Progress In Electromagnetics Research M, Vol. 33, 185–196, 2013

A STUDY ON THE NUMERICAL ACCURACY OF THE MATRIX ELEMENTS IN A TIME DOMAIN MOD METHODOLOGY

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Abstract—In a time domain Marching-on-in-degree (MOD) solver based on a Galerkin implementation of the Method of Moments (MoM), it is observed that the matrix elements for the matrix to be inverted contain integrals that are similar to the ones encountered in a frequency domain MoM solver using the piecewise triangular patch basis functions. It is also observed that the error in the evaluation of the matrix elements involving these integrals are larger in the time domain than those involved in the frequency domain MoM solvers. The objective of this paper is to explain this dichotomy and how to improve upon them when using the triangular patch basis functions for both the time and the frequency domain techniques. When the distance between the two triangular patches involved in the evaluation of the matrix elements, are close to each other or when the degree of the Laguerre polynomial in a MOD method is high, the integral accuracy will be compromised and the number of sampling points to evaluate the integrals need to be increased. Numerical results are presented to illustrate this point.

Received 3 July 2013, Accepted 16 October 2013, Scheduled 17 October 2013

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1. INTRODUCTION

The time domain marching-on-in-degree (MOD) method has been used for the solution of the transient Method of Moments (MoM) problems [1–11]. In the MOD method, the unknown variables, such as the current or the potential functions related to the integral equation associated with the problem of interest are expanded by a set of both spatial and temporal basis functions. The spatial basis functions are generally chosen as the piecewise triangular patch functions known as the RWG basis [12, 13] whereas the temporal basis functions are chosen as the Laguerre functions in a Galerkin methodology. In a Galerkin time domain methodology in the MoM context, associated with the MOD method, the time variable is analytically integrated out. So the final equations that are used in the computations have only the spatial variables. In that context, the expressions for the matrix elements look very similar to the expressions used in a frequency domain MoM problem using the same triangular patch basis. The interesting feature is that even though the expressions for the matrix elements over the spatial basis functions are similar both in the time and in the frequency domain the Green's functions involved are different. Due to a difference in the Green's functions it is seen that the matrix elements for the time domain problem need to be evaluated more accurately using an increased number of quadrature sampling points for integration than its frequency domain counterpart. This paper is focused on the numerical accuracy of the matrix elements in a time domain MOD method. Many practical examples, such as a tank, a Fokker aircraft, an AS-322 helicopter, and a Boeing-737 aircraft, have been presented in [9, 10].

In this work, it is illustrated that the spatial integral accuracy in the MOD method needs more sampling points to evaluate the numerical integrals than the frequency domain counterpart in order to maintain the same integral accuracy. In Section 2 the expressions for the Green's functions are presented along with a few numerical results to illustrate the hypothesis in Section 3 and then followed by conclusions in Section 4.

2. GREEN'S FUNCTIONS IN THE MATRIX ELEMENTS

For solving currents or potentials on the surfaces of objects, basis functions and unknown coefficients are needed to represent them. The scattered fields are presented by an integral of current or potential, which also equal to an integral of basis functions. If we apply Galerkin's method to solve the unknown coefficients, inner products are performed by multiplying another basis function to the equation and integrating them over the whole domain. Then a double integral of two spatial basis function is major part in the equation to solve the problem.

In the MOD method [9, 10], the spatial integrals involved in the evaluation of the expressions for the matrix elements are in the form of

$$A^{TD}_{mnab} = \frac{1}{4\pi}\int_S \mathbf{f}_m(\mathbf{r}) \cdot \int_S \mathbf{f}_n(\mathbf{r}') \frac{I_{ab}(sR/c)}{R} dS' dS \tag{1a}$$

$$B^{TD}_{mnab} = \frac{1}{4\pi}\int_S \nabla \cdot \mathbf{f}_m(\mathbf{r}) \int_S \nabla' \cdot \mathbf{f}_n(\mathbf{r}') \frac{I_{ab}(sR/c)}{R} dS' dS \tag{1b}$$

in which, $\mathbf{f}_m$ and $\mathbf{f}_n$ are the spatial RWG basis functions, s is a scaling factor, R is the distance between the field and the source points, $\mathbf{r}$ and $\mathbf{r}'$, respectively, and c is the velocity of propagation in free space, and $I_{ab}(sR/c)$ is defined by

$$I_{ab}(sR/c) = \begin{cases} 0 & b > a \\ e^{(-sR/(2c))} & b = a \\ e^{(-sR/(2c))}\left(L_{a-b}(sR/c) - L_{a-b-1}(sR/c)\right) & b < a \end{cases} \tag{2}$$

in which the L_i is the Laguerre polynomial of the degree i [14, 15]. In the frequency domain MoM [12, 13], the spatial integrals are defined in the form of

$$A^{FD}_{mn} = \frac{1}{4\pi}\int_S \mathbf{f}_m(\mathbf{r}) \cdot \int_S \mathbf{f}_n(\mathbf{r}') \frac{e^{-jkR}}{R} dS' dS \tag{3a}$$

$$B^{FD}_{mn} = \frac{1}{4\pi}\int_S \nabla \cdot \mathbf{f}_m(\mathbf{r}) \int_S \nabla' \cdot \mathbf{f}_n(\mathbf{r}') \frac{e^{-jkR}}{R} dS' dS \tag{3b}$$

in which, j is the imaginary unit and k the wave number.

Between the equations of (1) and (3), the only difference is in the Green's function. For the time domain MOD method, the Green's function is $G_{TD} = I_{ab}(sR/c)/R$ and for the frequency domain MoM it is $G_{FD} = e^{-jkR}/R$. The derivatives of these two Green's functions with respect to R can be obtained as

$$\frac{\partial G_{TD}}{\partial R} = -\frac{1}{R^2} I_{ab}\left(\frac{sR}{c}\right) - \frac{s}{2cR} e^{-\frac{sR}{2c}}\left[L_{a-b}\left(\frac{sR}{c}\right) + L_{a-b-1}\left(\frac{sR}{c}\right)\right], \quad b < a \tag{4a}$$

$$\frac{\partial G_{FD}}{\partial R} = -\frac{e^{-jkR}}{R^2} - \frac{jke^{-jkR}}{R}. \tag{4b}$$

And also the ratio between the spatial derivatives of the Green's functions with respect to the Green's functions are also calculated as

$$\left|\frac{\partial G_{TD}/\partial R}{G_{TD}}\right| = \left|\frac{1}{R} + \frac{s}{2c}\frac{L_{a-b}(sR/c) + L_{a-b-1}(sR/c)}{L_{a-b}(sR/c) - L_{a-b-1}(sR/c)}\right| \tag{5a}$$

$$\left|\frac{\partial G_{FD}/\partial R}{G_{FD}}\right| = \sqrt{\frac{1}{R^2} + k^2} \tag{5b}$$

Equation (5b) is a monotonically decaying function with respect to the spatial variables and it does not have any singularities in the domain $R \in (0, +\infty)$. But for Eq. (5a), the denominator term $L_{a-b}(sR/c) - L_{a-b-1}(sR/c)$ does have zeros in the domain $R \in (0, +\infty)$ when $b < a$. Therefore, Eq. (5a) has some singularities in this region. Consider a very small error ΔR associated with the evaluation of the spatial variable R, and this will result in an error in the value of the Green's function ΔG_{TD}. The error ΔG_{TD} is given by

$$\Delta G_{TD} = \Delta R \, \frac{\partial G_{TD}}{\partial R} \tag{6}$$

and the corresponding relative error is given by

$$\left|\frac{\Delta G_{TD}}{G_{TD}}\right| = \Delta R \, \left|\frac{\partial G_{TD}/\partial R}{G_{TD}}\right|. \tag{7}$$

When the value of R is such that the denominator of $(5a)$ is close to zero, so the Green's function has a pole, a very small error in R can result in a large relative error in the value of the Green's function. In conclusion, the Green's function encountered in the MOD method is more sensitive to the error in the evaluation of R. Generally, the integrals encountered in (1) and (3) for both a time domain and a frequency domain problem cannot be handled analytically and a numerical technique needs to be employed to evaluate them over the surfaces involved. Since, in the time domain the functions associated with the integrals have singularities, more sampling points need to be used in the evaluation of the integrals than in the frequency domain. So in the evaluation of the integrals in Eq. (1), one will need more sampling points in the evaluation of the Green's function than in the frequency domain in order to maintain similar accuracy in the final results.

A plot of the two Green's functions for the time and frequency domain are displayed in Fig. 1. For the time domain Green's function in Fig. 1(a), s is chosen as 5×10^9 and the degree $a - b$ is 50 and 150,

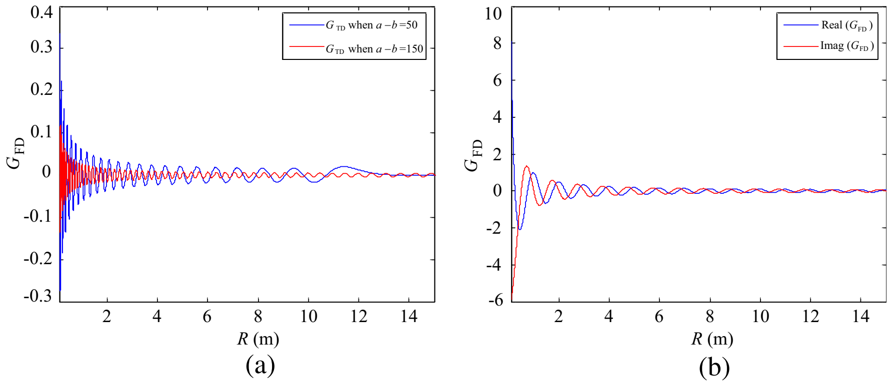


Figure 1. Comparison of Green's functions. (a) The time domain Green's function G_{TD}. (b) The frequency domain Green's function G_{FD}.

respectively, which are common values for most practical problems [9]. For the frequency domain Green's function in Fig. 1(b), k is chosen as 2π. From Fig. 1, we can see that the time domain Green's function oscillates more than the frequency domain one, especially when R is small or the degree $a - b$ is high. Therefore, when R has a small error, the time domain Green's function will result in a larger computational error than the other when using the same number of sample points to evaluate the integrals. Numerical examples in the evaluation of Eqs. (1) and (3) will be shown to illustrate this point.

3. NUMERICAL EXAMPLES

In our study, both the time and frequency domain integrals are carried out using the Gaussian quadrature rules for a triangular region [16] using the RWG basis functions [13]. We vary the number of sampling points from 1 to 79 in the evaluation of the integrals encountered in (1) and (3).

Example 1 considers two triangular spatial basis functions parallel to each other as shown in Fig. 2. For the time domain Green's function, s is chosen as 5×10^9 and $a - b = 50$. For the frequency domain Green's function, k is chosen to be 2π. The integral values in the evaluation of the expressions in Eqs. (1a) and (3a) are listed in the Table 1 and Table 2. Both the expressions converge when one increases the number of the sampling quadrature points. Because we don't know a priori, what are the exact values for the integrals, we consider the results

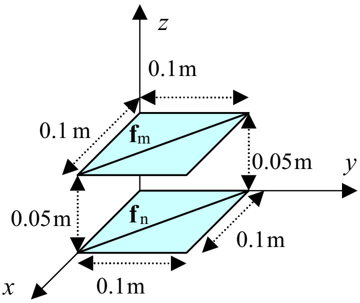
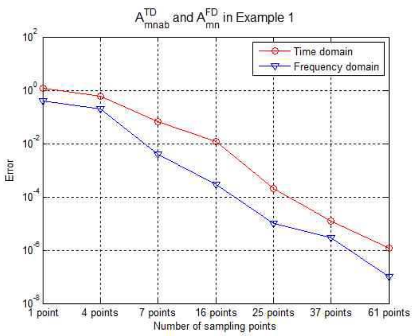
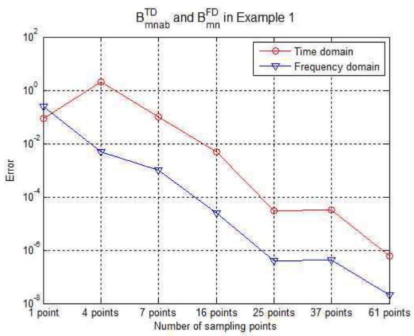


Figure 2. The two basis functions for Example 1.



(a)



(b)

Figure 3. Relative errors in the evaluation of integrals in Example 1. (a) A_{mnab}^{TD} and A_{mn}^{FD}, (b) B_{mnab}^{TD} and B_{mn}^{FD}.

when using 79 points as the one closest to the accurate values. And the relative error at this value is plotted in Fig. 3. The relative errors in the time and frequency domains are defined as

$$\text{Error}^{TD}(i) = \frac{\left|A_{mnab}^{TD}(79) - A_{mnab}^{TD}(i)\right|}{\left|A_{mnab}^{TD}(79)\right|} \tag{8}$$

and

$$\text{Error}^{FD}(i) = \frac{\left|A_{mn}^{FD}(79) - A_{mn}^{FD}(i)\right|}{\left|A_{mn}^{FD}(79)\right|} \tag{9}$$

in which A_{mnab}^{TD} and A_{mn}^{FD} are the time and frequency domain integrals in Eqs. (1a) or (3a) computed with i sampling points, respectively, and the operator $|\cdot|$ is the absolute value of the function. From Fig. 3(a), one can observe that when one uses 7 sampling points, which is a very common case for the computations, the frequency domain integral can

Table 1. The integral values in the evaluation of Eq. (1a) in Example 1.

Sampling points i	A_{mnab}^{TD}
1	+1.196862990017365E-006
4	–9.185507229096584E-008
7	–1.415802785348749E-007
16	–1.520710249941450E-007
25	–1.503210213166758E-007
37	–1.502934135708395E-007
61	–1.502913746203929E-007
79	–1.502915693675874E-007

Table 2. The integral values in the evaluation of Eq. (3a) in Example 1.

Sampling points i	A_{mn}^{FD}
1	0.1026460570347953E-04 – j 0.1601586946542305E-06
4	0.7931528008435943E-05 – j 0.1788829860551588E-06
7	0.6072444309197887E-05 – j 0.1781290363986577E-06
16	0.6105805417065320E-05 – j 0.1781298635103971E-06
25	0.6104483239659091E-05 – j 0.1781298635107554E-06
37	0.6104447874541564E-05 – j 0.1781298635107529E-06
61	0.6104432328759541E-05 – j 0.1781298635107444E-06
79	0.6104431829695501E-05 – j 0.1781298635106877E-06

get an error of less than 1% while the error in the time domain is around 10%. Similar phenomenon also appears in the evaluation of the integrals of B_{mnab}^{TD} and B_{mn}^{FD} in the Eqs. (1b) and (3b) and their relative errors are plotted in Fig. 3(b). In this figure, when the number of sampling point is chosen as 7, the frequency domain integrals has an error of around 0.1% while the time domain expressions provide a relative error of about 10%.

Example 2 uses two triangular patch basis functions that are perpendicular to each other as shown in Fig. 4. The parameters of s, k, a, b are the same as in Example 1. The relative error in the evaluations for the quantities A^{TD}_{mnab} and A^{FD}_{mn} are listed in Tables 3 and 4. The relative errors of A^{TD}_{mnab} and A^{FD}_{mn} are plotted in Fig. 5(a) and the relative errors of B^{TD}_{mnab} and B^{FD}_{mn} are plotted in Fig. 5(b). From Fig. 5, one can observe that the errors are much larger than the ones from Example 1. This is because the time domain Green's functions vary over a larger value when R is small as shown in the Fig. 1. In the first example, R is greater than 0.05 m, but in this example some of the values of R are close to zero. As implied in Fig. 1, the error in the evaluation of the Green's function is more sensitive to the error in the evaluation of R for both time and frequency domain cases and in order to get an accurate value for the integral, more sampling points

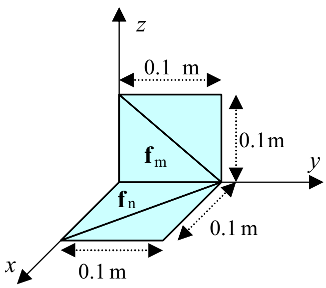
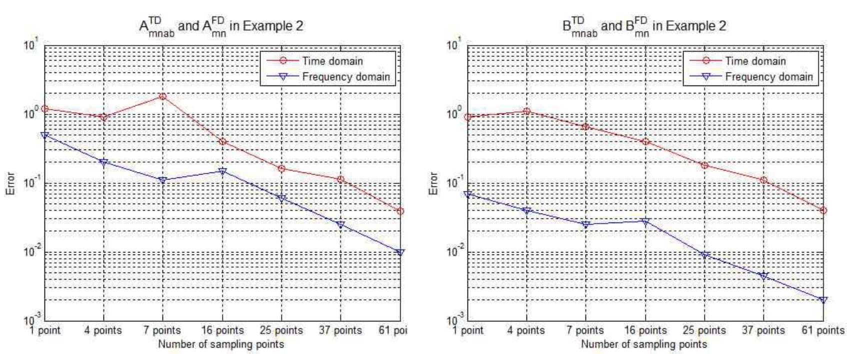


Figure 4. The two basis functions for Example 2.



(a) (b)

Figure 5. Relative errors in the evaluation of integrals in Example 2. (a) A^{TD}_{mnab} and A^{FD}_{mn}, (b) B^{TD}_{mnab} and B^{FD}_{mn}.

Table 3. The integral values in the evaluation of Eq. (1a) in Example 2.

Sampling points i	A_{mnab}^{TD}
1	+3.761504994513267E-007
4	–1.152579677805389E-006
7	+9.593733385070720E-008
16	–1.317525027949061E-007
25	–9.084526029380924E-008
37	–8.595394401284385E-008
61	–7.911876314641855E-008
79	–7.616760175651086E-008

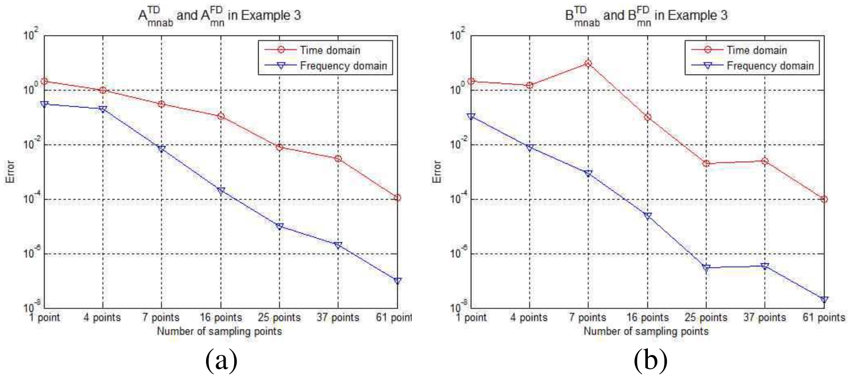


Figure 6. Relative errors in the evaluation of integrals in Example 3. (a) A_{mnab}^{TD} and A_{mn}^{FD}, (b) B_{mnab}^{TD} and B_{mn}^{FD}.

are needed. From Fig. 5, we can see that if we use only 7 sampling points in the time domain solver, the errors are around 110% and 60%, respectively. These errors are so large that the results of the solver are unreliable and more sampling points are necessary.

Example 3 is the same as Example 1 except that the degree of $a-b$ is changed to 150. All other parameters are the same. The relative errors in the evaluation of A_{mnab}^{TD} and A_{mn}^{FD} are plotted in Fig. 6(a) and the relative errors in the evaluation of B_{mnab}^{TD} and B_{mn}^{FD} are plotted in Fig. 6(b). Comparing these two figures with Figs. 3(a) and 3(b), one can see that as the degree increases, the errors associated with

Table 4. The integral values in the evaluation of Eq. (3a) in Example 2.

Sampling points i	A_{mn}^{FD}
1	0.8504782549007540E-05 − j 0.1596313625583668E-06
4	0.1563952782733322E-04 − j 0.1788662643803946E-06
7	0.1418624593893488E-04 − j 0.1784863698059096E-06
16	0.1489178854301966E-04 − j 0.1784867859716771E-06
25	0.1345273330446101E-04 − j 0.1784867859718516E-06
37	0.1300656612436935E-04 − j 0.1784867859718702E-06
61	0.1282336903240997E-04 − j 0.1784867859718813E-06
79	0.1269753806813858E-04 − j 0.1784867859718796E-06

the integrals associated with the time domain Green's function also increase. This is because the time domain Green's function will vary more rapidly as the degree gets large as seen in Fig. 1. If we do not increase the number of sampling points with the increase of degree, the error will be larger when one uses the MOD solution procedure.

4. CONCLUSIONS

Compared to a frequency domain solver, the error associated with the evaluation of the matrix elements in a time domain MOD solver is more sensitive to the error in the evaluation of R than the frequency domain one. This is because the Green's function varies faster with respect to R. Therefore, one needs more sampling points in the numerical evaluation of the integrals in order to obtain an accurate result. If one uses the same number of sampling points as in a frequency domain solver, the errors will be much larger. This is especially important when the distance between two triangular patches is close to each other or when the degree of the associated Laguerre polynomials is large.

ACKNOWLEDGMENT

This work was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2010-013-D00057).

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