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Inhomogeneous and Homogeneous Losses and Magnetic Field Effect in Planar Undulator Radiation

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Abstract—We construct an analytical model for the description of emission of undulator radiation (UR) harmonics with account for several sources of line broadening, including the effect of a constant magnetic constituent. We compare it with that of the beam energy spread, emittance and focusing components. The analytical expressions obtained for the UR intensity and spectrum allow for profound analysis of homogeneous and inhomogeneous losses in their explicit form. We analyse the contributions to the fundamental frequency as well as to higher harmonics in long undulators. We study a possibility to compensate for the off-axis effects in undulators by a properly imposed constant magnetic field and obtain an expression for the intensity of such compensating effect. The results obtained are discussed in the context of their possible applications to free electron lasers (FEL). Recommendations for improvement of an UR harmonic line quality, profitable for FEL, are also proposed.

1. INTRODUCTION

Sources of synchrotron radiation (SR) and of undulator radiation (UR) are nowadays commonly used when the interaction of radiation and matter is studied. Synchrotron radiation was predicted in 1944 by Ivanenko and Pomeranchyk [1] and discovered three years later in Brookhaven National Laboratory [2]. Angular and spectral properties of SR were explored in [3]. At that time Schwinger worked out his seminal papers (see [4]), repeating much of Schott's early studies [5], put in a useful formalism and completed with numerical calculations. The developed formulation of Schwinger's works can be found, for example, in [6, 7]. The physical nature of undulator radiation that arises in periodic magnetic structures is similar to that of SR, since both are due to photon emission by accelerated ultra-relativistic electrons [8]. This idea was first advanced by Ginzburg [9], and several years later a first undulator was built and tested by Motz et al. [10]. More theoretical studies of radiation of fast particles in magnetic fields can be found, for example, in [11–14]. Theoretical studies and technological development drove each other to the best and yielded great improvement in the design of wigglers and undulator magnets [13]. Numerical and analytical methods were proposed to model properties of UR, accounting for the complex of physical phenomena in undulators [15]. SR and UR are known for high beam intensity and narrow cone of radiation emission. They provided an impulse for the appearance and the development of free electron lasers (FEL). FEL contain undulators and make use of the UR. In the simplest case, an undulator is placed in an optical resonator, consisting of two mirrors. One of the problems with exploiting such a device consists in the deterioration of reflecting mirrors by the hard component of the emission. More sophisticated FEL constructions are used nowadays, working in Röntgen range with ultra fast and high coherent electron beams. All that maintains interest to undulator radiation studies and, in particular, to undulators of non-standard configurations [16, 17]. Modern undulators allow for efficient regulation of harmonic emission. However, distortions of the periodic magnetic field (that are due to non-homogeneity of the periodic structure, but not only)

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can significantly affect operation of such devices. In particular, it concerns undulators with complex magnetic fields and with many periods. High gain in undulators and high quality of the electron beams are particularly important for new generations of sources of radiation — FEL with self amplified spontaneous emission (SASE) [18, 19], high gain harmonic generation (HGHG) [20] and other modern schemes, frequently including two-frequency undulators (see, for example, [21–23]). In this context the quality of the undulator device can be evaluated according to its UR spectrum deterioration, which, if not ascribed to inhomogeneous broadening or to beam transport problems, may be viewed as the consequence of the distorted structure of the undulator magnetic field. Deviation from the ideal periodic form of the field [24] in permanent magnet undulators is usually present in the form of a constant field component, superimposed on the on-axis periodic field. In what follows, we will explore its effect on the undulator performance, comparing it with that due to electron energy spread and some beam transport losses. In particular, we will demonstrate how long undulators with many periods may be sensitive to it and how properly imposed constant field can partially compensate for off-axis effects. We will include both homogeneous and inhomogeneous effects, recognizing enlightening studies of the effect of inhomogeneous field error in undulators [25, 26], performed in the framework of the phase error concept.

Our analysis will be largely based upon the technique of generalized special functions and on mathematical results, obtained earlier in [27]. We recall that for a planar undulator with N periods of the length λ_u along the z-axis, the periodic magnetic field amplitude H_0 and the constant magnetic field H_d, superimposed on it reads

$$\vec{H} = H_0\left(\rho,\, \kappa + \sin(k_\lambda z),\, \delta\right), \quad k_\lambda = 2\pi/\lambda_u, \quad H_d = H_0\kappa_1, \tag{1}$$

where

$$\kappa_1 = \sqrt{\kappa^2 + \rho^2}, \tag{2}$$

ρ and κ are the factors for the transversal components of the constant magnetic field H_d, and the UR intensity depends on their combination (2), rather than on each of them separately [28]. Moreover, let us demonstrate that for a high energy electron beam, longitudinal component δ does not play any significant role. Ultrarelativistic ($\gamma \gg 1$) electrons move in undulators with very small transverse speed $\beta_\perp$ in a pure magnetic field

$$\beta_\perp \ll 1, \quad \beta_\perp H_= \ll H_\perp, \quad \vec{E} = 0. \tag{3}$$

Then the longitudinal component of the constant magnetic field, interacting with the transversal component of the electron velocity, effects in higher orders of k/γ, rather than do the transversal magnetic components, interacting with the relativistic drift of the electron $\beta_z^0 \approx 1$. Straightforward integration of the equations of motion leads to the system of differential equations, which, in its turn, yields the following law of motion in the lowest order of k/γ:

$$\vec{r}\,(t) \cong \begin{pmatrix} \dfrac{c}{\omega_0}\dfrac{k}{\gamma}\left(\sin(\omega_0 t) - \dfrac{\kappa}{2}(\omega_0 t)^2\right) \\ \dfrac{c}{\omega_0}\dfrac{k}{\gamma}\dfrac{\rho}{2}(\omega_0 t)^2 \\ c\beta_z^0 t - \dfrac{c}{\omega_0}\left(\dfrac{k}{\gamma}\right)^2\left[\dfrac{\kappa_1^2}{6}(\omega_0 t)^3 + \dfrac{1}{8}\sin(2\omega_0 t) - \kappa\cos(\omega_0 t) - \kappa\omega_0 t\sin(\omega_0 t)\right] \end{pmatrix}. \tag{4}$$

The longitudinal field component δ plays its role in the terms of the second order of k/γ, such as $\frac{1}{6}(\frac{k}{\gamma})^2\rho\delta\frac{c}{\omega_0}(\omega_0 t)^3$, $(\frac{k}{\gamma})^2\delta\frac{c}{\omega_0}(\cos\omega_0 t - 1)$ etc., whereas the transversal constituents play a role in the 1st order of k/γ. We omit here the complete expression for the sake of conciseness. We only note that, in the case of ultrarelativistic motion of the high energy electrons and relatively weak constant components $H_0\rho$, $H_0\kappa$ and $H_0\delta$ in undulators, which we consider, the effect of the longitudinal field, factorized by $\rho\,(k/\gamma)^2$ and similar products, is negligible. In low energy applications, on the contrary, the longitudinal component plays an important role. Some undulator designs even include longitudinal fields for focusing in low energy FELs. Analytical computation of the UR intensity in the case of $k/\gamma < 10$ goes beyond the usual $(k/\gamma)^2$ order, accounting for the corrections of $(k/\gamma)^2$ approximation for UR spectrum frequencies. The relativistic electron drift β_z^0, the undulator frequency ω_0 and the undulator parameter k are given as always:

$$\beta_z^0 = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{k^2}{2}\right), \tag{5}$$

$$\omega_0 = k_\lambda \beta_z^0 c, \quad k_\lambda = \frac{2\pi}{\lambda}, \quad k = \frac{e}{mc^2}\frac{H_0}{k_\lambda}. \tag{6}$$

The shift of the electron trajectory due to a constant magnetic field in an undulator was discussed in [27]. We just underline that for e.g., $\kappa = \rho = 10^{-4}$, $\gamma = 10^3$, $k = 1$, the energies $\sim 500\,\text{MeV}$ and $\lambda_u \approx 6\,\text{mm}$, and the deviation will be only $10^{-3}\,\text{mm}$. However, the shift along the x-direction after 100 periods is 10 times the oscillation amplitude, and after 150 periods the electron trajectory shift in x- and y-directions becomes 20 times its oscillation amplitude. This indicates possible changes in UR characteristics.

2. UR INTENSITY WITH ACCOUNT FOR OFF-AXIS EFFECTS AND FOR THE CONSTANT MAGNETIC FIELD

The intensity of the electron emission in the farfield zone [29] is given by the well-known expression for the radiation integral [30]

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c}\left|\omega \int_{-\infty}^{\infty} dt \left[\vec{n}\times\left[\vec{n}\times\vec{\beta}\right]\right] \exp\left[i\omega\left(t - \vec{n}\vec{r}/c\right)\right]\right|^2. \tag{7}$$

This is a generic classical expression for radiation intensity of an accelerated charged particle, where $\vec{n}$ is the observation vector, which has the following approximate form for$\gamma \gg 1$:

$$\vec{n} \cong \left(\psi\cos\varphi,\, \psi\sin\varphi,\, 1 - \psi^2/2\right). \tag{8}$$

Following the lines of [27], we now include off-axis effects and a 2-component constant field to end up with a complicated expression, similar to those, described in detail in [27], but with many new terms in the exponential of the radiation integral. We omit it for brevity and proceed upon the realistic supposition of a weak constant field $\kappa_1 \ll 1$ to simplify complicated and cumbersome exact analytical expression, eventually obtaining for the UR intensity

$$\frac{d^2 I}{d\omega d\Omega} \cong \frac{e^2 N^2 \gamma^2}{c}\frac{k^2}{(1+k^2/2)^2}\sum_{n=-\infty}^{\infty} n^2 \left\{\begin{array}{l} \left[S\left(\nu_n,\beta,\eta\right)\left(J_{\frac{n+1}{2}}\left(-\frac{\xi}{8}\right) + J_{\frac{n-1}{2}}\left(-\frac{\xi}{8}\right)\right)\right]^2 \\ +(4\pi N\kappa_1)^2\left[\frac{\partial S(\nu_n,\beta,\eta)}{\partial \nu_n} J_{\frac{n}{2}}\left(-\frac{\xi}{8}\right)\right]^2 \end{array}\right\}, \tag{9}$$

where $\xi = \frac{\omega}{\omega_0}(\frac{k}{\gamma})^2$ is the argument of the Bessel function $J_n(-\xi/8)$. The second argument of the function $S(\alpha,\beta,\eta)$ reads as follows:

$$\beta = (2\pi n N + \nu_n)\frac{(\gamma\theta_H)^2}{1 + k^2/2 + (\gamma\theta_H)^2}, \tag{10}$$

θ_H is the effective bending angle

$$\theta_H = \frac{2}{\sqrt{3}}\frac{k}{\gamma}\pi N \kappa_1 \tag{11}$$

and ν_n is the detuning parameter, describing the deviation from the central frequency ω_n of the harmonic n

$$\nu_n = 2\pi N n\left(\frac{\omega}{\omega_n} - 1\right). \tag{12}$$

The off-axis effects are accounted for in the argument

$$\eta = 2\pi^2 N^2\left(\kappa\cos\varphi - \rho\sin\varphi\right)\frac{\omega}{\omega_0}\left(\frac{k}{\gamma}\right)\psi \tag{13}$$

of the generalized Airy-type special function

$$S\left(\alpha,\beta,\eta\right) \equiv \int_0^1 d\tau e^{i\left(\alpha\tau + \eta\tau^2 + \beta\tau^3\right)}, \quad S\left(\alpha,\beta,0\right) = \int_0^1 e^{i\left(\alpha\tau+\beta\tau^3\right)}d\tau, \quad S\left(\alpha,0,0\right) = e^{i\alpha/2}\,\mathrm{sinc}\,\alpha/2. \tag{14}$$

They now play the role of a sinc function and determine the shape of the UR harmonics, which read for a common undulator as follows:

$$\omega_{R0} = \frac{2\omega_0\gamma^2}{1 + k^2/2}, \quad \omega_{n0} = n\omega_{R0}. \tag{15}$$

For a 2-component transversal magnetic field, Eqs. (9)–(14) depend purely on the intensity of the constant transversal constituent, related to that of the periodic undulator field: $\kappa_1 = (\kappa^2 + \rho^2)^{1/2}$, where $\kappa_1 H_0$ and ρH_0 are the transversal components of H_d. Thus, the direction of the transversal component of H_d does not matter, and its longitudinal part is irrelevant as discussed above (see also [28, 31]). This important observation allows choosing the direction of the constant field at our convenience when treating more complicated mathematical problems of radiation of charges in multi-component fields.

So far as the shape of the harmonics is concerned, we find from Fig. 1 and from the analysis of (14) that the emission line has a discrete and evident peak along the line of the values

$$\alpha \approx -\beta, \quad \text{if } \alpha, \beta \in [-8,\, 8]. \tag{16}$$

Beyond this range, the local side-maxima of S become stronger, while the main peak of the function fades out. It does not follow Eq. (16) any more, remaining at $\alpha \approx -5$ and thus the single discrete harmonic spreads into a wide band of the emission. The derivative of $S(\alpha, \beta)$ exhibits similar behaviour. With account for the off-axis effects and the constant magnetic component, we obtain the spectrum for the undulator (1) in the form of peaks with the following central frequencies ω_n:

$$\omega_n|_{\psi\neq 0,\, B_d \neq 0} = n\omega_R = \frac{2n\omega_0\gamma^2}{\left(1 + \frac{k^2}{2}\right) + (\gamma\psi)^2 + (\gamma\theta_H)^2 - \sqrt{3}(\gamma\theta_H)(\gamma\Omega)}, \tag{17}$$

where $\Omega = \psi(\rho\sin\varphi - \kappa\cos\varphi)/\kappa_1$. Eq. (17) generalizes the expression previously obtained in [27] and demonstrates that the off-axis effect can be partially compensated by proper configuration of the constant magnetic field H_d. It follows from (17) that the angular divergence is mostly compensated by the effective bending angle $\tilde{\theta}_H$ due to the constant magnetic field, such that

$$\tilde{\theta}_H = \frac{\sqrt{3}}{2}\Omega, \quad \Omega = \psi\frac{\rho\sin\varphi - \kappa\cos\varphi}{\kappa_1}, \quad \kappa_1 = \sqrt{\rho^2 + \kappa^2}. \tag{18}$$

The horizontal divergence (i.e., $\varphi = 0, \pi$) is compensated by the vertical field component

$$\tilde{\theta}_H = \mp\psi\frac{\sqrt{3}}{2}\frac{\kappa}{\kappa_1}, \quad \varphi = 0, \pi, \tag{19}$$

and the vertical divergence (i.e., $\varphi = \pm\,\pi/2$) is compensated by the horizontal field component

$$\tilde{\theta}_H = \pm\psi\frac{\sqrt{3}}{2}\frac{\rho}{\kappa_1}, \quad \varphi = \pm\frac{\pi}{2} \tag{20}$$

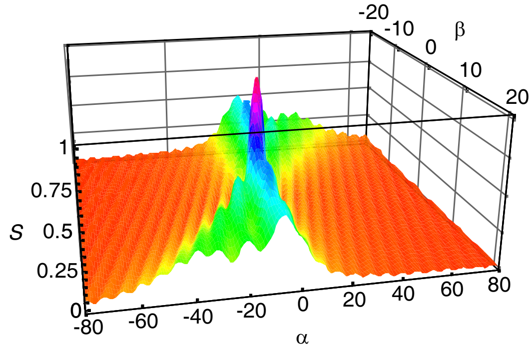

Figure 1. Absolute value of the function $S(\alpha, \beta, 0)$ for the parameters α and β.

with the respective sign choice. The best compensation by $\tilde{\theta}_H$ in the proper half-plane can reduce the divergence angle ψ to its half: $\psi/2$. Thus, if we induce constant magnetic fields opposite each other in two semi planes for $\varphi = \pi/2$ and $\varphi = -\pi/2$ and choose the intensity accordingly from (20), so that the bending angle $\tilde{\theta}_H$ is produced, we will compensate (in part) for vertical divergency! For the radial divergency compensation refer to (19) with $\varphi = 0,\ \pi$.

3. SPECTRUM DISTORTIONS AND THE BROADENING PARAMETERS CONCEPT

It is useful to recall some basic notes of the focusing concept in undulators, beam emittance, electron energy losses, UR spectrum corrections and the line broadening associated with the above mentioned phenomena. To this end we refer, for example, to [32], although this topic was broadly discussed by other researchers as well. It is well known that the ratio of the half-width of an UR harmonic to the harmonic frequency is inversely proportional to the number of the undulator periods N

$$\frac{\Delta\omega}{\omega_{n0}} = \frac{\omega - \omega_{n0}}{\omega_{n0}} = \frac{1}{nN}, \tag{21}$$

where ω_{n0} are the frequencies of the ideal planar undulator (15). Obviously, we are at the half-height of the spectrum line for $\Delta\omega/\omega_{n0} = 1/(2nN)$; $\Delta\omega/\omega_{n0} \ll 1$ for any realistic number of undulator periods. Good shape of the spectrum line is insured by (16).

The deviation of the peak harmonic frequencies from (15) $\Delta\omega_R = \omega_n - \omega_{n0}$, caused by the field H_d, has a negative sign and it can be expressed as follows:

$$\Delta\omega_R = -\frac{\omega_{n0}}{1 + \left(1 + k^2/2\right)/\left(\gamma\theta_H\right)^2}. \tag{22}$$

Thus, the constant magnetic field shifts the UR frequencies down. The value of the detuning parameter (12) $\nu_n = \nu_{n\,\mathrm{Res}}$ at the peak frequency of the n-th harmonic depends on the constant field intensity and on the number of periods as follows:

$$\nu_{n\,\mathrm{Res}} = -\frac{2\pi Nn\left(\gamma\theta_H\right)^2}{1 + k^2/2} = -\frac{n}{3}\frac{(2\pi N)^3\kappa_1^2}{1/2 + 1/k^2}. \tag{23}$$

Evidently, in the absence of the constant field we have $\nu_{n\,\mathrm{Res}} = 0$, and then the UR spectrum goes over to (15). From (16) the following upper limit for the ratio H_d/H_0 arises to preserve the line width and distinct harmonic shape:

$$\kappa_1 \le \kappa_{\max} = \frac{1}{(\pi N)^{3/2}}\sqrt{\frac{3}{n}\left(\frac{1}{2} + \frac{1}{k^2}\right)}. \tag{24}$$

Thus (24) can be viewed as the condition for the constant magnetic field intensity, under which (17) determines harmonic frequencies. When Eq. (24) is not fulfilled, $\nu_{n\,\mathrm{Res}}$ approximately corresponds to the middle of the wide frequency band $\nu_n \in [0,\ 2\,\nu_{n\,\mathrm{Res}}]$, that remained from the n-th harmonic line. The larger the induced bending angle, the wider the frequency spread becomes

$$\left|\frac{\Delta\omega}{\omega_n}\right|\Bigg|_{\nu_{n\,\mathrm{Res}}>10} \approx \frac{2(\gamma\theta_H)^2}{1 + k^2/2} = \frac{2}{3}\frac{(2\pi N\kappa_1)^2}{1/2 + 1/k^2}. \tag{25}$$

For example, for $|\nu_{n\,\mathrm{Res}}| \approx 20$ we obtain the harmonic line spread in a wide range $\Delta\nu_n \approx 40$, which is seven times wider, as compared with $\Delta\nu_n = 2\pi$, the half-width of a common undulator harmonic.

As regards the odd harmonics — planar UR feature arising in a constant magnetic field — their amplitude strongly depends on N and on the intensity of the constant field via κ_1 (see (9)), growing with their increase. When κ_1 is below $\kappa_{\max}$, specified by (24), they are negligible; otherwise, when $|\alpha|,\ |\beta| > 10$, odd harmonics have the following frequencies:

$$\omega_n|_{\psi=0} = n\omega_R = \frac{2n\omega_0\gamma^2}{\left(1 + \frac{k^2}{2}\right) + 2\left(\gamma\theta_H\right)^2}, \quad n = \text{ odd}, \quad (|\alpha|, |\beta| > 10). \tag{26}$$

We can heuristically introduce broadening parameters μ_i corresponding to each broadening factor "i", such as the off-axis effects, the energy spread, the field effect etc.. The constant field broadening (22), normalized to (21), yields the following broadening parameter:

$$\mu_H \equiv \frac{\Delta\omega_R/\omega_n}{\Delta\omega/\omega_{n0}} = \frac{Nn(\gamma\theta_H)^2}{1+k^2/2}. \tag{27}$$

We can evaluate the maximum value of the frequency detuning $|\Delta\omega_R|/\omega_n$, corresponding to the maximum intensity of the constant field $\kappa_1 \cong \kappa_{\max}$, when the harmonic still has a distinctive shape. This reads $|\frac{\Delta\omega_R}{\omega_n}|_{\max} = \frac{4}{\pi n N}$, and the maximum value $\mu_{H\ \max}$ of the constant field broadening parameter μ_H for a distinct line unsurprisingly becomes

$$\mu_{H\ \max} \equiv \frac{(\Delta\omega_R/\omega_n)_{\max}}{\Delta\omega/\omega_{n0}} = \frac{4}{\pi} \approx 1.3. \tag{28}$$

In this context it is interesting to make a comparison with the broadening effect of the emittance of the beam. We recall expressions (22) for the emission line shift by a constant field $\Delta\omega_R/\omega_n$ and the constant field broadening parameter μ_H (27) and compare them with the relevant expressions for the frequency shift due to the emittance effects

$$\left|\left\langle\frac{\Delta\omega}{\omega}\right\rangle\right|_{x,y} = \frac{\gamma^2}{1+k^2/2}\Theta_{x,y}^2 \tag{29}$$

and

$$\mu_{x,y} = nN\left|\left\langle\frac{\Delta\omega}{\omega}\right\rangle\right|_{x,y}, \tag{30}$$

where $\Theta_{x,y} = \varepsilon_{x,y}/\sigma_{x,y}$ are the electron beam divergences in the undulator, related to $\varepsilon_{x,y}$, the horizontal and the vertical emittances of the beam, and to $\sigma_{x,y}$, the beam size. Whether the broadening is dominated by the angular divergence of the beam or by the constant magnetic field depends on which bending angle is bigger: $\Theta_{x,y} > \theta_H$ or vice versa.

As regards the focusing effects in undulators, first of all, note that the magnetic field in its form (1) does not satisfy Maxwell equations and is quite accurate only in a small region in the vicinity of the undulator axis. Proper discussion of this topic and of necessary corrections for all three field components can be found in [32]. Not intending to go into details of the focusing concept, we just estimate the order of the magnitude of the effect produced in order to understand under which conditions the effect of our constant magnetic field B_d can be compared with that of the focusing field in a planar undulator. Following [32], for the amplitude of the magnetic field component, in charge of the focusing in the undulator, we derive the following maximum value:

$$B_{\max\text{-}f} = \frac{2B_0L^2}{k^2\beta_x^2}\gamma^2\Theta_x\Theta_y = 2B_0\left(\frac{\gamma}{k}\right)^2 L^2\frac{\varepsilon_x^3}{\sigma_x^5}\frac{\varepsilon_y}{\sigma_y}, \tag{31}$$

where β_x is the Twiss coefficient.

Note that the above value essentially depends on the ratio of the beam divergence angles to the SR characteristic angle, which is of the order of $1/\gamma$. Upon the comparison with ρ in (1), the following equivalent value of the horizontal constant field intensity (2) arises:

$$\tilde{\kappa} \equiv \tilde{\rho} = 2B_0\left(\frac{\gamma}{k}\right)^2\frac{\varepsilon_x^3}{\sigma_x^5}L^2\Theta_y. \tag{32}$$

Consider an undulator of a total length $L = 2.1$ m with 300 periods, $k = 0.5$ and on-axis (!) field amplitude 7.5 kG at the storage ring Siberia-2 in Novosibirsk [33] in the regime of a "standard" beam of 2.5 GeV, whose size is 1.5 mm × 0.078 mm, horizontal emittance is $\varepsilon_x = 98$ nm rad, evaluated for $\varepsilon_y = 49$ pm rad vertical emittance. We obtain $\tilde{\kappa}_1 = 7.2\cdot10^{-5}$ and $B_{\max\text{-}f} = 0.5$ G , which is of the order of the value of the magnetic field of the Earth! In a "bright" regime its beam energy is 1.3 GeV, its sizes are 0.363 mm × 0.017 mm and a horizontal emittance equals $\varepsilon_x = 4.9$ nm rad. For the same undulator we have $\tilde{\kappa}_1 = 1.28\cdot10^{-5}$ and $B_{\max\text{-}f} = 0.1$ G. With the same "standard" beam of 2.5 GeV in Novosibirsk and a 2 m long undulator with $k = 1$ and $\Theta_y = 0.013$ m rad, we obtain the following values: $\tilde{\kappa}_1 = 3.4 \cdot 10^{-4}$ and for 2.5 kG on-axis field we have $B_{\max\_f} = 0.86$ G. If the beam energy is 0.51 GeV, for a 1 m long undulator with 2.5 kG on-axis field, $k = 1$ and $\Theta_y = 0.013$ m rad we have $\tilde{\kappa}_1 = 3.4 \cdot 10^{-6}$ and the respective field value $B_{\max\_f} = 8.6$ mG. These examples show that a constant ambient (or of another origin) field can reach and even exceed the order of the magnitude of focusing field components in undulators. As far as the divergency is concerned, horizontal and vertical spreads of a 1.3 GeV beam of Siberia installation are 0.09 m rad and 0.013 m rad, respectively. The transversal constant magnetic field of 0.5 G induces a bending angle of 0.014 m rad. This is almost exactly the value of the vertical beam divergence of the device (!) and 0.5 G is approximately equal to the value of the strength of the magnetic field of the Earth. Thus, it is evident that the effect of a constant field of $B_d \approx 0.1 \div 0.5$ G (and even lower) across the axis should not be neglected at least because it is comparable in our example with the contribution of the terms, responsible for inhomogeneous broadening, such as beam emittance ($\mu_x = 0.30$), and with the focusing effects. Moreover, another important observation is that the constant magnetic component can be exploited to compensate for the angular divergence (in part). Indeed, we know that the off-axis divergence in the angle ψ produces broadening and the shift $\omega_n|_{\psi\neq 0} = n\omega_R = \frac{2n\omega_0\gamma^2}{1+k^2/2+(\gamma\psi)^2}$ (see, for example, [32]) and it can be partially compensated for by properly chosen field H_d as discussed in the end of the previous Section (see Eqs. (17)–(20)). It should be noted that we also have to account for the other source of broadening in undulators, namely, for the electron energy beam spread.

With the assumption of Gaussian energy distribution in the beam, the relevant line broadening is $(\Delta\omega/\omega)_\varepsilon \approx 2\sqrt{\sigma_e}$ with zero average frequency shift. Normalized to $(\Delta\omega/\omega)_0 = 1/2nN$, it becomes

$$\mu_\varepsilon \equiv \frac{(\Delta\omega/\omega)_\varepsilon}{(\Delta\omega/\omega)_0} \approx 4Nn\sqrt{\sigma_e}. \tag{33}$$

Evidently, the broadening effects, induced by the energy spread in beams, are negligible when $\sqrt{\sigma_e} \ll 1/(4nN)$. For $N \sim 100$ and $n = 1$, we obtain $\sqrt{\sigma_e} \ll 2.5 \cdot 10^{-3}$ and we conclude that the value $\sigma_e < 10^{-6}$ can be neglected. This was confirmed by numerical code (see, for example, in [27, 28]). Comparing the constant field effect with the energy spread effect, we relate (27) to (33):

$$\frac{\mu_H}{\mu_\varepsilon} = \frac{(\gamma\theta_H)^2}{4\sqrt{\sigma_e}(1+k^2/2)} = \frac{\pi^2}{3}\frac{k^2}{1+k^2/2}\frac{(N\kappa_1)^2}{\sqrt{\sigma_e}}. \tag{34}$$

Let us estimate the condition for the effects of the constant field and of the energy spread to be of the same order, i.e., we set $\mu_H \approx \mu_\varepsilon$ to obtain the following value for the squared intensity of the constant magnetic component $\kappa_{1\,\varepsilon}$, imposed across the undulator in this case:

$$\kappa_{1\,\varepsilon}^2 \approx \frac{3}{\pi^2}\frac{\sqrt{\sigma_e}}{N^2}\frac{1+k^2/2}{k^2}, \quad \text{note that } \kappa_{1\,\varepsilon} \propto \frac{1}{N}. \tag{35}$$

For example: for the undulator **1** with $N = 100$, $k = 2$ and $\sigma_e = 10^{-6}$ we obtain the value $\kappa_{1\,\varepsilon} \cong 1.6 \cdot 10^{-4}$, for which the constant field and the beam energy spread contributions are approximately the same. For the undulator **2** with $N = 200$ we have $\kappa_{1\,\varepsilon} \cong 0.8 \cdot 10^{-4}$. Note that the Earth magnetic field of $B_d = 0.5$ G produces $\kappa_1 = 2 \cdot 10^{-4}$ in an undulator with the on-axis field $B_0 = 2.5$ kG. For the undulator **1** the Earth magnetic field bends $\gamma\theta_H \cong 0.14$, for the undulator **2** $\gamma\theta_H \cong 0.28$. The last example **3** is of an undulator at the Siberia-2 installation in Novosibirsk with the on-axis field amplitude of 0.75 T, $\sqrt{\sigma_e} = 0.5 \cdot 10^{-4}$, $N = 300$ and $k = 0.5$ [33]. In this case the energy spread broadening coefficient amounts to the small value $\mu_\varepsilon \cong 0.06$, for which we find $\kappa_{1\,\varepsilon} \cong 0.28 \cdot 10^{-4}$, corresponding to $B_d = B_0\kappa_{1\,\varepsilon} = 0.21$ G, which produces the same order effect as the energy spread does. The value of B_d appears of the order of, but somewhat lower than 0.5 G (that of the Earth), and respectively $\kappa_1 = 0.667 \cdot 10^{-4}$ ($\kappa_1 \cong 2.4\,\kappa_{1\,\varepsilon}$). Although bending due to 0.5 G field in this undulator is small, i.e., $\gamma\theta_H \cong 0.036$, it produces respectable $\mu_H \cong 0.35$. Detailed consideration of the emittance of the electron beam in an undulator, of the focusing and the defocusing etc., remains beyond the scopes of the present work. Here we only note that with account for the energy spread and the emittance of the beam, the total spectrum bandwidth is given by

$$\left[\frac{\Delta\omega}{\omega}\right]_{Tot} = \frac{\Delta\omega}{\omega_{n,0}}\sqrt{1+\mu_\varepsilon^2+\mu_H^2+\left(\mu_x^2+\mu_y^2\right)}. \tag{36}$$

In a similar way the near field effects can be heuristically included, which, combined with emittance, can produce on the axis appreciable reduction of the peak UR intensity.

Above we applied a simple approach based on $\Delta\omega_i/\omega$ and on corresponding broadening parameters μ_i, which gives correct qualitative results for the UR intensity. To describe the exact behaviour of the function, we need generalized Airy functions, which at the beginning just shift the frequency with little broadening and only then flatten the peak. In order to account better for the energy spread we refer, for example, to [32] and write the following simple convolution:

$$I = \int_{-\infty}^{\infty} \left(\left|S\left(\nu_n + 4\pi n N\varepsilon, \beta\right)\right|\right)^2 \exp\left[-\varepsilon^2/2\sigma_e\right] / \left(\sqrt{2\pi\sigma_\varepsilon}\right) d\varepsilon, \tag{37}$$

where σ_ε is the energy spread, ν_n is the detuning parameter. Now we turn back to the undulator with 300 periods at the installation Siberia-2 in Novosibirsk and complete our consideration with account for the energy spread, off-axis effect and its compensation by the constant magnetic field in Fig. 2.

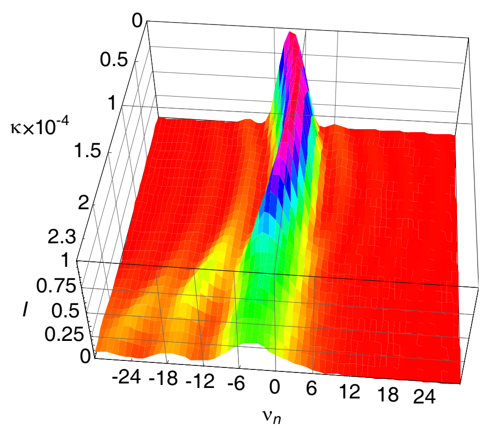

Figure 2. Emission line for $N = 300$, $n = 1$, $k = 0.5$, $\sigma = 10^{-8}$, $B_0 = 7.5\,\text{kG}$, off-axis angle: $\gamma\psi \sim 0.04$.

Note that at $\kappa \cong 0.7 \times 10^{-4}$ almost complete compensation of the shift, originated from the off-axis effects, does occur, which means that $B_d \sim 0.7 \cdot 10^{-4} B_0 \approx 0.5\,\text{G}$ is the required intensity value of the constant field (evidently directed opposite to each other for $\varphi = 0$ and for $\varphi = \pi$). Accidentally, it is of the order of the strength of the magnetic field of the Earth. It has more impact on the emission line than the energy spread has (see discussion above). In a stronger constant magnetic field $B_d > 10^{-4} B_0$ incoherence prevails; it broadens the line and the intensity of the fundamental harmonic reduces to $1/4$ of its initial value for $\kappa \cong 2.5 \times 10^{-4}$, as observed in Fig. 2.

4. BROADENING OF HIGHER UR HARMONICS

As we have underlined above, a constant magnetic field affects the UR spectrum, i.e., odd harmonics appear on the axis and even harmonics shift down, and the reduction of the UR intensity occurs, which should be properly accounted for, since users of UR require high brightness and intense beams. Corrections due to various sources of broadening were discussed above. In what follows, we will give some examples of the influence of a constant magnetic constituent on higher UR harmonics. Consider, for example, undulators with the periodic magnetic field $H_0 = 2.5\,\text{kG}$. Let us evaluate the effect of the magnetic field of the Earth (not accounting for the magnetic screening though) $H_d = 0.5\,\text{G}$, i.e., $\kappa_1 \cong 2 \cdot 10^{-4}$. Its effect on the main harmonic of the undulator with $k = 2$ and $N = 100$ was shown to be unnoticeable for $\kappa_1 < 1.5 \cdot 10^{-4}$ in [27]. Recent development of new sources of high frequency and high coherency radiation, such as FEL with SASE, HGHG and others, require high quality beams in precision made undulators. They frequently employ higher UR harmonics. Therefore, it is important to know how non-periodic magnetic components, present in undulators, influence them. Modelling of the UR in such sources should always include losses, associated with homogeneous and inhomogeneous effects. It is particularly important for the higher harmonics, as we will demonstrate in what follows.

Separating the constant field effect on the 3rd harmonic line of the undulator with $N = 100$, $k = 2$ in additional constant magnetic field $\kappa_1 H_0$ as shown in Fig. 3, we conclude that the magnetic field H_d has no effect on it for $\kappa_1 < 0.5 \cdot 10^{-4}$. It causes none or little detuning and fading for $\kappa_1 < 1.0 \cdot 10^{-4}$, and for $\kappa_1 > 1.5 \cdot 10^{-4}$ it already decreases the UR intensity three times with the detuning value $\nu_{n\,\mathrm{Res}} \approx 6$. For $\kappa_1 > 2 \cdot 10^{-4}$ we see the emission in a wide frequency band instead of the line of the 3rd harmonic. Thus $\kappa_1 \approx 1.0 \cdot 10^{-4}$ is the maximum possible value, when the 3rd harmonic is intended for use. If, on the contrary, the 3rd harmonic is undesirable, then it can be reduced to half of the main harmonic intensity by imposing the field with $\kappa_1 > 1.5 \cdot 10^{-4}$. Even stronger is the effect of H_d on the 5th and higher harmonics. Thus, even if H_d is as weak as 0.01% of H_0, it matters for the higher harmonics.

The constant magnetic constituent H_d also gives rise to odd harmonics on the axis. The example of the second harmonic is demonstrated in Fig. 4.

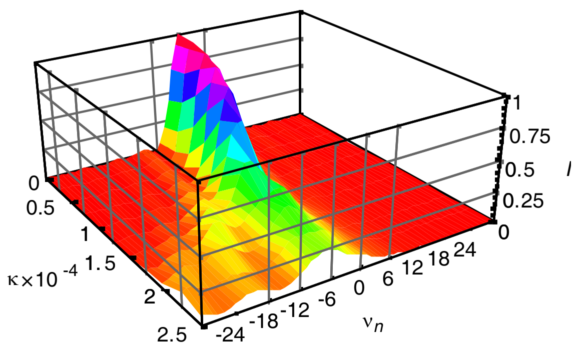

Figure 3. Broadening of the 3rd ($n = 3$) harmonic of the undulator with $N = 100$, $k = 2$, due to the constant magnetic field $\kappa_1 H_0$.

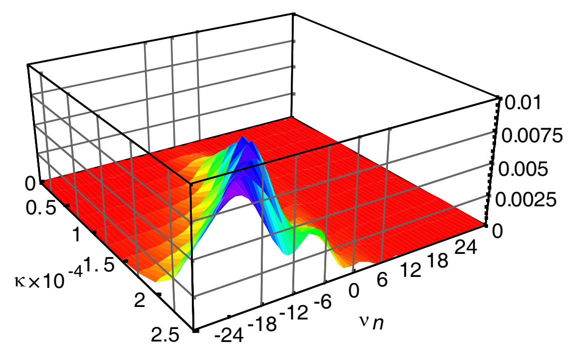

Figure 4. Broadening of the 2nd ($n = 2$) harmonic of the undulator with $N = 100$, $k = 2$, due to the constant magnetic field $\kappa_1 H_0$.

However, the intensity of odd harmonics for the chosen values of the magnetic fields and the undulator periods is below 1% of even harmonics intensity. It grows with the increase of κ_1 and N, but for reasonably low values of the constant field, such that $\kappa_1 < 2.5 \cdot 10^{-4}$, it remains at most few percent even for $N \approx 200$.

We complete our consideration of broadening effects due to constant magnetic field with the study of inhomogeneous broadening contributions due to the electron beam energy spread. It causes serious reduction of the UR intensity, in particular, for higher harmonics, as demonstrated in Fig. 5 for the 3rd harmonic of the undulator with $N = 100$, $k = 2$ (the plot is scaled with $e^2\gamma^2/c$ and with the Bessel functions factor, independent of N and κ). For $\sigma_e \approx 10^{-6}$ the beam energy spread alone reduces the 3rd harmonic with $30 \div 50\%$ and only $\sigma_e \approx 10^{-7}$ (and less) preserves the line shape and the intensity. Thus for the effective radiation of the 3rd harmonic independently of the constant field broadening contribution, the quality of the undulator itself and of the beam should be high enough to allow for the value of σ_e at least one order smaller than that needed for the main harmonic radiation (see [27] to compare).

Broadening effects for higher harmonics are even more evident. At the same time, the energy spread does not change the UR line of the first 2 harmonics of the undulator with $N = 100$ and $k = 2$ in the presence of a constant field and just reduces the fundamental harmonic intensity by less than 10%. The 2nd harmonic emission is totally determined by the intensity of H_d. For $N = 100$ and $H_d = 2 \cdot 10^{-4} H_0$ it is as weak as $\approx 1\%$ of the main harmonic of the undulator. We omit proper figures for brevity.

Eventually, we study the dependence of the harmonic intensity on the number of periods in undulators N. Calculating (9) for ν_n and for N, we can make an optimal choice for the undulator in terms of maximum output of a certain harmonic, respectively to another one, and obtain best emission intensity and minimal line broadening. The example of the 3rd harmonic intensity, factorized by $(5 \cdot 10^3 e^2\gamma^2/c)^{-1}$, is given in Fig. 6.

The 3rd harmonic growth, presented in Fig. 6, gives evidence that in order to exploit it in the undulator with $k = 2$, where the constant magnetic field constituent $H_d = H_0 \cdot 10^{-4}$ is present and $\sigma_e = 10^{-6}$ is the value of a beam energy spread, $N \approx 100$ is the preferable choice. Indeed, the intensity

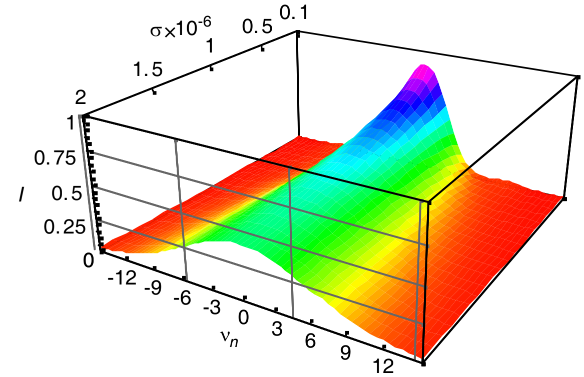

Figure 5. Pure inhomogeneous effects of the 3rd harmonic of the undulator with $k = 2$, $N = 100$.

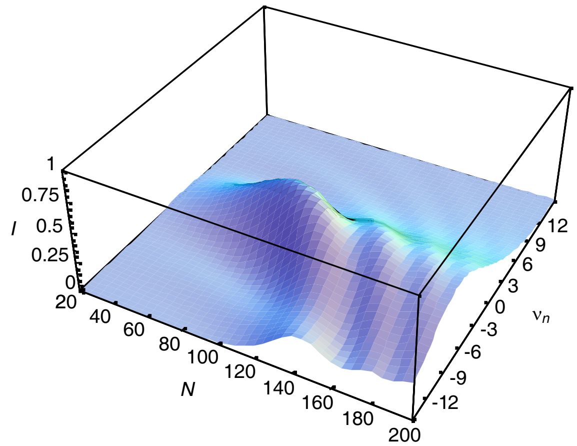

Figure 6. Broadening of the spectral line of the 3rd ($n = 3$) harmonic of the undulator with $k = 2$ with the energy spread $\sigma_e = 10^{-6}$ and $\kappa = 10^{-4}$ (due to constant magnetic field).

has a maximum at $N \approx 100$ and any further increase of N just broadens the line. To give an idea of the value of the total losses for the 3rd harmonic, we can refer to [32] and proper figures, where the beam transport and energy losses are zero. The ideal intensity there is ≈ 1.5 and ≈ 6 times higher for $N = 100$ and $N = 200$ periods respectively. This study complements the discussion of the UR spectrum features in an extra constant magnetic field, presented in the previous Sections, and demonstrates how the UR intensity should be carefully evaluated with account for non-periodic magnetic fields, which may appear in undulators either due to structural imperfections, manufacturing defects or magnetic fields external to an undulator, like that of the Earth.

5. CONCLUSIONS

We have analyzed the contribution of various sources of the undulator line broadening to the total UR intensity, employing the technique of generalized special functions and accounting for the broadening effects in an analytical form. The obtained expressions allow for clear distinction of the terms, responsible for spectrum modification, broadening and shift. In particular, we have demonstrated that a constant magnetic component of a proper intensity and space orientation can compensate for negative off-axis effects. Analytical expressions were obtained, examples of some undulators considered and values of the field intensity, best for such compensation, were derived. If too strong, it impairs the coherency of electron oscillations in an undulator and deteriorates the UR emission. It appeared that the Earth can induce a magnetic field of comparable order, which, however, is partially screened out by the undulator itself. Odd harmonics appear on the axis, but their intensity remains less than 1% of the intensity of even harmonics. We have obtained analytical expressions for the spectrum shift and emission line broadening due to a constant magnetic field and obtained the ratio of constant and periodic field intensities $\kappa_{\max} = H_d/H_0$, under which the line shape is preserved.

We also compared the effect of a constant magnetic constituent with the focusing effects in undulators, included the UR line broadening due to the electron beam energy spread, the emittance of the beam and other losses. Several examples of undulators were studied. We found that in long undulators the effect of the constant field can reach and even exceed that of the above mentioned factors, and at the same time, the Earth also creates a non-negligible disturbance. The UR characteristics depend on the absolute value of the constant magnetic field constituent κ_1 in the plane transversal to the undulator axis. Longitudinal component of the magnetic field was demonstrated to be not important for high energy beam undulators; in this case external to the undulator harmful constant magnetic component can be redirected along the undulator axis, and its influence, if any, virtually eliminated.

Dependently on the undulator characteristics, an increase of the number of periods above $N \approx 100 \div 150$ (depends on the undulator parameters and the constant field strength), may give no gain for high harmonics in particular, but may lead to serious line broadening. In long undulators, higher harmonics are especially subjected to broadening effects. It may be considered as negative feature, but, also, as positive, when high harmonics are undesired. Proper choice of the field strength, values of k and N, can reduce high harmonics radiation several times. It may be important for some FEL applications, where hard components of radiation damage the mirrors. Limitations of the harmonics gain are also important in FEL applications, where high gain is essential, e.g., in SASE schemes and in HGHG FEL. They may involve two-frequency undulators for high harmonic generation. If the periodic field H_2 with the period λ_2 in such devices is significantly weaker than the main field H_1, then the planned effect of the field H_2 may be reduced and even eliminated by the constant magnetic constituent $H_d \approx 10^{-4} H_2$. Moreover, for non sinusoidal periodic fields in an undulator, higher terms of their Fourier expansions, being smaller than the main terms, can be masked by a constant magnetic field, which is likely to appear in complex magnetic structures. In this context UR can be used as a control tool for the undulator device itself, when intended as a part of a FEL system. Our study also demonstrates that particular attention should be paid to inhomogeneous effects, which, together with extra fields at least as strong as 10^{-4} of the periodic field, can equalize even significant theoretical gain of a high harmonic with the radiation of the fundamental frequency in long undulators.

We hope that the results of our research reported above can find direct application in studies of the performance of synchrotron insertion devices and used for tuning of SR, UR sources and FEL devices.

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