Complex Resonances of a Rectangular Patch in a Multilayered Medium: A New Accurate and Efficient Analytical Technique

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Abstract—A new analytical technique to study the complex resonances of a rectangular patch in a multilayered medium is introduced. The problem is formulated as an electric field integral equation (EFIE) in the spectral domain and discretized by means of products of Chebyshev polynomials of first and second kind multiplied by their orthogonal weights in a Galerkin's scheme. The method is fast convergent, i.e., few expansion functions are needed to achieve accurate results, but leads to the numerical evaluation of infinite double integrals of oscillating and slowly decaying functions. To overcome this problem, suitable half-space contributions are pulled out of the kernels of such integrals in order to obtain exponentially decaying integrands. Moreover, the slowly converging integrals of the extracted contributions are expressed as combinations of quickly converging integrals by means of algebraic manipulations and an appropriate integration procedure in the complex plane.

1. INTRODUCTION

For several decades, the study of microwave antennas and resonators based on microstrip technology has been an important issue. This can be explained by their conformability and minimized shape, low weight and cost, and due to the high number of applications including, just for examples, mobile satellite communications, direct broadcast satellite services and non-destructive testing sensors in permittivity and porosity measurements [1–6].

Furthermore, the use of multilayered dielectric substrates allows to obtain many advantages as evidenced by the many works devoted to this subject [7–19]. It has been verified that multilayered dielectric substrates can be used to reduce the radiation losses by surface waves to enhance the efficiency. On the other hand, a dielectric substrate placed on top of a microstrip antenna can provide protection against environmental hazards and, appropriately located, can increase the gain of the antenna. Moreover, multilayered microstrip circuits allow for more versatile designs and offer the advantage of greater compactness.

What emerges from the literature devoted to analyze microstrip circuits and antennas in multilayered media, is that full-wave methods [20–22], taking into account the effects of the electromagnetic coupling, surface waves, and radiation loss, are more accurate than traditional quasi-static methods [23] and equivalent waveguide models [24]. Moreover, the most preferred full-wave techniques to analyze non-shielded structures are integral equation formulations, allowing to express the fields as functions of unknowns on finite support, discretized by means of the variational method of moments.

Among them, the method introduced in [14, 25–39] to analyse propagation, radiation and scattering by polygonal cross-section cylinders and one-dimensional perfectly conducting plates (disks, annular rings, ...) in a homogeneous or a layered medium has been shown to be a very accurate and efficient approach. The problem is formulated in terms of surface integral equations in the spectral domain

and discretized by means of Galerkin's method with analytically Fourier transformable orthonormal expansion basis reconstructing the behaviour of the fields at the edges [40]. Moreover, suitable analytical acceleration techniques to fast evaluate the obtained improper integrals of oscillating and slowly decaying functions has been developed. Even if this method can be immediately extended to the analysis of the radiation and the scattering by a perfectly conducting rectangular plate in a homogeneous or a layered medium, the proposed acceleration techniques reveals to be not effective when applied to the resulting infinite double integrals of oscillating and slowly decaying functions. In order to overcome this problem, in [41] the range of integration has been subdivided into annular sectors and the region external to the maximum sector divided into three subregions over which the generic integrand has been approximated with a suitable asymptotic behaviour so to reduce the corresponding double integral to a single one. Recently, a new technique, drastically outperforming the acceleration technique presented in [41], has been introduced in [42]: a new quickly converging representation of the infinite double integrals resulting from the analysis of the scattering by a rectangular plate in a homogeneous medium has been obtained by means of a suitable integration procedure in the complex plane. Unfortunately, in case of a multilayered medium, such technique cannot be immediately applied due to the oscillating nature of the Green's functions of the problem.

The aim of this paper is the introduction of a new accurate and efficient analytical technique to study the complex resonances of a rectangular patch in a multilayered medium. The problem formulated as an EFIE in the spectral domain is discretized by means of products of Chebyshev polynomials of first and second kind multiplied by their orthogonal weights in a Galerkin's scheme. In order to fast evaluate the obtained infinite double integrals of oscillating and slowly decaying functions, suitable half-space contributions are pulled out of the kernels of such integrals obtaining exponentially decaying integrands. Moreover, the slowly converging integrals of the extracted contributions are expressed as combinations of proper and fast converging improper integrals of non-oscillating and exponentially decaying functions by means of suitable algebraic manipulations and using an appropriate integration procedure in the complex plane generalizing the one proposed in [42].

In Sections 2 the formulation of the problem and the discretization of the integral equations are presented. The new acceleration technique is illustrated in Section 3. Section 4 is devoted to show the accuracy and the efficiency of the presented technique and the conclusions are summarized in Section 5.

2. FORMULATION AND SOLUTION OF THE PROBLEM

The geometry of the problem is sketched in Figure 1. A planar layered medium of $L+1$ homogeneous and isotropic layers of dielectric permittivity $\varepsilon_q = \varepsilon_0 \varepsilon_{rq}$, magnetic permeability $\mu_q = \mu_0 \mu_{rq}$ and wave number $k_q = \omega \sqrt{\varepsilon_q \mu_q}$ with $q \in \{1, \ldots, L+1\}$ is represented, where ε_0 and μ_0 are the dielectric permittivity and
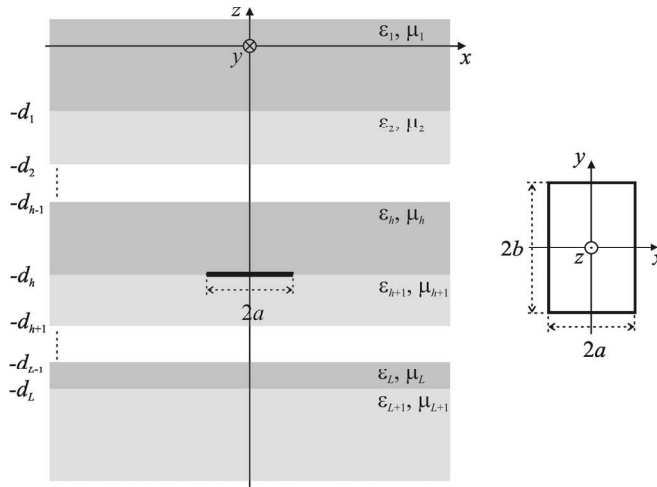

Figure 1. Geometry of the problem.

the magnetic permeability of the vacuum and ω is the angular frequency. Moreover, a coordinate system with the z axis orthogonal to the discontinuity surfaces is introduced, and at the interface $z = -d_q$, a perfectly conducting rectangular patch of dimensions $2a$ and $2b$, with the sides parallel to the x and y axes and the centre at the point $x = y = 0$, is located.

Resonant solutions can be obtained by imposing the transversal component of the electric field with respect to the z axis to be vanishing on the patch's surface, i.e., [43]

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underline{\tilde{\underline{G}}}^{(q)}(k_x, k_y) \cdot \underline{\tilde{J}}(k_x, k_y) e^{-j(k_x x + k_y y)} dk_x dk_y = 0 \tag{1}$$

with $|x| \le a$ and $|y| \le b$, where $\underline{\tilde{J}}(\cdot)$ is the double Fourier transform with respect to the x and y axes of the surface current density on the rectangular patch and [44, 45]

$$\underline{\tilde{\underline{G}}}^{(q)}(k_x, k_y) = \frac{1}{8\pi^2\left(k_x^2 + k_y^2\right)}\left[\frac{\underline{\underline{M}}^{TE} Z_q^{TE}\left(1 + e_q^2\tilde{R}_{q,-}^{TE}\right)\left(1 + \tilde{R}_{q,+}^{TE}\right)}{1 - \tilde{R}_{q,+}^{TE}\tilde{R}_{q,-}^{TE}e_q^2} - \frac{\underline{\underline{M}}^{TM} Z_q^{TM}\left(1 - e_q^2\tilde{R}_{q,-}^{TM}\right)\left(1 - \tilde{R}_{q,+}^{TM}\right)}{1 - \tilde{R}_{q,+}^{TM}\tilde{R}_{q,-}^{TM}e_q^2}\right] \tag{2a}$$

$$\underline{\underline{M}}^{TE} = \begin{pmatrix} -k_y^2 & k_x k_y \\ k_x k_y & -k_x^2 \end{pmatrix}, \tag{2b}$$

$$\underline{\underline{M}}^{TM} = \begin{pmatrix} k_x^2 & k_x k_y \\ k_x k_y & k_y^2 \end{pmatrix}, \tag{2c}$$

$$Z_q^{TE} = \frac{\omega\mu_q}{k_{q,z}}, \tag{2d}$$

$$Z_q^{TM} = \frac{k_{q,z}}{\omega\varepsilon_q}, \tag{2e}$$

$$\tilde{R}_{1,-}^{TE,TM} = \tilde{R}_{L+1,+}^{TE,TM} = 0, \tag{2f}$$

$$\tilde{R}_{q,\pm}^{TE,TM} = \frac{R_{q,\pm}^{TE,TM} + \tilde{R}_{q\pm1,\pm}^{TE,TM} e_{q\pm1}^2}{1 + R_{q,\pm}^{TE,TM} \tilde{R}_{q\pm1,\pm}^{TE,TM} e_{q\pm1}^2}, \tag{2g}$$

$$R_{1,-}^{TE,TM} = R_{L+1,+}^{TE,TM} = 0, \tag{2h}$$

$$R_{q,\pm}^{TE} = \frac{\mu_{q\pm1}k_{q,z} - \mu_q k_{q\pm1,z}}{\mu_{q\pm1}k_{q,z} + \mu_q k_{q\pm1,z}}, \tag{2i}$$

$$R_{q,\pm}^{TM} = \frac{\varepsilon_{q\pm1}k_{q,z} - \varepsilon_q k_{q\pm1,z}}{\varepsilon_{q\pm1}k_{q,z} + \varepsilon_q k_{q\pm1,z}}, \tag{2j}$$

$$e_q = e^{-jk_{q,z}(d_q - d_{q-1})}, \tag{2k}$$

$$k_{q,z} = \sqrt{k_q^2 - k_y^2 - k_x^2}. \tag{2l}$$

The system of integral Equation (1) can be reduced to a symmetric matrix equation by means of Galerkin's method with products of Chebyshev polynomials of first and second kind multiplied by their orthogonal weights, i.e., factorizing the behaviour of the surface current density at the edges [40], as expansion functions. Therefore, the elements of the coefficient matrix are proportional to the following double integrals

$$M_{rs_{n,h,m,k}}^{(q)} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{G}_{rs}^{(q)}(k_x, k_y) \, \tilde{\psi}_{sr_{n,h,m,k}}(ak_x, bk_y) \, dk_x dk_y \tag{3}$$

with $r, s \in \{x, y\}$ and n, h, m, k nonnegative integers, where

$$\tilde{\psi}_{sr_{n,h,m,k}}(\alpha, \beta) = \tilde{\varphi}_{s_{n,h}}(\alpha, \beta) \, \tilde{\varphi}_{r_{m,k}}(-\alpha, -\beta), \tag{4a}$$

$$\tilde{\varphi}_{s_{n,h}}(\alpha, \beta) = \frac{J_{n+p_s}(\alpha)}{\alpha^{p_s}}\frac{J_{h+1-p_s}(\beta)}{\beta^{1-p_s}} \tag{4b}$$

is proportional to the Fourier transform of the generic expansion function, $p_x = 1$, $p_y = 0$, and $J_\nu(\cdot)$ is the Bessel function of first kind and order ν. Moreover, the complex resonant frequencies of the problem can be readily obtained by enforcing the determinant of the truncated coefficient matrix to be zero.

3. A NEW ACCELERATION TECHNIQUE TO EFFICIENTLY EVALUATE THE ELEMENTS OF THE COEFFICIENTS MATRIX

As a first task, by means of the change of variables $k_x = \rho c_\psi$ and $k_y = \rho s_\psi$, where $c_\psi = \cos\psi$ and $s_\psi = \sin\psi$, it is simple to rewrite (3) as follows

$$M^{(q)}_{rs_{n,h,m,k}} = \int_0^{2\pi} \int_0^{+\infty} \tilde{G}^{(q)}_{rs}(\rho c_\psi, \rho s_\psi) \, \tilde{\psi}_{sr_{n,h,m,k}}(a\rho c_\psi, b\rho s_\psi) \, \rho d\rho d\psi. \tag{5}$$

Observing that [46]

$$J_\nu(z) \overset{|z|\to+\infty}{\sim} \sqrt{\frac{2}{\pi z}} \cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) \quad \text{for} -\pi < \arg(z) < \pi, \tag{6}$$

it can be concluded that the integrand of the integral in (5) is an oscillating function which decays asymptotically as $1/\rho^2$. Therefore, the computation time rapidly increases with the accuracy required for the solution.

In order to overcome this problem, a new acceleration technique is proposed here.

The expression in (2a) can be reduced to the following non-oscillating function by posing $(\varepsilon_l, \mu_l) = (\varepsilon_q, \mu_q)$ for $l \in \{1, \ldots, q-1\}$ and $(\varepsilon_l, \mu_l) = (\varepsilon_{q+1}, \mu_{q+1})$ for $l \in \{q+2, \ldots, L+1\}$, i.e., when only two half-spaces are involved,

$$\underline{\underline{\tilde{G}}}'^{(q)}(k_x, k_y) = \frac{\underline{\underline{M}}^{TE} Z_q^{TE}\left(1 + R_{q,+}^{TE}\right) - \underline{\underline{M}}^{TM} Z_q^{TM}\left(1 - R_{q,+}^{TM}\right)}{8\pi^2\left(k_x^2 + k_y^2\right)}. \tag{7}$$

It is simple to verify that the difference between the functions in (2a) and (7) is

$$\begin{aligned}
\underline{\underline{\tilde{G}}}''^{(q)}(k_x, k_y) &= \underline{\underline{\tilde{G}}}^{(q)}(k_x, k_y) - \underline{\underline{\tilde{G}}}'^{(q)}(k_x, k_y) \\
&= \frac{1}{8\pi^2\left(k_x^2 + k_y^2\right)} \cdot \left[e_q^2\left(\underline{\underline{M}}^{TE} Z_q^{TE} G_{q,-}^{TE} + \underline{\underline{M}}^{TM} Z_q^{TM} G_{q,-}^{TM}\right)\right. \\
&\quad \left. + e_{q+1}^2\left(\underline{\underline{M}}^{TE} Z_q^{TE} G_{q,+}^{TE} + \underline{\underline{M}}^{TM} Z_q^{TM} G_{q,+}^{TM}\right)\right],
\end{aligned} \tag{8}$$

where

$$G_{q,-}^{TE,TM} = \frac{\tilde{R}_{q,-}^{TE,TM}\left(1 \pm 2\tilde{R}_{q,+}^{TE,TM} + R_{q,+}^{TE,TM}\tilde{R}_{q,+}^{TE,TM}\right)}{1 - \tilde{R}_{q,+}^{TE,TM}\tilde{R}_{q,-}^{TE,TM}e_q^2}, \tag{9a}$$

$$G_{q,+}^{TE,TM} = \frac{\tilde{R}_{q+1,+}^{TE,TM}\left(1 - R_{q,+}^{TE,TM}\tilde{R}_{q,+}^{TE,TM}\right)}{1 - \tilde{R}_{q,+}^{TE,TM}\tilde{R}_{q,-}^{TE,TM}e_q^2}, \tag{9b}$$

and the upper sign has to be taken for the TE case whilst the lower sign has to be taken for the TM case.

Therefore, by rewriting the double integral in (5) as

$$\begin{aligned}
M^{(q)}_{rs_{n,h,m,k}} &= M'^{(q)}_{rs_{n,h,m,k}} + M''^{(q)}_{rs_{n,h,m,k}} \\
&= \int_0^{2\pi} \int_0^{+\infty} \tilde{G}'^{(q)}_{rs}(\rho c_\psi, \rho s_\psi) \, \tilde{\psi}_{sr_{n,h,m,k}}(a\rho c_\psi, b\rho s_\psi) \, \rho d\rho d\psi \\
&\quad + \int_0^{2\pi} \int_0^{+\infty} \tilde{G}''^{(q)}_{rs}(\rho c_\psi, \rho s_\psi) \, \tilde{\psi}_{sr_{n,h,m,k}}(a\rho c_\psi, b\rho s_\psi) \, \rho d\rho d\psi
\end{aligned} \tag{10}$$

and remembering (6), it can be concluded that the integrand of the second double integral (integral of the accelerated contribution) has an exponential asymptotic decay of the kind $e^{-2\rho\Delta_q}/\rho^2$ where

$$\Delta_q = \begin{cases} d_2 - d_1 & \text{for } q = 1 \\ \min\{d_q - d_{q-1}, d_{q+1} - d_q\} & \text{for } q \in \{2, \ldots, L-1\} \\ d_L - d_{L-1} & \text{for } q = L \end{cases} . \tag{11}$$

On the other hand, as can be shown to follow, the slowly converging first double integral in (10) (integral of the extracted contribution) can be rewritten as a combination of very quickly converging integrals taking advantage of the non-oscillating nature of the kernel.

Firstly, by means of algebraic manipulations, the recurrence formula [46]

$$2\nu J_\nu(z) = z\left[J_{\nu-1}(z) + J_{\nu+1}(z)\right], \tag{12}$$

and the property [46]

$$J_\nu\left(ze^{jp\pi}\right) = e^{jp\nu\pi} J_\nu(z) \tag{13}$$

with p integer, it is possible to rewrite the first integral in (10) as follows

$$M'^{(q)}_{xx_{n,h,m,k}} = \frac{\left(\mu_q^2 - \mu_{q+1}^2\right) I^{(q)}_{\mu_{n+1,h,m+1,k}} + \left(\varepsilon_{q+1}\mu_{q+1} - \varepsilon_q\mu_q\right) I^{(q)}_{\varepsilon_{n+1,h,m+1,k}}}{4\pi^2\omega a^2 \left(\varepsilon_q\mu_{q+1} - \varepsilon_{q+1}\mu_q\right)}$$
$$- \frac{\omega\mu_q\mu_{q+1}\left(I^{(q)}_{\mu_{n,h,m,k}} + I^{(q)}_{\mu_{n,h,m+2,k}} + I^{(q)}_{\mu_{n+2,h,m,k}} + I^{(q)}_{\mu_{n+2,h,m+2,k}}\right)}{16\pi^2 (n+1)(m+1)}, \tag{14a}$$

$$M'^{(q)}_{xy_{n,h,m,k}} = M'^{(q)}_{yx_{m,k,n,h}} = \frac{\left(\mu_q^2 - \mu_{q+1}^2\right) I^{(q)}_{\mu_{n,h+1,m+1,k}} + \left(\varepsilon_{q+1}\mu_{q+1} - \varepsilon_q\mu_q\right) I^{(q)}_{\varepsilon_{n,h+1,m+1,k}}}{4\pi^2\omega ab \left(\varepsilon_q\mu_{q+1} - \varepsilon_{q+1}\mu_q\right)}, \tag{14b}$$

$$M'^{(q)}_{yy_{n,h,m,k}} = \frac{\left(\mu_q^2 - \mu_{q+1}^2\right) I^{(q)}_{\mu_{n,h+1,m,k+1}} + \left(\varepsilon_{q+1}\mu_{q+1} - \varepsilon_q\mu_q\right) I^{(q)}_{\varepsilon_{n,h+1,m,k+1}}}{4\pi^2\omega b^2 \left(\varepsilon_q\mu_{q+1} - \varepsilon_{q+1}\mu_q\right)}$$
$$- \frac{\omega\mu_q\mu_{q+1}\left(I^{(q)}_{\mu_{n,h,m,k}} + I^{(q)}_{\mu_{n,h,m,k+2}} + I^{(q)}_{\mu_{n,h+2,m,k}} + I^{(q)}_{\mu_{n,h+2,m,k+2}}\right)}{16\pi^2 (h+1)(k+1)}, \tag{14c}$$

where

$$I^{(q)}_{\eta_{n,h,m,k}} = \left[(-1)^n + (-1)^m\right]\left[(-1)^h + (-1)^k\right] \int_0^{\pi/2} \int_0^{+\infty} F^{(q)}_\eta\left(\rho^2\right) f_{n,h,m,k}\left(a\rho c_\psi, b\rho s_\psi\right) \rho\, d\rho\, d\psi, \tag{15a}$$

$$f_{n,h,m,k}(\alpha, \beta) = J_n(\alpha) J_h(\beta) J_m(\alpha) J_k(\beta), \tag{15b}$$

$$F^{(q)}_\eta\left(\rho^2\right) = \frac{1}{\eta_{q+1}\sqrt{k_q^2 - \rho^2} + \eta_q\sqrt{k_{q+1}^2 - \rho^2}} \tag{15c}$$

with $\eta \in \{\varepsilon, \mu\}$. It is immediate to establish that $I^{(q)}_{\eta_{n,h,m,k}} = 0$ for $n+m$ or $h+k$ odd, and $I^{(q)}_{\eta_{n,h,m,k}} = I^{(q)}_{\eta_{n,k,m,h}} = I^{(q)}_{\eta_{m,h,n,k}} = I^{(q)}_{\eta_{m,k,n,h}}$ that allow to consider only the cases for $n+m$ and $h+k$ even, with $n \geq m$ and $h \geq k$ (henceforth, these assumptions will be implicitly done).

Let us focus on the single integral over the semi-infinite range in (15a) that can be rewritten as follows

$$\int_0^{+\infty} F^{(q)}_\eta\left(\rho^2\right) f_{n,h,m,k}\left(a\rho c_\psi, b\rho s_\psi\right) \rho\, d\rho$$
$$= \int_0^d F^{(q)}_\eta\left(\rho^2\right) f_{n,h,m,k}\left(a\rho c_\psi, b\rho s_\psi\right) \rho\, d\rho + \int_d^{+\infty} F^{(q)}_\eta\left(\rho^2\right) f_{n,h,m,k}\left(a\rho c_\psi, b\rho s_\psi\right) \rho\, d\rho, \tag{16}$$

where the choice of $d > \max\{k_q, k_{q+1}\}$ has been discussed in [42]. The first integral at the second member of the previous formula is a proper integral. Moreover, supposing that $ac_\psi \geq bs_\psi$ (analogous considerations can be done for $ac_\psi < bs_\psi$ by reversing the role of the sides), the following alternative expression for the second integral in (16) can be found by generalizing the integration procedure in the complex plane detailed in [42]

$$
\int\limits_d^{+\infty} F_\eta^{(q)}\left(\rho^2\right) f_{n,h,m,k}\left(a\rho c_\psi, b\rho s_\psi\right) \rho d\rho = j \int\limits_d^{+\infty} F_\eta^{(q)}\left(-\sigma^2\right) g_{n,h,m,k}\left(a\sigma c_\psi, b\sigma s_\psi\right) \sigma d\sigma
$$

$$
+ j \int\limits_0^{\pi/2} \Re\left\{ F_\eta^{(q)}\left(d^2 e^{2jt}\right) h_{n,h,m,k}\left(ade^{jt}c_\psi, bde^{jt}s_\psi\right) d^2 e^{2jt} \right\} dt, \quad (17)
$$

where

$$
g_{n,h,m,k}\left(\alpha, \beta\right) = \frac{4}{\pi^3}\left(-1\right)^{(n-h+m+k)/2} K_n\left(\alpha\right) I_h\left(\beta\right) K_m\left(\alpha\right) K_k\left(\beta\right), \tag{18a}
$$

$$
h_{n,h,m,k}\left(\alpha, \beta\right) = \left(h_{n,h,m,k}^{(1)}\left(\alpha, \beta\right) + h_{n,h,m,k}^{(2)}\left(\alpha, \beta\right) \right)\Big/2, \tag{18b}
$$

$$
h_{n,h,m,k}^{(1)}\left(\alpha, \beta\right) = J_n\left(\alpha\right) J_h\left(\beta\right) H_m^{(1)}\left(\alpha\right) H_k^{(1)}\left(\beta\right), \tag{18c}
$$

$$
h_{n,h,m,k}^{(2)}\left(\alpha, \beta\right) = H_n^{(1)}\left(\alpha\right) J_h\left(\beta\right) \left[J_m\left(\alpha\right) J_k\left(\beta\right) + Y_m\left(\alpha\right) Y_k\left(\beta\right)\right], \tag{18d}
$$

$I_\nu(\cdot)$ and $K_\nu(\cdot)$ are the modified Bessel functions of first and second kind and order ν respectively, $Y_\nu(\cdot)$ is the Bessel function of second kind and order ν, $H_\nu^{(1)}(\cdot) = J_\nu(\cdot) + jY_\nu(\cdot)$ is the Hankel function of first kind and order ν, and $\Re\{\cdot\}$ denotes the real part of a complex number. Observing that [46]

$$
I_\nu\left(z\right) \overset{|z| \to +\infty}{\sim} \frac{e^z}{\sqrt{2\pi z}} \qquad \text{for } -\frac{\pi}{2} < \arg\left(z\right) < \frac{\pi}{2}, \tag{19a}
$$

$$
K_\nu\left(z\right) \overset{|z| \to +\infty}{\sim} \sqrt{\frac{2}{\pi z}} e^{-z} \qquad \text{for } -\frac{3\pi}{2} < \arg\left(z\right) < \frac{3\pi}{2}, \tag{19b}
$$

it is possible to conclude that the first integral at the second member of (17) is an improper integral of a non-oscillating function which decays asymptotically as $e^{-2a\sigma c_\psi}/\sigma^2$, while the last one is a proper integral.

Therefore, the integral at the first member of (16) (and, then, the double integral in (15a)) has been expressed as a combination of quickly converging integrals.

4. NUMERICAL RESULTS

The aim of this section is to show the accuracy and the efficiency of the presented technique in terms of calculation time and storage requirement.

It is worth noting that more than 20 integrals per second can be computed by means of an adaptive Gauss-Legendre cubature routine on a laptop equipped with an Intel Core 2 Duo CPU T9600 2.8 GHz, 3 GB RAM, running Windows XP.

The overall number of matrix coefficients is $4N^2M^2$, where N and M are the number of expansion functions used along the x and y axes respectively for each component of the surface current density. However, due to the properties detailed above, the number of integrals of the form $M''^{(q)}_{rs_{n,h,m,k}}$, $I^{(q)}_{\varepsilon_{n,h,m,k}}$ and $I^{(q)}_{\mu_{n,h,m,k}}$ that has to be computed is respectively reduced to $NM(2NM+1)$, $N^+N^-M^+M^-/16$ and $N^+(N^-+2)M^+(M^-+2)/16$, where $P^\pm = P + 2 \pm \mod(P,2)$ with $P \in \{N, M\}$ and $\mod(\cdot,\cdot)$ is the modulus operation.

Moreover, few expansion functions are needed to achieve accurate results. It can be appreciated in Table 1 where the complex resonant frequency of the square patch in a grounded one-layer dielectric medium sketched in Figure 2(a) is shown as a function of the number of expansion functions used

Table 1. Complex resonant frequency of the square patch in a grounded one-layer medium sketched in Figure 2(a) for different numbers of basis functions used. $2a = 2b = 10\,\mathrm{mm}$, $\varepsilon_{r1} = 1$, $\varepsilon_{r2} = 1.046$, $d_2 - d_1 = 0.98\,\mathrm{mm}$.

$N = M$	f_r (GHz)
1	-
2	$12.50867 + j0.416032$
3	$12.49978 + j0.441532$
4	$12.49636 + j0.441182$
5	$12.49576 + j0.441112$
6	$12.49522 + j0.441061$
7	$12.49491 + j0.441032$
8	$12.49474 + j0.441015$
9	$12.49463 + j0.441004$
10	$12.49456 + j0.440997$
11	$12.49451 + j0.440992$
12	$12.49447 + j0.440989$
13	$12.49445 + j0.440987$
14	$12.49444 + j0.440986$
15	$12.49444 + j0.440986$

Table 2. Resonant frequencies of the square patch in a grounded two-layer dielectric medium sketched in Figure 2(b). $2a = 2b = 10\,\mathrm{mm}$, $\varepsilon_{r1} = 1$, $\varepsilon_{r3} = 1.046$, $d_3 - d_2 = 0.98\,\mathrm{mm}$.

ε_{r2}	$d_2 - d_1$ (mm)	Li et al. [18]			This method		
		f_r (GHz)	*Deviation* (%)		f_r (GHz)	*Deviation* (%)	
			HFSS	*Meas.*		*HFSS*	*Meas.*
1	-	12.333	-0.54	0.06	$12.495(+j0.441)$	0.77	1.38
3	0.508	11.259	-0.58	-0.58	$11.356(+j0.354)$	0.27	0.27
3.27	0.508	11.195	0.18	-0.04	$11.245(+j0.347)$	0.63	0.40
4.5	0.508	10.921	1.59	-0.26	$10.800(+j0.317)$	0.47	-1.37
6	0.635	10.502	4.76	1.96	$10.099(+j0.279)$	0.74	-1.95
6.15	0.635	10.471	4.45	3.67	$10.055(+j0.277)$	0.30	-0.45
9.2	0.635	9.930	7.35	4.25	$9.292(+j0.233)$	0.45	-2.45
9.8	0.635	9.838	7.52	3.83	$9.165(+j0.226)$	0.16	-3.27
10.2	0.635	9.780	7.77	7.77	$9.085(+j0.221)$	0.11	0.11

supposing that $N = M$. It is clear as the accuracy on the real part of the resonant frequency reaches 3, 4, 5 and 6 significant figures by using, respectively, only 2, 7, 12 and 14 expansion functions along the x and y axes for each current component.

To conclude, in Tables 2 and 3, the resonant frequencies of the square patches in grounded two-layer and three-layer dielectric media sketched in Figures 2(b) and 2(c) are shown. The results obtained with this method (by using only 6 expansion functions along the x and y axes for each current component) and by means of a suitable application of the conformal mapping technique [18] are compared with the experimental data and the results achieved by using the commercial software HFSS. As can be seen, the presented method results to be more accurate than the formulas in [18]. Indeed, the resonant frequencies obtained with the presented method differ always less than 1% from the ones obtained by using HFSS, and only in two cases differ more than 2% and 3% from the experimental data (although it should be noted that, in such cases, the agreement with HFSS is excellent). Conversely, the percentage deviation of the resonant frequencies obtained with the formulas in [18] from those provided by HFSS and the experimental data is strictly dependent on the dielectric permittivities of the layers (just for an example, the percentage deviation showed in Table 2 increases more and more as ε_{r2} increases).

Table 3. Resonant frequencies of the square patch in a grounded three-layer dielectric medium sketched in Figure 2(c). $2a = 2b = 10$ mm, $\varepsilon_{r1} = 1$, $\varepsilon_{r4} = 1.046$, $d_4 - d_3 = 0.99$ mm.

| ε_{r2}	$d_2 - d_1$ (mm)	ε_{r3}	$d_3 - d_2$ (mm)	Li et al. [18]			This method		
				f_r (GHz)	Deviation (%)		f_r (GHz)	Deviation (%)	
					HFSS	Meas.		HFSS	Meas.
6	0.635	3.27	0.508	9.963	-3.74	-2.83	$10.060(+j0.340)$	-0.10	0.85
9.2	0.635	3.27	0.508	9.408	-2.25	-0.44	$9.550(+j0.345)$	-0.78	1.06
3.27	0.508	6	0.635	9.437	-2.46	-2.71	$9.687(+j0.273)$	0.12	-0.13
9.2	0.635	6	0.635	8.494	-3.75	-4.56	$8.745(+j0.277)$	-0.91	-1.74
3.27	0.508	9.2	0.635	8.922	-0.31	-0.87	$8.952(+j0.226)$	0.02	-0.53
6	0.635	9.2	0.635	8.447	-1.20	-1.78	$8.519(+j0.224)$	-0.36	-0.94
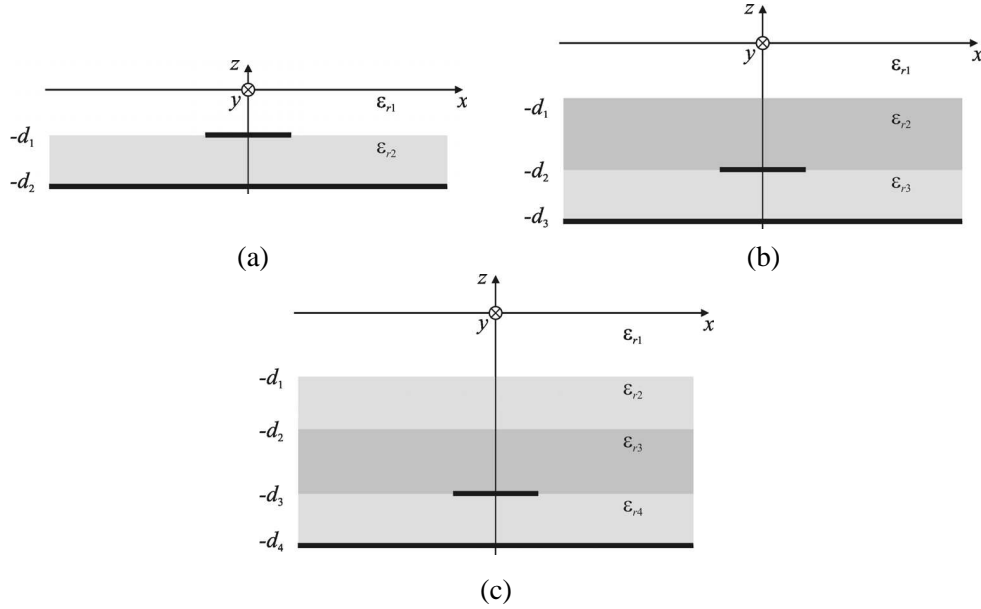
Figure 2. Geometries analyzed throughout the paper. Square patches in a grounded (a) one-layer, (b) two-layer, and (c) three layer dielectric media.

5. CONCLUSION

In this paper, a new accurate and efficient technique to analyze the complex resonances of a rectangular patch in a multilayered medium has been introduced. Future perspective is the generalization of the method to cases in which anisotropic media are involved.

ACKNOWLEDGMENT

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