega$, $L_R^c = 3.37\,\text{nH}$, $C_L^c = 1.63\,\text{pF}$, $L_R^d = 1.65\,\text{nH}$, $C_L^d = 3.35\,\text{pF}$, $L_L^c = 4.08\,\text{nH}$, $C_R^c = 1.34\,\text{pF}$, $L_L^d = 8.34\,\text{nH}$, $C_R^d = 0.66\,\text{pF}$.

Figure 2 depicts the dispersion and the characteristic impedance of an E-CRLH structure. It is clear from this figure, that the wave can not propagate in (at most) three frequency bands. These frequency bands define the cutoff frequency bands of the E-CRLH structure. Also, it is clear that the dispersion diagram of E-CRLH-TL has two left-handed bands and two right-handed bands.

On the other hand, E-CRLH structures can be excited in the *balanced* mode. The balanced conditions for these structures are defined as [1]

$$\omega_{se}^c = \omega_{sh}^c \triangleq \omega_0^c \quad \& \quad \omega_{se}^d = \omega_{sh}^d \triangleq \omega_0^d \quad \& \quad \omega_0^c = \omega_0^d \triangleq \omega_0 \tag{8}$$

In the balanced mode, the first and third cutoff frequency bands vanish (Figure 2). Applying the above conditions to Eqs. (5) and (7), we obtain the propagation constant and characteristic impedance of the balanced E-CRLH structure as

$$\beta_{\text{E-CRLH}} = \frac{\sqrt{L_R^c C_R^c}}{\omega} \cdot \frac{\left(\omega^2 - \omega_{1,0}'^2\right)\left(\omega^2 - \omega_{2,0}'^2\right)}{\left(\omega^2 - \omega_0^2\right)}, \quad \omega_{1,0}' \triangleq \omega_{1,0} = \omega_{2,0} \quad \omega_{2,0}' \triangleq \omega_{3,0} = \omega_{4,0} \tag{9}$$

$$Z_{0,\text{E-CRLH}} \equiv Z_0 = \sqrt{\frac{L_R^c}{C_R^c}} = \sqrt{\frac{L_L^c}{C_L^c}} = \sqrt{\frac{L_R^d}{C_R^d}} = \sqrt{\frac{L_L^d}{C_L^d}}. \tag{10}$$

Figure 3 illustrates the dispersion and the characteristic impedance of the E-CRLH structure in the balanced mode. Similar to Figure 2(a), it can be seen that, even in the balanced mode, the dispersion diagram of E-CRLH-TL has two left-handed bands and two right-handed bands. It should be emphasized that in the balanced mode, there are only two frequencies that provide zero phase shift. The result of Figure 3, backs up the systematic design procedure for our novel methodology to obtain zero phase shift at *two* arbitrary frequencies using a proper systematic method; it also paves the way, as we will see in the next section, for extending our methodology to obtain zero phase shift in the E-CRLH structures, at *four* arbitrary frequencies.

2.2. Design of an E-CRLH Structure with Zero Phase Shifts at Four Arbitrary Frequencies

Following the equations derived in the previous section, in order to design an E-CRLH structure with zero phase shift at four arbitrary, frequencies, $f_{1,0}$, $f_{2,0}$, $f_{3,0}$ and $f_{4,0}$, we need to set Eq. (2) to zero. Therefore, the values of the lumped elements can be evaluated using Eq. (2) and can be written as

$$L_R^c = \frac{Z_0}{\omega_R^c} \tag{11a}$$

$$L_R^d = \frac{K_1 Z_0}{\omega_R^c \left(\omega_{se}^d\right)^2} \tag{11b}$$

$$L_L^c = \frac{Z_0 \omega_R^c}{\left(\omega_{sh}^c\right)^2} \tag{11c}$$

$$L_L^d = \frac{Z_0 \omega_R^c}{K_2} \tag{11d}$$

$$C_R^c = \frac{1}{Z_0 \omega_R^c} \tag{11e}$$

$$C_R^d = \frac{K_2}{Z_0 \omega_R^c \left(\omega_{sh}^d\right)^2} \tag{11f}$$

$$C_L^c = \frac{\omega_R^c}{Z_0 \left(\omega_{se}^c\right)^2} \tag{11g}$$

$$C_L^d = \frac{\omega_R^c}{K_1 Z_0} \tag{11h}$$

where ω_{se}^c, ω_{sh}^c, K_1 and K_2 are given as Eq. (12) and ω_{se}^d, ω_{sh}^d, ω_R^c and Z_0 are the known *design* parameters. Parameters ω_{se}^d, ω_{sh}^d and ω_R^c are notion of bandwidth in the operating frequencies of the E-CRLH structure.

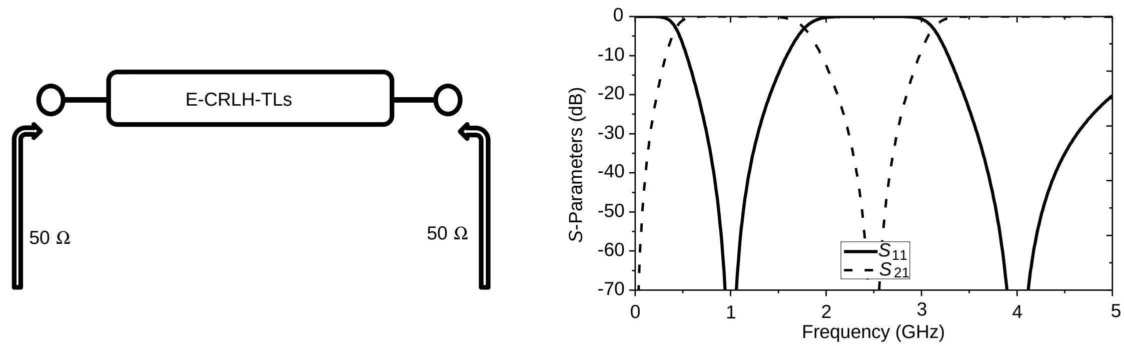
$$\omega_{sh}^c = \frac{\omega_{2,0}\omega_{4,0}}{\omega_{sh}^d} \tag{12a}$$

$$\omega_{se}^c = \frac{\omega_{1,0}\omega_{3,0}}{\omega_{se}^d} \tag{12b}$$

$$K_1 \triangleq \frac{1}{L_R^c C_L^d} = \omega_{1,0}^2 + \omega_{3,0}^2 - \omega_{se}^{c^2} - \omega_{se}^{d^2} = \omega_{se}^{d^2} \left[\left(\frac{\omega_{se}^c}{\omega_{1,0}\omega_{3,0}} \right)^2 \left(\omega_{1,0}^2 + \omega_{3,0}^2 - \omega_{se}^{c^2}\right) - 1 \right] \tag{12c}$$

$$K_2 \triangleq \frac{1}{L_L^d C_R^c} = \omega_{2,0}^2 + \omega_{4,0}^2 - \omega_{sh}^{c^2} - \omega_{sh}^{d^2} = \omega_{sh}^{d^2} \left[\left(\frac{\omega_{sh}^c}{\omega_{2,0}\omega_{4,0}} \right)^2 \left(\omega_{2,0}^2 + \omega_{4,0}^2 - \omega_{sh}^{c^2}\right) - 1 \right] \tag{12d}$$

As an example of application, a dual-band E-CRLH filter is presented in Figure 4(a). This filter is designed in order to work at $f_1 = 1\,\text{GHz}$ and $f_2 = 4\,\text{GHz}$. Also, a demonstration of the principle of this paper is depicted in Figure 4(b) by plotting the scattering parameters of the two-port filters.



(a) (b)

Figure 4. (a) Dispersion and (b) Characteristic impedance of the balanced E-CRLH structure with $f_{1,0} = f_{2,0} = 1\,\text{GHz}$, $f_{3,0} = f_{4,0} = 4\,\text{GHz}$, $f_R^c = 2.5\,\text{GHz}$, $f_{se}^d = 2.5\,\text{GHz}$, $f_{sh}^d = 2.5\,\text{GHz}$, $Z_0 = 50\,\Omega$, $L_R^c = 3.18\,\text{nH}$, $C_L^c = 3.10\,\text{pF}$, $L_R^d = 4.17\,\text{nH}$, $C_L^d = 0.97\,\text{pF}$, $L_L^c = 7.77\,\text{nH}$, $C_R^c = 1.27\,\text{pF}$, $L_L^d = 2.42\,\text{nH}$, $C_R^d = 1.67\,\text{pF}$.

2.3. Design of an E-CRLH Structure with Arbitrary Phase Shifts at Four Arbitrary Frequencies

In this section, we show how the E-CRLH structures can be used to exhibit four arbitrary phase shifts, ϕ_1, ϕ_2, ϕ_3 and ϕ_4, at four arbitrary frequencies, f_1, f_2, f_3 and f_4, respectively. (To avoid complexity, only the balanced E-CRLH structures are considered in this section while applying the proposed analysis to the general case is straightforward.) As the length of the E-CRLH unit cell is fixed, the four aforementioned phase shifts directly translate into four propagation constants, β_1, β_2, β_3 and β_4, respectively. Hence, according to Eq. (9), we can write

$$\frac{\left(\omega_i^2 - A^2\right)\left(\omega_i^2 - B^2\right)}{D\omega_i\left(\omega_i^2 - C^2\right)} = \beta_i; \quad i = 1, 2, 3, 4. \tag{13}$$

where A, B, C and D are positive parameters and can be computed from

$$A \triangleq \omega'_{1,0} \quad \& \quad B \triangleq \omega'_{2,0} \quad \& \quad C \triangleq \omega_0 \quad \& \quad D \triangleq \omega_R^c. \tag{14}$$

In what follows, we try to express A, B, C and D in terms of ω_i and β_i $i = 1, 2, 3, 4$. Using Eq. (13), the parameter D can be written in terms of A, B and C, in the following forms

$$D = \frac{\left(\omega_1^2 - A^2\right)\left(\omega_1^2 - B^2\right)}{\beta_1\omega_1\left(\omega_1^2 - C^2\right)} \tag{15a}$$

$$D = \frac{\left(\omega_2^2 - A^2\right)\left(\omega_2^2 - B^2\right)}{\beta_2\omega_2\left(\omega_2^2 - C^2\right)}. \tag{15b}$$

Next, dividing Eq. (15a) by Eq. (15b) leads to

$$\frac{\omega_1\left(\omega_2^2 - A^2\right)\left(\omega_2^2 - B^2\right)\left(\omega_1^2 - C^2\right)}{\omega_2\left(\omega_1^2 - A^2\right)\left(\omega_1^2 - B^2\right)\left(\omega_2^2 - C^2\right)} = \frac{\beta_2}{\beta_1}. \tag{16}$$

Therefore, by manipulating Eq. (16), the value of C can be written as

$$C^2 = \frac{\beta_2\omega_2^3\left(\omega_1^2 - A^2\right)\left(\omega_1^2 - B^2\right) - \beta_1\omega_1^3\left(\omega_2^2 - A^2\right)\left(\omega_2^2 - B^2\right)}{\beta_2\omega_2\left(\omega_1^2 - A^2\right)\left(\omega_1^2 - B^2\right) - \beta_1\omega_1\left(\omega_2^2 - A^2\right)\left(\omega_2^2 - B^2\right)}. \tag{17}$$

Now, by using Eq. (13) and Eq. (17), the parameter D can be expressed as

$$D = \frac{\beta_2\omega_2\left(\omega_1^2 - A^2\right)\left(\omega_1^2 - B^2\right) - \beta_1\omega_1\left(\omega_2^2 - A^2\right)\left(\omega_2^2 - B^2\right)}{\beta_1\beta_2\omega_1\omega_2\left(\omega_1^2 - \omega_2^2\right)}. \tag{18}$$

On the other hand, substituting Eq. (17) and Eq. (18) into Eq. (13), with respect to $i = 3$, we can write

$$\frac{\alpha_1\left(\omega_3^2 - A^2\right)\left(\omega_3^2 - B^2\right)}{\alpha_2\left(\omega_1^2 - A^2\right)\left(\omega_1^2 - B^2\right) - \alpha_3\left(\omega_2^2 - A^2\right)\left(\omega_2^2 - B^2\right)} = 1. \tag{19}$$

where, α_1, α_2 and α_3 are defined as

$$\alpha_1 \triangleq \frac{\beta_1\beta_2\omega_1\omega_2}{\beta_3\omega_3}(\omega_1^2 - \omega_2^2) \tag{20a}$$

$$\alpha_2 \triangleq \beta_2\omega_2(\omega_3^2 - \omega_2^2) \tag{20b}$$

$$\alpha_3 \triangleq \beta_1\omega_1(\omega_3^2 - \omega_1^2) \tag{20c}$$

Now, B^2 can be written in terms of A^2 as

$$B^2 = \frac{\alpha_1\omega_3^2\left(\omega_3^2 - A^2\right) - \alpha_2\omega_1^2\left(\omega_1^2 - A^2\right) + \alpha_3\omega_2^2\left(\omega_2^2 - A^2\right)}{\alpha_1\left(\omega_3^2 - A^2\right) - \alpha_2\left(\omega_1^2 - A^2\right) + \alpha_3\left(\omega_2^2 - A^2\right)}. \tag{21}$$

Substituting the above relation into Eq. (13), along with the change of variables $\omega_3 \to \omega_4, \beta_3 \to \beta_4$ and $x \triangleq A^2$, we obtain

$$\delta_0\frac{\delta_1 + \delta_2 x + \delta_3 x^2}{\delta_4 + \delta_5 x + \delta_6 x^2} = 1. \tag{22}$$

where δ_is are given by

$$\delta_0 \triangleq \alpha_1\frac{\beta_3\omega_3}{\beta_4\omega_4} \tag{23a}$$

$$\delta_1 \triangleq b_0\omega_4^2 \tag{23b}$$

$$\delta_2 \triangleq b_1\omega_4^2 - b_0 \tag{23c}$$

$$\delta_3 \triangleq -b_1 \tag{23d}$$

$$\delta_4 \triangleq \alpha_2\omega_1^2 c_0\frac{\omega_4^2 - \omega_2^2}{\omega_3^2 - \omega_2^2} - \alpha_3\omega_2^2 d_0\frac{\omega_4^2 - \omega_1^2}{\omega_3^2 - \omega_1^2} \tag{23e}$$

$$\delta_5 \triangleq \alpha_2\left(\omega_1^2 c_1 - c_0\right)\frac{\omega_4^2 - \omega_2^2}{\omega_3^2 - \omega_2^2} - \alpha_3\left(\omega_2^2 d_1 - d_0\right)\frac{\omega_4^2 - \omega_1^2}{\omega_3^2 - \omega_1^2} \tag{23f}$$

$$\delta_6 \triangleq -\alpha_2 c_1\frac{\omega_4^2 - \omega_2^2}{\omega_3^2 - \omega_2^2} + \alpha_3 d_1\frac{\omega_4^2 - \omega_1^2}{\omega_3^2 - \omega_1^2} \tag{23g}$$

and

$$b_0 \triangleq \alpha_1\omega_4^2\omega_3^2 - \alpha_2\omega_4^2\omega_1^2 + \alpha_3\omega_4^2\omega_2^2 - \alpha_1\omega_3^4 + \alpha_2\omega_1^4 - \alpha_3\omega_2^4 \tag{24a}$$

$$b_1 \triangleq -\alpha_1\omega_4^2 + \alpha_2\omega_4^2 - \alpha_3\omega_4^2 + \alpha_1\omega_3^2 - \alpha_2\omega_1^2 + \alpha_3\omega_2^2 \tag{24b}$$

$$c_0 \triangleq \alpha_1\omega_1^2\omega_3^2 - \alpha_2\omega_1^2 + \alpha_3\omega_1^2\omega_2^2 - \alpha_1\omega_3^4 + \alpha_2\omega_1^4 - \alpha_3\omega_2^4 \tag{24c}$$

$$c_1 \triangleq -\alpha_1\omega_1^2 + \alpha_2\omega_1^2 - \alpha_3\omega_1^2 + \alpha_1\omega_3^2 - \alpha_2\omega_1^2 + \alpha_3\omega_2^2 \tag{24d}$$

$$d_0 \triangleq \alpha_1\omega_2^2\omega_3^2 - \alpha_2\omega_1^2\omega_2^2 + \alpha_3\omega_2^4 - \alpha_1\omega_3^4 + \alpha_2\omega_1^4 - \alpha_3\omega_2^4 \tag{24e}$$

$$d_1 \triangleq -\alpha_1\omega_2^2 + \alpha_2\omega_2^2 - \alpha_3\omega_2^2 + \alpha_1\omega_3^2 - \alpha_2\omega_1^2 + \alpha_3\omega_2^2 \tag{24f}$$

At the same time, Eq. (22) can be re-written as

$$\left(\delta_6 - \delta_0\delta_3\right)x^2 + \left(\delta_5 - \delta_0\delta_2\right)x + \delta_4 - \delta_0\delta_1 = 0. \tag{25}$$

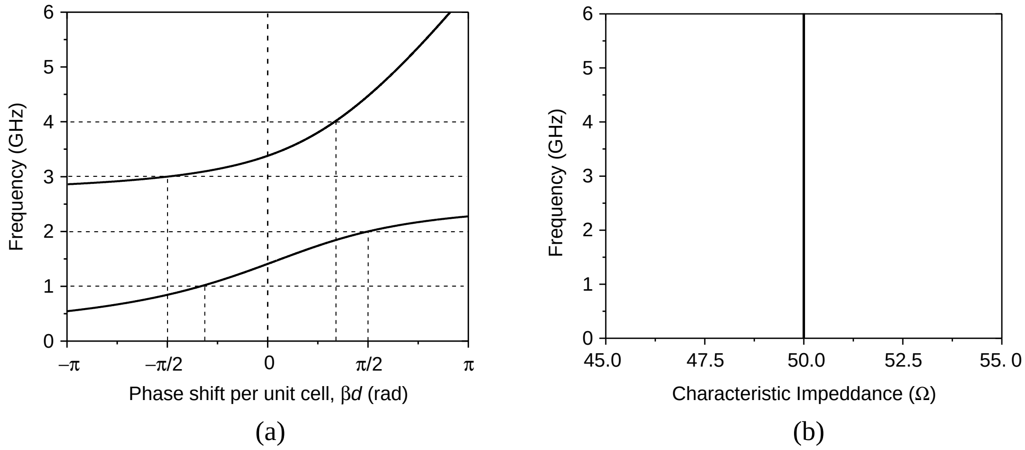


Figure 5. (a) Dispersion and (b) Characteristic impedance of the E-CRLH structure with $Z_0 = 50\,\Omega$, $L_R^c = 4.75\,\text{nH}$, $C_L^c = 1.61\,\text{pF}$, $L_R^d = 2.25\,\text{nH}$, $C_L^d = 1.63\,\text{pF}$, $L_L^c = 4.02\,\text{nH}$, $C_R^c = 1.90\,\text{pF}$, $L_L^d = 4.09\,\text{nH}$, $C_R^d = 0.90\,\text{pF}$ to exhibit phase shifts $\phi_1 = -\pi/3$, $\phi_2 = \pi/2$, $\phi_3 = -\pi/2$ and $\phi_4 = \pi/3$ at four arbitrary frequencies, say here, $f_1 = 1\,\text{GHz}$, $f_2 = 2\,\text{GHz}$, $f_3 = 3\,\text{GHz}$ and $f_4 = 4\,\text{GHz}$, respectively.

Therefore, by solving above equation for $A = x^{1/2}$, the value of A can be calculated and expressed as Eq. (26).

$$A = x^{1/2} = \sqrt{\frac{-(\delta_5 - \delta_0\delta_2) \pm \sqrt{(\delta_5 - \delta_0\delta_2)^2 - 4(\delta_6 - \delta_0\delta_3)(\delta_4 - \delta_0\delta_1)}}{2(\delta_6 - \delta_0\delta_3)}}. \tag{26}$$

Thus, by calculating A, the parameters B, C and D are obtained from Eqs. (21), (17) and (18), respectively. It must be emphasized that as indicated by Figure 3(a), we require that $\omega'_{2,0} \geq \omega'_{1,0}$. Hence, according to Eq. (14), when using Eq. (26) to calculate A and B, the plus or minus sign must be chosen accordingly so as to meet the aforementioned requirement. Now, the rest of the design can be accomplished by the methodology of Section 2.2. In fact, we *assume* that our goal is to design a dual-band balanced E-CRLH structure which exhibits zero phase shifts at frequencies $f'_{1,0} = \omega'_{1,0}/2\pi = A/2\pi$ and $f'_{2,0} = \omega'_{2,0}/2\pi = B/2\pi$ and the *design* parameters $\omega_{se}^d = \omega_{sh}^d = C$ and $\omega_R^c = D$. Figure 5 provides us with an example of an E-CRLH structure to exhibit phase shifts $\phi_1 = -\pi/3, \phi_2 = \pi/2, \phi_3 = -\pi/2$ and $\phi_4 = \pi/3$ at four arbitrary frequencies $f_1 = 1\,\text{GHz}$, $f_2 = 2\,\text{GHz}$, $f_3 = 3\,\text{GHz}$ and $f_4 = 4\,\text{GHz}$, respectively. Also, as can bee seen in this figure, the characteristic impedance of the E-CRLH-TL, for all frequencies, is almost $50\,\Omega$. As discussed in the previous section, the result of Figure 5, backs up the ease, being systematic and suitableness of our proposed methodology to obtain zero phase shift at four arbitrary frequencies, which in turn, helps fill the aforementioned gap in the literature.

3. CONCLUSION

In this paper we present a novel design methodology with closed-form formulas for dual- and quad-band E-CRLH based transmission lines exhibiting four arbitrary phases, ϕ_1, ϕ_2, ϕ_3 and ϕ_4, at four arbitrary frequencies, f_1, f_2, f_3 and f_4, respectively. The dispersion and characteristic impedance diagrams of the E-CRLH-TLs are derived and plotted. The results of this paper can be applied to various types of dual- and quad-band TL-based microwave devices.

APPENDIX A. PROOF OF THE LEMMA

Assume without loss of generality that $b \geq a > 0$. Note that $P_2(x)$ is a polynomial hence continuous on $\mathbb{R}$. Since

$$P_2(0) = +ab > 0$$

$$P_2(a) = -ca < 0$$
$$P_2(b) = -cb < 0$$
$$P_2(M \to \infty) = +\infty > 0$$

according to the Intermediate Value Theorem, $P_2(x)$ has *at least* one root in each of the intervals, $(0, a)$ and (b, M), for sufficently large M (e.g., $M = a + b + c$). Meanwhile we know that each polynomial of degree n has *at most* n distinct (complex) roots. Thus, $P_2(x)$ has exactly two positive distinct roots, each lying in the intervals, $(0, a)$ and (b, M), respectively.

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