

## GENERAL SOLUTION FOR WAVEGUIDE MODES IN FRACTIONAL SPACE

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**Abstract**—In this paper, general solution for the electric and magnetic fields are developed using the vector potentials  $\mathbf{A}$  and  $\mathbf{F}$  when the wave is propagating in fractional dimensional space. Different field configurations can be analyzed using the developed expressions for electric and magnetic fields, here we have analyzed  $TE^z$  and  $TM^z$  modes when the wave propagates in fractional space inside a rectangular waveguide. It is observed that wave propagation behavior in fractional space changes substantially from the non-fractional space. It is also observed that the obtained results show generalization of the concept of solutions for wave propagation from integer to fractional space. As a special case, when all the dimensions are considered integer, then all classical results are recovered.

### 1. INTRODUCTION

Most objects have irregular shapes and cannot be modeled using Euclidean geometry. For example, the branching of trees, waves in ocean, cloud, dust particles and many more. As a result of the abundance of irregular geometrical objects in nature, the idea of fractals and non-fractals originated [1–3]. First proper classification of mediums into fractals and non-fractals was done by Mandelbrot [4]. The abundance of fractals in nature resulted in research into the formulation of laws of physics for fractional spaces. Palmer and Stavrinou worked on equations of motion in a non-integer dimensional

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space [5]. Similarly, Tarasov analyzed Electromagnetic fields on fractals [6]. And other researchers also made their efforts in explaining physical phenomena in fractals [7, 8].

Generalization of electromagnetics in fractals has been studied at length. Resultantly solutions of Laplace and Poisson's equations have been reformulated in fractional dimensional space [9, 10]. Similarly Faraday's law and Ampere's law as well as Maxwell's electromagnetic stress tensor have been reformulated in fractals by Martin et al. [11].

Differential electromagnetic equations are also developed [12] resulting in wave equation and general plane wave solutions in fractals [14]. Cylindrical and Spherical wave equations have also been expressed in fractals [15, 16]. Vector potentials have been also developed in fractional space, which further made the analysis of electromagnetics easy. Antenna radiations can be analyzed using these vector potentials. Mughal and Zubair first remodeled this idea in fractional space and then analyzed dipole antenna placed in fractional space and many parameters like directivity and radiation pattern of dipole antenna were analyzed in the domain of fractional space [20]. Proceeding the journey of electromagnetic in fractals, reflection and transmission co-efficient in many scenarios have been studied at length. When a wave passes through a layer of fractal sandwiched between non-fractals then reflection and transmission will occur which was analytically modeled by Attiya [18]. Similarly reflection co-efficient was analyzed at an interface which is developed when a half space of fractal meet another half space of non-fractal [19]. Also reflection from a fractal-fractal interface analyzed by Omar and Mughal [26]. This idea was utilized further by them in developing reflection and transmission coefficients for chiral-fractal dielectric interface where half space was assumed chiral and second half space as fractal. It was observed in all these cases that the dimension plays a very important role in the reflection and transmission of a wave from such interfaces. Electromagnetic radiations from fractal structures have been an area of interest in past few years [21–24]. Most of the concepts of wave propagation phenomena in fractional space have been compiled together by Zubair et al. in [13, 17]. Green's function for fractional space was developed by Asad in fractional space [27].

In this paper, general solutions for electromagnetic wave is developed and electromagnetic field configurations are analyzed in fractional space. In first Section, analytical expressions for general solution is constructed in fractional space. Both electric field and magnetic field are expressed in terms of vector potentials in fractional space. Many field configurations can be generated and analyzed, here  $TE^z$  and  $TM^z$  have been modeled mathematically. In Section 3,

it is observed that all these solutions and field configurations are a generalization of these results from integer dimensional space to fractional space. When we consider a special case of integer dimensions in the expressions developed here, then all results in Euclidean space can be recovered which shows complete agreement with the classical results. Further  $TM^z$  and  $TE^z$  are analyzed in a rectangular waveguide and observed that there are substantial changes in wave propagation in fractal media from that in non-fractal media. And when we consider integer dimensions for the case of rectangular waveguide, then it is again observed that all classical results can be recovered. In the last section, conclusions are drawn.

## 2. CONSTRUCTION OF GENERAL SOLUTIONS IN FRACTIONAL SPACE

As we already know, total electric field in fractional space in terms of vector potentials  $\mathbf{A}$  and  $\mathbf{F}$  can be written as [12],

$$\mathbf{E} = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\epsilon}\nabla_D(\nabla_D \cdot \mathbf{A}) - \frac{1}{\epsilon}\nabla_D \times \mathbf{F} \quad (1)$$

and similarly total magnetic field in fractional space in terms of vector potentials  $\mathbf{A}$  and  $\mathbf{F}$  can be written as [12],

$$\mathbf{H} = -j\omega\mathbf{F} - j\frac{1}{\omega\mu\epsilon}\nabla_D(\nabla_D \cdot \mathbf{F}) + \frac{1}{\mu}\nabla_D \times \mathbf{A} \quad (2)$$

where the subscript  $D$  in gradient and divergence operators signifies that these are the operators used in fractional space as developed by Zubair et al. [12]. The vector potentials  $A$  and  $F$  can have the form

$$\mathbf{A}(x, y, z) = \hat{a}_x A_x(x, y, z) + \hat{a}_y A_y(x, y, z) + \hat{a}_z A_z(x, y, z) \quad (3)$$

$$\mathbf{F}(x, y, z) = \hat{a}_x F_x(x, y, z) + \hat{a}_y F_y(x, y, z) + \hat{a}_z F_z(x, y, z) \quad (4)$$

which must satisfy the following equations in source free region [20],

$$\nabla_D^2 \mathbf{A} + \beta^2 \mathbf{A} = 0 \quad (5)$$

$$\nabla_D^2 \mathbf{F} + \beta^2 \mathbf{F} = 0 \quad (6)$$

Here  $\nabla_D^2$  represents the Laplacian operator in fractional space [12]. It is also observed that the gradient operator in fractional space takes the form as [28],

$$\nabla_D f = \hat{x} \left[1 + \frac{x}{\ell_o}\right]^{1-\zeta_x} \frac{\partial}{\partial x} f + \hat{y} \left[1 + \frac{y}{\ell_o}\right]^{1-\zeta_y} \frac{\partial}{\partial y} f + \hat{z} \left[1 + \frac{z}{\ell_o}\right]^{1-\zeta_z} \frac{\partial}{\partial z} f \quad (7)$$

whereas divergence operator in fractional space is defined as

$$\nabla_D \cdot f = \left[1 + \frac{x}{\ell_o}\right]^{1-\zeta_x} \frac{\partial}{\partial x} f_x + \left[1 + \frac{y}{\ell_o}\right]^{1-\zeta_y} \frac{\partial}{\partial y} f_y + \left[1 + \frac{z}{\ell_o}\right]^{1-\zeta_z} \frac{\partial}{\partial z} f_z \quad (8)$$

Here  $\zeta_x$ ,  $\zeta_y$ , and  $\zeta_z$  are used to measure the extent of fractionality along three dimensions. Their values lie in the range of 0–1, where 1 represents integer dimension and any value less than 1 signifies fractionality along the corresponding dimension. Equation (1) when expanded using Equations (3), (4) and Equations (7), (8) can be written as

$$\begin{aligned}
\mathbf{E} = & \hat{x} \left[ -j\omega A_x - j \frac{1}{\omega\mu\epsilon} \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} \left( \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} A_x \right. \right. \\
& \left. \left. + \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} A_y + \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} A_z \right) - \frac{1}{\epsilon} (\nabla_D \times F) \cdot \hat{a}_x \right] \\
& + \hat{y} \left[ -j\omega A_y - j \frac{1}{\omega\mu\epsilon} \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} \left( \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} A_x \right. \right. \\
& \left. \left. + \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} A_y + \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} A_z \right) - \frac{1}{\epsilon} (\nabla_D \times F) \cdot \hat{a}_y \right] \\
& + \hat{z} \left[ -j\omega A_z - j \frac{1}{\omega\mu\epsilon} \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} \left( \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} A_x \right. \right. \\
& \left. \left. + \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} A_y + \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} A_z \right) - \frac{1}{\epsilon} (\nabla_D \times F) \cdot \hat{a}_z \right] \quad (9)
\end{aligned}$$

and similarly  $\mathbf{H}$  can be expanded from Equation (2), using Equations (3), (4) and Equations (7), (8), to the following form

$$\begin{aligned}
\mathbf{H} = & \hat{x} \left[ -j\omega F_x - j \frac{1}{\omega\mu\epsilon} \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} \left( \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} F_x \right. \right. \\
& \left. \left. + \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} F_y + \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} F_z \right) - \frac{1}{\epsilon} (\nabla_D \times A) \cdot \hat{a}_x \right] \\
& + \hat{y} \left[ -j\omega F_y - j \frac{1}{\omega\mu\epsilon} \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} \left( \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} F_x \right. \right. \\
& \left. \left. + \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} F_y + \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} F_z \right) - \frac{1}{\epsilon} (\nabla_D \times A) \cdot \hat{a}_y \right] \\
& + \hat{z} \left[ -j\omega F_z - j \frac{1}{\omega\mu\epsilon} \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} \left( \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} F_x \right. \right.
\end{aligned}$$

$$+ \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} F_y + \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} F_z \Big) - \frac{1}{\epsilon} (\nabla_D \times A) \cdot \hat{a}_z \Big] \quad (10)$$

Here we can clearly see that the fractionality terms are introduced into the classical solutions for the field expressions in fractional space. They show generalization from integer to non-integer dimensional space. It is obvious from these expressions that classical results as obtained by Balanis [25] can be recovered upon inserting the values of dimensions as integers. Different field configurations can be analyzed in fractional space using the expressions developed for  $\mathbf{E}$  and  $\mathbf{H}$ . Following Sections cover  $TM^z$  and  $TE^z$  modes. Other modes can be analyzed accordingly.

### 2.1. Transverse Magnetic Mode in Fractal Media, $TM^z$ : Source Free Region

$TM$  mode are field configurations in which the magnetic field components are lying in a plane that is transverse to a given direction. For example, if the fields are  $TM$  to  $z$  (or  $TM^z$ ), this means that  $H_z = 0$ . And the remaining two magnetic field components, ( $H_y$  and  $H_x$ ), and the three electric field components ( $E_z$ ,  $E_y$  and  $E_x$ ) may or may not exist.

As it is obvious that to derive the field expressions for a field configuration that are transverse magnetic to a given direction, irrespective of the coordinate system, it is a sufficient assumption to let the vector potential  $\mathbf{A}$  to have only a component in that direction in which the fields are considered to be transverse magnetic. The remaining components of the vector potential  $\mathbf{A}$  as well as all components of  $\mathbf{F}$  are set equal to zero. Therefore for  $TM^z$  modes, we let

$$A = \hat{a}_z A_z(x, y, z) \quad (11)$$

and

$$F = 0 \quad (12)$$

where  $\mathbf{A}$  must satisfy the following reduced equation.

$$\nabla_D^2 A_z + \beta^2 A_z = 0 \quad (13)$$

First we need to find the solution for  $A_z$  from Equation (13). Once the solution for  $A_z$  is found, then the next step is to use that solution to find the  $E$  and  $H$  field components from Equations (9) and (10) as,

$$E_x = -j \frac{1}{\omega \mu \epsilon} \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} \left( \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} A_z \right) \quad (14)$$

$$E_y = -j \frac{1}{\omega \mu \epsilon} \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} \left( \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} A_z \right) \quad (15)$$

$$E_z = -j\omega A_z - j\frac{1}{\omega\mu\epsilon} \left[1 + \frac{z}{\ell_o}\right]^{1-\zeta_z} \frac{\partial}{\partial z} \left( \left[1 + \frac{z}{\ell_o}\right]^{1-\zeta_z} \frac{\partial}{\partial z} A_z \right) \quad (16)$$

and

$$H_x = \frac{1}{\mu} \left[1 + \frac{y}{\ell_o}\right]^{1-\zeta_y} \frac{\partial}{\partial y} A_z \quad (17)$$

$$H_y = -\frac{1}{\mu} \left[1 + \frac{x}{\ell_o}\right]^{1-\zeta_x} \frac{\partial}{\partial x} A_z \quad (18)$$

$$H_z = 0 \quad (19)$$

For a specific problem, we need to find out the solution for vector potential  $A_z$ , and then applying proper boundary conditions after evaluating Equations (14) to (19) will lead us to the final expressions for  $E$  and  $H$  field components. This is demonstrated further in last Section. Following similar approach, we can reach to the electromagnetic wave solutions for  $TM^x$  and  $TM^y$  modes of field configurations.

## 2.2. Transverse Electric Mode in Fractal Media, $TE^z$ : Source Free Region

$TE$  mode are field configurations in which the electric field components are lying in a plane that is transverse to a given direction. For example, if the fields are  $TE$  to  $z$  (or  $TE^z$ ), this means that  $E_z = 0$ . And the remaining two electric field components, ( $E_y$  and  $E_x$ ), and the three magnetic field components ( $H_z$ ,  $H_y$  and  $H_x$ ) may or may not exist.

As it is obvious that to derive the field expressions for a field configuration that are transverse electric to a given direction, irrespective of the coordinate system, it is a sufficient assumption to let the vector potential  $\mathbf{F}$  to have only a component in that direction in which the fields are considered to be transverse electric. The remaining components of the vector potential  $\mathbf{F}$  as well as all components of  $\mathbf{A}$  are set equal to zero. Therefore for  $TE^z$  modes, we let

$$A = 0 \quad (20)$$

and

$$F = \hat{a}_z F_z(x, y, z) \quad (21)$$

where  $F$  must satisfy the following reduced equation

$$\nabla_D^2 F_z + \beta^2 F_z = 0 \quad (22)$$

Once  $F_z$  is found, then the next step is to find the  $E$  and  $H$  field components by using the following equations reduced from

Equations (9) and (10),

$$E_x = -\frac{1}{\epsilon} \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} F_z \quad (23)$$

$$E_y = \frac{1}{\epsilon} \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} F_z \quad (24)$$

$$E_z = 0 \quad (25)$$

$$H_x = -j \frac{1}{\omega \mu \epsilon} \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} \left( \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} F_z \right) \quad (26)$$

$$H_y = -j \frac{1}{\omega \mu \epsilon} \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} \left( \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} F_z \right) \quad (27)$$

$$H_z = -j \omega F_z - j \frac{1}{\omega \mu \epsilon} \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} \left( \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} F_z \right) \quad (28)$$

For a specific problem, we need to find out the solution for vector potential  $F_z$ , and then applying proper boundary conditions after evaluating Equations (23) to (28) will lead us to the final expressions for  $E$  and  $H$  field components. This is demonstrated further in last Section. Following similar approach, we can reach to the electromagnetic wave solutions for  $TE^x$  and  $TE^y$  modes of field configurations.

### 3. RESULTS AND DISCUSSION

All the equations developed so far in this paper represent generalization of the wave propagation concept to fractional space from integer dimensional space. As a special case, considering all dimensions integer, this problem shrinks to the classical wave propagation concept. That is, if we consider  $\zeta_x = \zeta_y = \zeta_z = 1$ , then

$$E = \hat{x} \left[ -j \omega A_x - j \frac{1}{\omega \mu \epsilon} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z \right) - \frac{1}{\epsilon} (\nabla \times F) \cdot \hat{a}_x \right] \quad (29)$$

$$+ \hat{y} \left[ -j \omega A_y - j \frac{1}{\omega \mu \epsilon} \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z \right) - \frac{1}{\epsilon} (\nabla \times F) \cdot \hat{a}_y \right] \quad (30)$$

$$+ \hat{z} \left[ -j \omega A_z - j \frac{1}{\omega \mu \epsilon} \frac{\partial}{\partial z} \left( \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z \right) - \frac{1}{\epsilon} (\nabla \times F) \cdot \hat{a}_z \right] \quad (31)$$

and

$$H = \hat{x} \left[ -j \omega F_x - j \frac{1}{\omega \mu \epsilon} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z \right) + \frac{1}{\mu} (\nabla \times A) \cdot \hat{a}_x \right] \quad (32)$$

$$+\hat{y}\left[-j\omega F_y-j\frac{1}{\omega\mu\epsilon}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}F_x+\frac{\partial}{\partial y}F_y+\frac{\partial}{\partial z}F_z\right)+\frac{1}{\mu}(\nabla\times A)\cdot\hat{a}_y\right] \quad (33)$$

$$+\hat{z}\left[-j\omega F_z-j\frac{1}{\omega\mu\epsilon}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}F_x+\frac{\partial}{\partial y}F_y+\frac{\partial}{\partial z}F_z\right)+\frac{1}{\mu}(\nabla\times A)\cdot\hat{a}_z\right] \quad (34)$$

which shows complete agreement with the classical expressions for the wave behavior as developed by Balanis [25].

As an example, consider a rectangular waveguide of dimensions  $a$  and  $b$  with  $a > b$ . Assuming the waveguide is filled with fractal media with all axis fractional. The fractionality along  $x$ -axis is represented by  $\zeta_x$ , the fractionality along  $y$ -axis is represented by  $\zeta_y$ , and along  $z$ -axis by  $\zeta_z$ . Then Transverse Electric and Transverse Magnetic modes can be analyzed as following.

### 3.1. TM

As already mentioned,  $TM^z$  electric and magnetic fields obey the following mathematical expressions:

$$E_x = -j\frac{1}{\omega\mu\epsilon}\left[1+\frac{x}{\ell_o}\right]^{1-\zeta_x}\frac{\partial}{\partial x}\left(\left[1+\frac{z}{\ell_o}\right]^{1-\zeta_z}\frac{\partial}{\partial z}A_z\right) \quad (35)$$

$$E_y = -j\frac{1}{\omega\mu\epsilon}\left[1+\frac{y}{\ell_o}\right]^{1-\zeta_y}\frac{\partial}{\partial y}\left(\left[1+\frac{z}{\ell_o}\right]^{1-\zeta_z}\frac{\partial}{\partial z}A_z\right) \quad (36)$$

$$E_z = -j\omega A_z - j\frac{1}{\omega\mu\epsilon}\left[1+\frac{z}{\ell_o}\right]^{1-\zeta_z}\frac{\partial}{\partial z}\left(\left[1+\frac{z}{\ell_o}\right]^{1-\zeta_z}\frac{\partial}{\partial z}A_z\right) \quad (37)$$

and

$$H_x = \frac{1}{\mu}\left[1+\frac{y}{\ell_o}\right]^{1-\zeta_y}\frac{\partial}{\partial y}A_z \quad (38)$$

$$H_y = -\frac{1}{\mu}\left[1+\frac{x}{\ell_o}\right]^{1-\zeta_x}\frac{\partial}{\partial x}A_z \quad (39)$$

$$H_z = 0 \quad (40)$$

The potential  $A$  must satisfy the following differential equation,

$$\nabla_D^2 A_z(x, y, z) + \beta^2 A_z(x, y, z) = 0 \quad (41)$$

The solution for the potential  $A$  can be obtained for considering only positive traveling waves using the analogy with the integer dimensional space as,

$$A_z(x, y, z) = x^{n_1} y^{n_2} z^{n_3} [C_1 J_{n_1}(\beta_x x)] [C_2 J_{n_2}(\beta_y y)] [C_3 H_{n_3}^{(2)}(\beta_z z)] \quad (42)$$



where

$$\begin{aligned} n_1 &= 1 - \frac{\zeta_x}{2} \\ n_2 &= 1 - \frac{\zeta_y}{2} \\ n_3 &= 1 - \frac{\zeta_z}{2} \end{aligned} \tag{43}$$

Using Equation (35) through (42) and applying the boundary conditions at the walls of the waveguide lead us to following electric field and magnetic field equations,

$$E_x^+ = -B'_{mn} \frac{j}{\omega\mu\epsilon} y^{n_2} J_{n_2}(\beta_y y) \left[1 + \frac{x}{\ell_o}\right]^{1-\zeta_x} \left[1 + \frac{z}{\ell_o}\right]^{1-\zeta_z} \frac{\partial^2}{\partial x \partial z} [x^{n_1} z^{n_3} J_{n_1}(\beta_x x) H_{n_3}(\beta_z z)] \tag{44}$$

$$E_y^+ = -B'_{mn} \frac{j}{\omega\mu\epsilon} \left[1 + \frac{y}{\ell_o}\right]^{1-\zeta_y} \left[1 + \frac{z}{\ell_o}\right]^{1-\zeta_z} x^{n_1} J_{n_1}(\beta_x x) \frac{\partial^2}{\partial y \partial z} (y^{n_2} z^{n_3} J_{n_2}(\beta_y y) H_{n_3}(\beta_z z)) \tag{45}$$

$$\begin{aligned} E_z^+ &= -j B'_{mn} \left[ \omega (x^{n_1} y^{n_2} z^{n_3} [J_{n_1}(\beta_x x)][J_{n_2}(\beta_y y)][H_{n_3}^{(2)}(\beta_z z)]) \right. \\ &\quad \left. + \frac{1}{\omega\mu\epsilon} x^{n_1} y^{n_2} J_{n_1}(\beta_x x) J_{n_2}(\beta_y y) \left[1 + \frac{z}{\ell_o}\right]^{1-\zeta_z} \frac{\partial}{\partial z} \left[ z^{n_3} \left[1 + \frac{z}{\ell_o}\right]^{1-\zeta_z} H_{n_3}(\beta_z z) \right] \right] \end{aligned} \tag{46}$$

$$H_x^+ = B'_{mn} \frac{1}{\mu} x^{n_1} z^{n_3} \left[1 + \frac{y}{\ell_o}\right]^{1-\zeta_y} J_{n_1}(\beta_x x) H_{n_3}^{(2)}(\beta_z z) \frac{\partial}{\partial y} (y^{n_2} J_{n_2}(\beta_y y)) \tag{47}$$

$$H_y^+ = -B'_{mn} \frac{1}{\mu} y^{n_2} z^{n_3} \left[1 + \frac{x}{\ell_o}\right]^{1-\zeta_x} J_{n_2}(\beta_y y) H_{n_3}^{(2)}(\beta_z z) \frac{\partial}{\partial x} (x^{n_1} J_{n_1}(\beta_x x)) \tag{48}$$

$$H_z^+ = 0 \tag{49}$$

and applying the boundary conditions on field expressions at the walls along  $y$ -axis for  $y = y_o$  and  $y = y_1$ , we can find out numerically the solution for  $\beta_y$  from the following simultaneous equations,

$$y_o^{n_2} J_{n_2}(\beta_y y_o) = 0 \tag{50}$$

and

$$y_1^{n_2} J_{n_2}(\beta_y y_1) = 0 \tag{51}$$

similarly applying the boundary conditions on field expressions at the walls along  $x$ -axis for  $x = x_o$  and  $x = x_1$ , we can numerically find the

solution for  $\beta_x$  from the following simultaneous equations,

$$x_0^{n_1} J_{n_1}(\beta_x x_0) = 0 \tag{52}$$

and

$$x_1^{n_1} J_{n_1}(\beta_x x_1) = 0 \tag{53}$$

The cutoff frequency can be computed from the following expression,

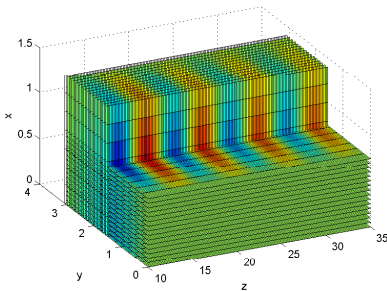
$$\beta_z^2 = \beta^2 - \beta_x^2 - \beta_y^2 \tag{54}$$

Since variations along all the three fractional axes are defined by Bessel functions and Hankel function, which are orthogonal hence verifying that different modes are orthogonal as well. For a waveguide filled with fractional space having dimensions of  $D = 2.76$ , the wave propagation is depicted in Figure 1. Its clear from the plot that the wave propagation is observing Hankel function variations along fractional  $z$ -axis. The planar view for the same waves propagating is shown in Figure 3. The effect of fractionality is obvious from the planar view as well.

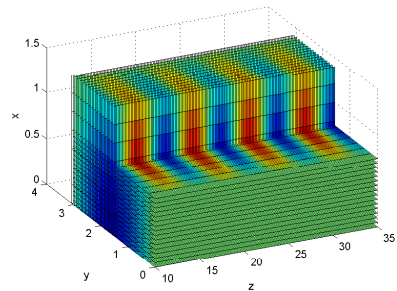
Upon considering all axis to be integer dimensional, that is focusing on the special classical case, then the order for each axis become equal to  $1/2$ , i.e.,  $n_1 = n_2 = n_3 = 1/2$  and the Bessel and Hankel functions get reduced to [25],

$$H_{\frac{1}{2}}^{(2)}(\beta_z z) = j \sqrt{\frac{2}{\pi \beta_z z}} e^{-j\beta_z z} \tag{55}$$

$$J_{\frac{1}{2}}(\beta_x x) = \sqrt{\frac{2}{\pi \beta_x x}} \sin(\beta_x x) \tag{56}$$



**Figure 1.** Cross section of a rectangular waveguide filled with fractal media, with dimension ( $D$ ) equal to 2.76, showing the transverse electric field propagation.



**Figure 2.** Cross section of a rectangular waveguide filled with non-fractal, with dimension ( $D$ ) equal to 3, media showing the transverse electric field propagation.

$$Y_{\frac{1}{2}}(\beta_y y) = \sqrt{\frac{2}{\pi\beta_y y}} \cos(\beta_y y) \tag{57}$$

and all the results obtained in this section gets reduced to the following equations:

$$E_x^+ = -B_{mn} \frac{\beta_x \beta_y}{\omega \mu \epsilon} \cos(\beta_x x) \sin(\beta_y y) e^{-j(\beta_z z)} \tag{58}$$

$$E_y^+ = -B_{mn} \frac{\beta_y \beta_z}{\omega \mu \epsilon} \sin(\beta_x x) \cos(\beta_y y) e^{-j(\beta_z z)} \tag{59}$$

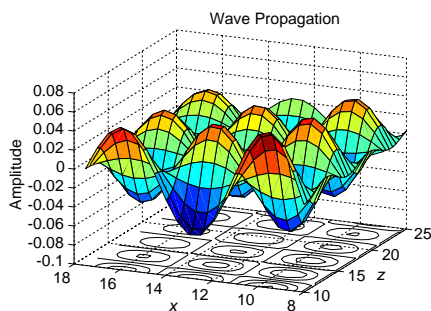
$$E_z^+ = -j B_{mn} \frac{\beta_c^2}{\omega \mu \epsilon} \sin(\beta_x x) \sin(\beta_y y) e^{-j(\beta_z z)} \tag{60}$$

$$H_x^+ = B_{mn} \frac{\beta_y}{\mu} \sin(\beta_x x) \cos(\beta_y y) e^{-j(\beta_z z)} \tag{61}$$

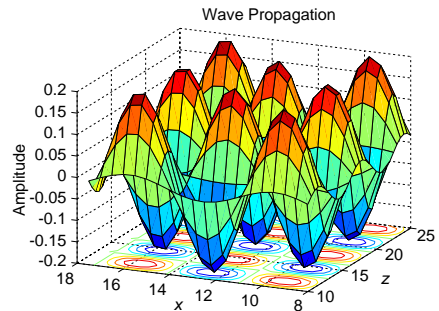
$$H_y^+ = -B_{mn} \frac{\beta_x}{\mu} \cos(\beta_x x) \sin(\beta_y y) e^{-j(\beta_z z)} \tag{62}$$

$$H_z^+ = 0 \tag{63}$$

The wave propagation in the waveguide filled with integer dimensional space, i.e.,  $D = 3$ , is graphically shown in Figure 2. Its planar view is drawn in Figure 4. Both of these plots are verifying that the wave propagation follows cosine function variation along the integer dimensional  $z$ -axis which shows complete agreement with the results obtained by Balanis [25].



**Figure 3.** Planar view of the wave propagation along  $z$ -axis in rectangular waveguide filled with fractal media with dimension ( $D$ ) equal to 2.76.



**Figure 4.** Planar view of the wave propagation along  $z$ -axis in rectangular waveguide filled with non-fractal media with dimension ( $D$ ) equal to 3.

### 3.2. TE

As already mentioned,  $TE^z$  electric and magnetic fields get satisfied by the following set of equations:

$$E_x = -\frac{1}{\epsilon} \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} F_z \quad (64)$$

$$E_y = \frac{1}{\epsilon} \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} F_z \quad (65)$$

$$E_z = 0 \quad (66)$$

$$H_x = -j \frac{1}{\omega \mu \epsilon} \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \frac{\partial}{\partial x} \left( \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} F_z \right) \quad (67)$$

$$H_y = -j \frac{1}{\omega \mu \epsilon} \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \frac{\partial}{\partial y} \left( \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} F_z \right) \quad (68)$$

$$H_z = -j \omega F_z - j \frac{1}{\omega \mu \epsilon} \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} \left( \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial}{\partial z} F_z \right) \quad (69)$$

where the potential  $F$  must satisfy the following differential equation,

$$\nabla_D^2 F_z(x, y, z) + \beta^2 F_z(x, y, z) = 0 \quad (70)$$

The solution for the potential  $F$  can be obtained for considering only positive traveling waves using the analogy with the integer dimensional space as,

$$F_z(x, y, z) = x^{n_1} y^{n_2} z^{n_3} [C_1 Y_{n_1}(\beta_x x)] [C_2 Y_{n_2}(\beta_y y)] [C_3 H_{n_3}^{(2)}(\beta_z z)] \quad (71)$$

Using Equation (64) through (71) and applying the boundary conditions at the walls of the waveguide lead us to following electric field and magnetic field equations,

$$E_x^+ = -A'_{mn} \frac{1}{\epsilon} x^{n_1} z^{n_3} \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} Y_{n_1}(\beta_x x) [n_2 y^{n_2-1} Y_{n_2}(\beta_y y) + \beta_y y^{n_2-1} Y'_{n_2}(\beta_y y)] H_{n_3}^{(2)}(\beta_z z) \quad (72)$$

$$E_x^+ = A'_{mn} \frac{1}{\epsilon} y^{n_2} z^{n_3} \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} Y_{n_2}(\beta_y y) [n_1 x^{n_1-1} Y_{n_1}(\beta_x x) + \beta_x x^{n_1-1} Y'_{n_1}(\beta_x x)] H_{n_3}^{(2)}(\beta_z z) \quad (73)$$

$$E_z^+ = 0 \quad (74)$$

$$H_x^+ = -A'_{mn} \frac{j}{\omega \mu \epsilon} \left[ 1 + \frac{x}{\ell_o} \right]^{1-\zeta_x} \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial^2}{\partial x \partial z}$$

$$\left[ x^{n_1} y^{n_2} z^{n_3} Y_{n_1}(\beta_x x) Y_{n_2}(\beta_y y) H_{n_3}^{(2)}(\beta_z z) \right] \quad (75)$$

$$H_y^+ = -A'_{mn} \frac{j}{\omega \mu \epsilon} \left[ 1 + \frac{y}{\ell_o} \right]^{1-\zeta_y} \left[ 1 + \frac{z}{\ell_o} \right]^{1-\zeta_z} \frac{\partial^2}{\partial y \partial z} \left[ x^{n_1} y^{n_2} z^{n_3} Y_{n_1}(\beta_x x) Y_{n_2}(\beta_y y) H_{n_3}^{(2)}(\beta_z z) \right] \quad (76)$$

$$H_z^+ = -j\omega F_z - j \frac{1}{\omega \mu \epsilon} x^{n_1} y^{n_2} Y_{n_1}(\beta_x x) Y_{n_2}(\beta_y y) \frac{\partial}{\partial z} \left( \left( 1 + \frac{z}{\ell_o} \right)^{1-\zeta_z} \frac{\partial}{\partial z} \left( z_3^n H_{n_3}^{(2)}(\beta_z z) \right) \right) \quad (77)$$

where  $\beta_y$  can be computed numerically from the following simultaneous equations which are derived from the boundary conditions,

$$n_2 y_o^{n_2-1} Y_{n_2}(\beta_y y_o) + \beta_y Y'_n(\beta_y y_o) = 0 \quad (78)$$

$$n_2 y_1^{n_2-1} Y_{n_2}(\beta_y y_1) + \beta_y Y'_n(\beta_y y_1) = 0 \quad (79)$$

where  $y_o$  and  $y_1$  represent the position of the walls of waveguide along the  $y$ -axis where the boundary conditions are applied. Similarly,  $\beta_x$  can be computed numerically from the following simultaneous equations as well,

$$n_1 x_o^{n_1-1} Y_{n_1}(\beta_x x_o) + \beta_x Y'_n(\beta_x x_o) = 0 \quad (80)$$

$$n_1 y_1^{n_1-1} Y_{n_1}(\beta_x x_1) + \beta_x Y'_n(\beta_x x_1) = 0 \quad (81)$$

where  $x_o$  and  $x_1$  represent the position of the walls of waveguide along the  $x$ -axis where the boundary conditions are applied. The cutoff frequency can be computed from the following expression,

$$\beta_z^2 = \beta^2 - \beta_x^2 - \beta_y^2 \quad (82)$$

Here orthogonality of the modes is also obvious from the Bessel function and Hankel function dependency of the wave propagation along fractional axes. Now if we consider  $\zeta_x = \zeta_y = \zeta_z = 1$ , that is we restrict our problem only to the classical one, then the Bessel and Hankel functions get reduced to Equations (55)–(57), and the fields expressions get evaluated as following,

$$E_x^+ = A_{mn} \frac{\beta_y}{\epsilon} \cos(\beta_x x) \sin(\beta_y y) e^{-j(\beta_z z)} \quad (83)$$

$$E_y^+ = -A_{mn} \frac{\beta_x}{\epsilon} \sin(\beta_x x) \cos(\beta_y y) e^{-j(\beta_z z)} \quad (84)$$

$$E_z^+ = 0 \quad (85)$$

$$H_x^+ = A_{mn} \frac{\beta_x \beta_z}{\omega \mu \epsilon} \sin(\beta_x x) \cos(\beta_y y) e^{-j(\beta_z z)} \quad (86)$$

$$H_y^+ = A_{mn} \frac{\beta_y \beta_z}{\omega \mu \epsilon} \cos(\beta_x x) \sin(\beta_y y) e^{-j(\beta_z z)} \quad (87)$$

$$H_z^+ = -j A_{mn} \frac{\beta_c^2}{\omega \mu \epsilon} \cos(\beta_x x) \cos(\beta_y y) e^{-j(\beta_z z)} \quad (88)$$

which shows complete correspondence with the one obtained by Balanis [25].

#### 4. CONCLUSIONS

General solutions for electromagnetic wave is developed and electromagnetic field configurations are analyzed in fractional space. Analytical expressions for general solution is constructed in fractional space. Both electric field and magnetic field are expressed mathematically in terms of vector potentials in fractional space. Based on the constructed solutions, many field configurations can be generated and analyzed, here we have discussed  $TE^z$  and  $TM^z$  modes. First we developed their mathematical models. Then further  $TM^z$  and  $TE^z$  are analyzed in a rectangular waveguide and observed that there are substantial changes in wave propagation in fractal media from that in non-fractal media. It is observed that all these solutions and field configurations are a generalization of these results from integer dimensional space to fractional space. When we consider a special case of integer dimensions in the expressions developed here, then all results in Euclidean space can be recovered which shows complete agreement with the classical results. Different waveguide systems filled with fractal media can be analyzed using techniques developed here. This work can be utilized in analyzing any field configuration. This idea can also be extended to other co-ordinate systems and can be utilized in constructing solution in cylindrical and spherical co-ordinate systems and analyzing their respective field configurations.

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