

## SECOND-ORDER FORMULATION FOR THE QUASI-STATIC FIELD FROM A HORIZONTAL ELECTRIC DIPOLE ON A LOSSY HALF-SPACE

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**Abstract**—In a recent study, we proposed improved quasi-static approximations for the electromagnetic field components excited by a vertical electric dipole lying on the surface of a flat and homogeneous lossy half-space. The present paper introduces an analytical approach to derive analogous formulas for the case of the horizontal electric dipole. The approach is based on the expansion of the integral representations for the fields into power series of the ratio between the wavenumbers in free-space and in the conducting medium. Later, only the terms up to the second-order are retained, and term-by-term analytical integration is performed. Numerical results demonstrate that using the derived second-order approximations for the fields in place of the zeroth-order ones allows to reduce the maximum relative error in the calculation of the fields from about 23% down to less than 7%.

### 1. INTRODUCTION

The study of the electromagnetic fields from vertical and horizontal electric dipole sources in presence of a plane conducting half-space has attracted the interest of researchers for many years. This is proved by the massive body of literature dedicated to the subject [1–17], and is justified by the applicability in a variety of engineering fields, including radio propagation and communication, exploration of the earth's subsurface structure, and even therapeutic heating and stimulation of biological tissues. An exhaustive review of research relating to this topic can be found in the classic monograph by Baños [1]. Many of the papers that have been published present simple expressions in the frequency-domain for the far-field, near-field and quasi-static

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field ranges. Due to the different mathematical approximations to be used, case by case, to evaluate the integral representations for the fields, these ranges have been treated separately. Contributions in this direction are the expressions presented by Wait [2], Moore and Blair [3], and Bannister [4, 5], which have been derived under the assumption that the ratio between the wavenumbers in free-space ( $k_0$ ) and in the conducting medium ( $k_1$ ) is small, a situation typically occurring when the conduction current in the lossy half-space predominates over the displacement current. Such zeroth-order formulas have been shown to be in agreement when compared with one another, even if, as noticed by Wait [2], they fail for poorly conducting media like, for instance, in the cases of frozen ground or soil composed of igneous or metamorphic rocks [18].

More recently, King [6–9] proposed improved expressions for the time-harmonic fields due to a dipole source in presence of a plane interface between two different media, which exhibit the following advantages. First, they are valid in a wide frequency range, this implying that, for each dipole configuration and orientation, the field components in the far-, intermediate- and near-field frequency ranges are described by a unique set of formulas. Second, if applied to the air-ground interface, they are accurate subject to the condition  $|\tau^2| = k_0^2/k_1^2 \ll 1$  (that is  $|k_1| \geq 3k_0$ ), which is less restrictive than the assumption  $|\tau| \ll 1$  at the basis of the previously published formulas on the same subject. This means theoretical possibility of application in a greater number of situations. Significant efforts have been also made to extend the theory developed for the homogeneous half-space problem to the multi-layer case. The research findings in this area of interest are summarized in the comprehensive books by Wait [19], King, Owens and Wu [6], and Li [20].

The present paper focuses on the field of a horizontal electric dipole (HED) lying on the surface of a homogeneous lossy ground. The scope of the study is two-fold. First, to show that, in the quasi-static limit, the relevant explicit expressions derived by King [6] effectively coincide with the zeroth-order quasi-static field expressions [2, 4] valid for highly conductive media ( $|\tau| \ll 1$ ) and, as a consequence, in the low-frequency range they do not constitute an improvement with respect to the previous approaches. Next, the objective is to derive second-order quasi-static approximations for the fields at the air-ground interface, which can be used even if  $|\tau^2|$  is not small, as long as  $|\tau^4| = k_0^4/k_1^4 \ll 1$ . This feature is what makes it possible to enhance the accuracy of the result of the computation in situations where the conductivity of the medium is low. Proceeding as done in an analogous study concerning the vertical electric dipole [21], the

field integrals are expanded into power series of  $\tau$ . Next, the terms up to the second-order are retained and cast into forms involving only known tabulated Sommerfeld Integrals. Numerical results show that, up to  $k_0\rho = 0.25$ , the level of accuracy exhibited by the second-order expressions is always higher than that pertaining to the zeroth-order ones, to a greater extent for smaller values of the conductivity of the half-space.

## 2. THEORY

Consider a HED of moment  $pe^{j\omega t}$  located on the surface of a homogeneous, isotropic and lossy half-space. The EM parameters of the medium are indicated in Fig. 1, and a cylindrical coordinate system  $(\rho, \varphi, z)$  is introduced. The quasi-static approximations for the EM field components generated on the surface of the medium ( $z \rightarrow 0^+$ ) were derived by Wait [2]. With the time-harmonic factor  $e^{j\omega t}$  suppressed for better clarity, they are given by

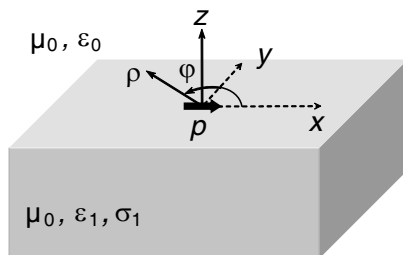
$$E_{\rho 0} = -\frac{j\omega\mu_0 p \cos \varphi}{2\pi k_1^2 \rho^3} \left[ 1 + (1 + jk_1\rho) e^{-jk_1\rho} \right], \quad (1)$$

$$E_{\varphi 0} = \frac{j\omega\mu_0 p \sin \varphi}{2\pi k_1^2 \rho^3} \left[ -2 + (1 + jk_1\rho) e^{-jk_1\rho} \right], \quad (2)$$

$$E_{z0} = \frac{j\omega\mu_0 p \cos \varphi}{2\pi\rho} K_1 I_1, \quad (3)$$

$$H_{\rho 0} = \frac{p \sin \varphi}{2\pi\rho^2} \left[ 3K_1 I_1 + \frac{jk_1\rho}{2} (K_0 I_1 - K_1 I_0) \right], \quad (4)$$

$$H_{\varphi 0} = -\frac{p \cos \varphi}{2\pi\rho^2} K_1 I_1, \quad (5)$$



**Figure 1.** Sketch of a horizontal electric dipole on a homogeneous lossy medium.

$$H_{z0} = -\frac{p \sin \varphi}{2\pi k_1^2 \rho^4} \left[ 3 - (3 + 3jk_1\rho - k_1^2\rho^2) e^{-jk_1\rho} \right], \quad (6)$$

where the argument of the modified Bessel functions  $I_\nu(\cdot)$  and  $K_\nu(\cdot)$  is  $jk_1\rho/2$ , and

$$k_n = \sqrt{\omega^2\mu_n\epsilon_n - j\omega\mu_n\sigma_n} \quad (7)$$

is the wavenumber in free-space ( $n = 0$ ) or in the conducting medium ( $n = 1$ ). Equations (1)–(6) are valid when the distance between source and field points is far less than a free-space wavelength ( $k_0\rho \ll 1$ ), and only if the material medium is much more dense than air ( $|k_1| \gg k_0$ ). The reason for the latter condition resides in the fact that (1)–(6) have been obtained by analytically evaluating the complete integral representations for the fields after setting  $k_0=0$  everywhere, thus implicitly assuming  $k_0$  negligible even if compared to  $|k_1|$ . Wait noticed that this may not be true for poorly conducting media and, in particular, when the ground is frozen [2].

Another, well-established, set of formulas for the same field components, valid in the near-, intermediate-, and far-field frequency ranges, has been proposed by King [6]. It can be easily verified that, in the quasi-static limit  $k_0\rho \ll 1$ , expressions [6, 5.5.57–5.5.62] for the  $E_\rho$ -,  $E_\varphi$ -, and  $H_z$ -fields reduce exactly to (1), (2), and (6), while expressions [6, 5.5.59–5.5.61] for the remaining field components simplify into

$$E_{z0} = \frac{\omega\mu_0 p \cos \varphi}{2\pi k_1 \rho^2} \left[ 1 - j \left( 1 + \frac{3}{2jk_1\rho} \right) e^{-jk_1\rho} \right], \quad (8)$$

$$H_{\rho 0} = \frac{p \sin \varphi}{2\pi \rho^2} \left[ \frac{2}{jk_1\rho} - j \left( 1 + \frac{7}{2jk_1\rho} \right) e^{-jk_1\rho} \right], \quad (9)$$

$$H_{\varphi 0} = \frac{p \cos \varphi}{2\pi k_1 \rho^3} \left[ j + \left( 1 + \frac{3}{2jk_1\rho} \right) e^{-jk_1\rho} \right], \quad (10)$$

which are, in fact, the asymptotic expansions of (3)–(5) for  $|k_1\rho| \gg 1$ . The reader may convince himself on this point by simply substituting the asymptotic behaviors of the modified Bessel functions

$$I_\nu \approx \frac{1}{\sqrt{\pi j k_1 \rho}} \left[ e^{\frac{jk_1\rho}{2}} \left( 1 - \frac{4\nu^2 - 1}{4jk_1\rho} \right) + (-1)^\nu j e^{-\frac{jk_1\rho}{2}} \left( 1 + \frac{4\nu^2 - 1}{4jk_1\rho} \right) \right], \quad (11)$$

$$K_\nu \approx \sqrt{\frac{\pi}{jk_1\rho}} e^{-\frac{jk_1\rho}{2}} \left[ 1 + \frac{4\nu^2 - 1}{4jk_1\rho} \right],$$

valid for large values of  $|k_1\rho|$ , into (3)–(5). In light of this comparison, it is concluded that formulation by King, limited to the sole low frequency

range, is a special case of the zeroth-order formulation by Wait (1)–(6), resulting from the hypothesis that the skin depth in the medium is much smaller than the source-receiver distance. The present study is aimed at deriving improved, second-order quasi-static approximations for the fields of a HED, which can be used when the zeroth-order ones fail, that is when the quantities  $\tau = k_0/k_1$  and  $\tau^2$  are not negligible with respect to unity. To this end, consider the complete integral representations for the field components at  $z = 0$  [22]

$$\begin{aligned}
 E_{\rho 0} &= \frac{j\omega\mu_0 p \cos \varphi}{2\pi k_1^2} \left( P_a - \frac{1}{\rho} Q_a \right), & H_{\rho 0} &= \frac{p \sin \varphi}{2\pi} \left( -\frac{1}{2} P_b + \frac{1}{k_1^2 \rho} Q_b \right), \\
 E_{\varphi 0} &= \frac{j\omega\mu_0 p \sin \varphi}{2\pi} \left( P_c - \frac{1}{k_1^2 \rho} Q_a \right), & H_{\varphi 0} &= \frac{p \cos \varphi}{2\pi} \left( \frac{1}{2} P_d - \frac{1}{k_1^2 \rho} Q_b \right), \quad (12) \\
 E_{z0} &= \frac{j\omega\mu_0 p \cos \varphi}{4\pi k_1^2} Q_d, & H_{z0} &= \frac{p \sin \varphi}{2\pi} Q_c,
 \end{aligned}$$

where the  $P_m$ 's and  $Q_m$ 's are, respectively, the  $J_0$ - and  $J_1$ -Hankel transforms defined as follows

$$\begin{aligned}
 P_a &= \int_0^\infty \frac{u_0 u_1}{u_0 + \tau^2 u_1} J_0(\lambda \rho) \lambda d\lambda & Q_a &= \int_0^\infty \frac{1}{u_0 + \tau^2 u_1} J_1(\lambda \rho) \lambda^2 d\lambda, \\
 P_b &= \int_0^\infty \frac{u_0 - u_1}{u_0 + u_1} J_0(\lambda \rho) \lambda d\lambda & Q_b &= \int_0^\infty \frac{u_0 - u_1}{u_0 + \tau^2 u_1} J_1(\lambda \rho) \lambda^2 d\lambda, \\
 P_c &= \int_0^\infty \frac{1}{u_0 + u_1} J_0(\lambda \rho) \lambda d\lambda & Q_c &= \int_0^\infty \frac{1}{u_0 + u_1} J_1(\lambda \rho) \lambda^2 d\lambda, \quad (13) \\
 P_d &= \int_0^\infty \frac{u_0 - \tau^2 u_1}{u_0 + \tau^2 u_1} J_0(\lambda \rho) \lambda d\lambda \\
 Q_d &= \frac{1}{\tau^2} \int_0^\infty \frac{u_0 - \tau^2 u_1}{u_0 + \tau^2 u_1} J_1(\lambda \rho) \lambda^2 d\lambda,
 \end{aligned}$$

being  $J_n(\cdot)$  the  $n$ th-order Bessel function, and  $u_n = \sqrt{\lambda^2 - k_n^2}$ . Next, substituting the power series expansions

$$\frac{1}{u_0 + u_1} = \frac{u_0 - u_1}{k_1^2(1 - \tau^2)} = \frac{1 + \tau^2}{k_1^2} (u_0 - u_1) + \mathcal{O}(\tau^4), \quad (14)$$

$$\frac{1}{u_0 + \tau^2 u_1} = \frac{1}{u_0} - \tau^2 \frac{u_1}{u_0^2} + \mathcal{O}(\tau^4), \quad (15)$$

into (13) provides the expressions

$$P_a = \int_0^\infty \left( u_1 - \tau^2 \frac{u_1^2}{u_0} \right) J_0(\lambda \rho) \lambda d\lambda + \mathcal{O}(\tau^4), \quad (16)$$

$$P_b = \frac{1 + \tau^2}{k_1^2} \int_0^\infty (u_0 - u_1)^2 J_0(\lambda \rho) \lambda d\lambda + \mathcal{O}(\tau^4), \quad (17)$$

$$P_c = \frac{1 + \tau^2}{k_1^2} \int_0^\infty (u_0 - u_1) J_0(\lambda\rho) \lambda d\lambda + \mathcal{O}(\tau^4), \tag{18}$$

$$P_d = \int_0^\infty \left(1 - 2\tau^2 \frac{u_1}{u_0}\right) J_0(\lambda\rho) \lambda d\lambda + \mathcal{O}(\tau^4), \tag{19}$$

$$Q_a = \int_0^\infty \left(\frac{1}{u_0} - \tau^2 \frac{u_1}{u_0^2}\right) J_1(\lambda\rho) \lambda^2 d\lambda + \mathcal{O}(\tau^4), \tag{20}$$

$$Q_b = \int_0^\infty \left[1 - \frac{u_1}{u_0} (1 + \tau^2) + \tau^2 \frac{u_1^2}{u_0^2}\right] J_1(\lambda\rho) \lambda^2 d\lambda + \mathcal{O}(\tau^4), \tag{21}$$

$$Q_c = \frac{1 + \tau^2}{k_1^2} \int_0^\infty (u_0 - u_1) J_1(\lambda\rho) \lambda^2 d\lambda + \mathcal{O}(\tau^4), \tag{22}$$

$$Q_d = \frac{1}{\tau^2} \int_0^\infty \left(1 - 2\tau^2 \frac{u_1}{u_0} + 2\tau^4 \frac{u_1^2}{u_0^2}\right) J_1(\lambda\rho) \lambda^2 d\lambda + \mathcal{O}(\tau^4), \tag{23}$$

in the last of which, because of the factor  $1/\tau^2$  outside the integral, the expansion (15) has been conveniently made explicit up to the fourth-order term, as follows

$$\frac{1}{u_0 + \tau^2 u_1} = \frac{1}{u_0} - \tau^2 \frac{u_1}{u_0^2} + \tau^4 \frac{u_1^2}{u_0^3} + \mathcal{O}(\tau^6). \tag{24}$$

As well known, asymptotic power series can be integrated termwise without additional restrictions or specific conditions on the validity of the result [23, p. 153, No. 8.31]. This means that the asymptotic expansions for the  $P_m$ 's and  $Q_m$ 's can be directly obtained from the term-by-term integration of the right-hand sides of (16)–(23). Hence, with the  $J_0$ - and  $J_1$ -transforms in (16)–(23) decomposed into sums of integrals, each term can be reduced to a known tabulated Sommerfeld Integral. The only exception is the second-order contribution to  $Q_a$  which, in the form appearing in (20), cannot be analytically evaluated. To overcome this problem, the asymptotic approximation

$$\frac{1}{u_0^2} = \frac{1}{\lambda^2 - \tau^2 k_1^2} = \frac{1}{\lambda^2} + \mathcal{O}(\tau^2) \tag{25}$$

is suitably introduced in (20) so as to obtain

$$Q_a = \int_0^\infty \frac{1}{u_0} J_1(\lambda\rho) \lambda^2 d\lambda - \tau^2 \int_0^\infty u_1 J_1(\lambda\rho) d\lambda + \mathcal{O}(\tau^4). \tag{26}$$

Use of the identity [24]

$$\lambda J_1(\lambda\rho) = -\frac{\partial J_0(\lambda\rho)}{\partial \rho} \tag{27}$$

and the relation [2]

$$u_n^2 J_\nu(\lambda\rho) = - \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{\nu^2}{\rho^2} + k_n^2 \right) J_\nu(\lambda\rho), \tag{28}$$

allows to rewrite the two terms in (26) as

$$\int_0^\infty \frac{1}{u_0} J_1(\lambda\rho) \lambda^2 d\lambda = - \frac{\partial}{\partial \rho} \int_0^\infty \frac{1}{u_0} J_0(\lambda\rho) \lambda d\lambda, \tag{29}$$

$$\int_0^\infty u_1 J_1(\lambda\rho) d\lambda = - (\nabla_t^2 - \frac{1}{\rho^2} + k_1^2) \int_0^\infty \frac{1}{u_1} J_1(\lambda\rho) d\lambda, \tag{30}$$

with

$$\nabla_t^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho}, \tag{31}$$

and where the interchange of derivative and integral is justified in virtue of the continuous dominated convergence theorem [21]. The method to decompose  $Q_a$  into the sum of simpler terms, and apply (27) and (28) to each of them, can be repeated for the remaining integrals in (16)–(23). It yields

$$P_a = - (\nabla_t^2 + k_1^2) (S_1 - \tau^2 S_0) + \mathcal{O}(\tau^4), \tag{32}$$

$$P_b = - \frac{2(1 + \tau^2)}{k_1^2} (\nabla_t^2 + k_0^2) (\nabla_t^2 + k_1^2) S_{01} + \mathcal{O}(\tau^4), \tag{33}$$

$$P_c = \frac{1 + \tau^2}{k_1^2} [(\nabla_t^2 + k_1^2) S_1 - (\nabla_t^2 + k_0^2) S_0] + \mathcal{O}(\tau^4), \tag{34}$$

$$P_d = 2\tau^2 (\nabla_t^2 + k_1^2) S_{01} + \mathcal{O}(\tau^4), \tag{35}$$

$$Q_a = - \frac{\partial S_0}{\partial \rho} + \tau^2 \left( \nabla_t^2 - \frac{1}{\rho^2} + k_1^2 \right) \hat{S}_1 + \mathcal{O}(\tau^4), \tag{36}$$

$$Q_b = \frac{\partial}{\partial \rho} (\nabla_t^2 + k_1^2) \left[ - (1 + \tau^2) S_{01} + \tau^2 \hat{S}_0 \right] + \mathcal{O}(\tau^4), \tag{37}$$

$$Q_c = - \frac{1 + \tau^2}{k_1^2} \frac{\partial}{\partial \rho} [(\nabla_t^2 + k_1^2) S_1 - (\nabla_t^2 + k_0^2) S_0] + \mathcal{O}(\tau^4), \tag{38}$$

$$Q_d = 2 \frac{\partial}{\partial \rho} (\nabla_t^2 + k_1^2) \left( -S_{01} + \tau^2 \hat{S}_0 \right) + \mathcal{O}(\tau^4), \tag{39}$$

with

$$S_n = \int_0^\infty \frac{1}{u_n} J_0(\lambda\rho) \lambda d\lambda, \quad n = 0, 1, \tag{40}$$

$$S_{01} = \int_0^\infty \frac{1}{u_0 u_1} J_0(\lambda\rho) \lambda d\lambda, \tag{41}$$

$$\hat{S}_0 = \int_0^\infty \frac{1}{u_0^2} J_0(\lambda\rho) \lambda d\lambda, \tag{42}$$

$$\hat{S}_1 = \int_0^\infty \frac{1}{u_1} J_1(\lambda\rho) d\lambda, \tag{43}$$

and where the identity [2]

$$\int_0^\infty J_0(\lambda\rho) \lambda d\lambda = 0 \tag{44}$$

has been accounted for. It should be observed that strictly the integral in (44) is divergent. As argued in [2], the equality above can be made rigorous by simply replacing the left-hand side with the quantity

$$\lim_{\zeta \rightarrow 0^+} \int_0^\infty e^{-u_n \zeta} J_0(\lambda\rho) \lambda d\lambda = - \left[ \frac{\partial}{\partial \zeta} \int_0^\infty \frac{e^{-u_n \zeta}}{u_n} J_0(\lambda\rho) \lambda d\lambda \right]_{\zeta=0}, \tag{45}$$

which is identically null according to [25, p. 9, No. 23]. The transforms on the right-hand side of (40)–(43) are well known Sommerfeld Integrals, tabulated in [25]. Thus, applying formulas [25, p. 7, No. 4], [25, p. 8, No. 17], [25, p. 11, No. 45], and [25, p. 18, No. 3], respectively, provides

$$S_n = \frac{e^{-jk_n\rho}}{\rho} \tag{46}$$

$$S_{01} = K_0(\alpha\rho) I_0(\beta\rho), \tag{47}$$

$$\hat{S}_0 = K_0(jk_0\rho), \tag{48}$$

$$\hat{S}_1 = \frac{1 - e^{-jk_1\rho}}{jk_1\rho}, \tag{49}$$

with

$$\alpha = \frac{1}{2}j(k_1 + k_0), \quad \beta = \frac{1}{2}j(k_1 - k_0). \tag{50}$$

After substituting (46)–(49) into (32)–(39), and performing all the derivatives, it is found that

$$P_a = \frac{1}{\rho^3} \left\{ [k_0^2\rho^2 + (1 + jk_0\rho - k_0^2\rho^2)\tau^2] e^{-jk_0\rho} - (1 + jk_1\rho) e^{-jk_1\rho} \right\}, \tag{51}$$

$$P_b = -\frac{1}{\rho^2} [4K_1 I_1 + jk_1\rho (K_0 I_1 - K_1 I_0) + jk_0\rho (K_0 I_1 + K_1 I_0)], \tag{52}$$

$$P_c = \frac{1}{k_1^2\rho^3} \left[ (1 + \tau^2) (-1 - jk_0\rho + 2k_0^2\rho^2) e^{-jk_0\rho} + (1 + jk_1\rho + jk_0\rho\tau + \tau^2) e^{-jk_1\rho} \right], \tag{53}$$



$$P_d = k_0^2 (K_0 I_0 + K_1 I_1), \tag{54}$$

$$Q_a = -\frac{1}{\rho^2} \left[ j k_0 \rho \tau - (1 + j k_0 \rho) e^{-j k_0 \rho} + \tau^2 e^{-j k_1 \rho} \right], \tag{55}$$

$$Q_b = \frac{k_1^2}{\rho} \left[ K_1 I_1 + \frac{j k_0 \rho}{2} (K_0 I_1 + K_1 I_0) - j k_0 \rho \tau^2 \tilde{K}_1 \right], \tag{56}$$

$$Q_c = -\frac{1}{k_1^2 \rho^4} \left[ (1 + \tau^2) (3 + 3 j k_0 \rho - k_0^2 \rho^2) e^{-j k_0 \rho} - (3 - k_0^2 \rho^2 + 3 j k_1 \rho - k_1^2 \rho^2 + 3 j k_0 \rho \tau + 3 \tau^2) e^{-j k_1 \rho} \right], \tag{57}$$

$$Q_d = \frac{k_1^2}{\rho} \left\{ (1 - \tau^2) [2 K_1 I_1 + j k_0 \rho (K_0 I_1 + K_1 I_0)] - 2 j k_0 \rho \tau^2 \tilde{K}_1 \right\}, \tag{58}$$

where only the terms on the order up to  $\tau^2$  have been retained. In the expressions above,  $K_n$  and  $I_n$  denote the  $n$ th-order modified Bessel functions calculated at  $\alpha\rho$  and  $\beta\rho$ , respectively, while the  $\sim$  symbol above  $K_1$  denotes calculation at  $j k_0 \rho$ . Finally, substitution of (51)–(58) into (12) provides

$$E_{\rho 0} = \frac{j \omega \mu_0 p \cos \varphi}{2 \pi k_1^2 \rho^3} \left[ j k_0 \rho \tau + (1 + j k_0 \rho - k_0^2 \rho^2) (-1 + \tau^2) e^{-j k_0 \rho} - (1 + j k_1 \rho - \tau^2) e^{-j k_1 \rho} \right], \tag{59}$$

$$E_{\varphi 0} = \frac{j \omega \mu_0 p \sin \varphi}{2 \pi k_1^2 \rho^3} \left\{ j k_0 \rho \tau + [(2 + \tau^2) (-1 - j k_0 \rho + k_0^2 \rho^2) + k_0^2 \rho^2 \tau^2] e^{-j k_0 \rho} + (1 + j k_1 \rho + j k_0 \rho \tau + 2 \tau^2) e^{-j k_1 \rho} \right\}, \tag{60}$$

$$E_{z 0} = \frac{j \omega \mu_0 p \cos \varphi}{4 \pi \rho} \left\{ (1 - \tau^2) [2 K_1 I_1 + j k_0 \rho (K_0 I_1 + K_1 I_0)] - 2 j k_0 \rho \tau^2 \tilde{K}_1 \right\}, \tag{61}$$

$$H_{\rho 0} = \frac{p \sin \varphi}{2 \pi \rho^2} \left[ 3 K_1 I_1 + \frac{j k_1 \rho}{2} (K_0 I_1 - K_1 I_0) + j k_0 \rho (K_0 I_1 + K_1 I_0) - j k_0 \rho \tau^2 \tilde{K}_1 \right], \tag{62}$$

$$H_{\varphi 0} = -\frac{p \cos \varphi}{2 \pi \rho^2} \left[ K_1 I_1 + \frac{j k_0 \rho}{2} (K_0 I_1 + K_1 I_0) - j k_0 \rho \tau^2 \tilde{K}_1 - \frac{k_0^2 \rho^2}{2} (K_0 I_0 + K_1 I_1) \right], \tag{63}$$

$$H_{z0} = -\frac{p \sin \varphi}{2\pi k_1^2 \rho^4} \left[ (1 + \tau^2) (3 + 3jk_0\rho - k_0^2\rho^2) e^{-jk_0\rho} - (3 - k_0^2\rho^2 + 3jk_1\rho - k_1^2\rho^2 + 3jk_0\rho\tau + 3\tau^2) e^{-jk_1\rho} \right]. \quad (64)$$

In the quasi-static limit  $k_0\rho \ll 1$  the function  $\tilde{K}_1$  may be replaced with its asymptotic behavior for small arguments [24, 26, 27]

$$\tilde{K}_1 \cong \frac{1}{jk_0\rho}, \quad (65)$$

while  $e^{-jk_0\rho}$  approaches unity. As a consequence, Equations (59)–(64) simplify into

$$E_{\rho 0} = \frac{j\omega\mu_0 p \cos \varphi}{2\pi k_1^2 \rho^3} \left[ -1 - (1 + jk_1\rho) e^{-jk_1\rho} + \tau^2 (1 + e^{-jk_1\rho}) \right], \quad (66)$$

$$E_{\varphi 0} = \frac{j\omega\mu_0 p \sin \varphi}{2\pi k_1^2 \rho^3} \left[ -2 + (1 + jk_1\rho) e^{-jk_1\rho} + \tau^2 (-1 + 2e^{-jk_1\rho}) \right], \quad (67)$$

$$E_{z0} = \frac{j\omega\mu_0 p \cos \varphi}{2\pi\rho} \left[ (1 - \tau^2) K_1 I_1 - \tau^2 \right], \quad (68)$$

$$H_{\rho 0} = \frac{p \sin \varphi}{2\pi\rho^2} \left[ 3K_1 I_1 + \frac{jk_1\rho}{2} (K_0 I_1 - K_1 I_0) - \tau^2 \right], \quad (69)$$

$$H_{\varphi 0} = \frac{p \cos \varphi}{2\pi\rho^2} (-K_1 I_1 + \tau^2), \quad (70)$$

$$H_{z0} = -\frac{p \sin \varphi}{2\pi k_1^2 \rho^4} \left[ 3 - (3 + 3jk_1\rho - k_1^2\rho^2) e^{-jk_1\rho} + 3\tau^2 (1 - e^{-jk_1\rho}) \right]. \quad (71)$$

The zeroth-order approximations for the fields (1)–(6) may be obtained directly from (66)–(71) by setting  $\tau=0$ .

### 3. DISCUSSION

In order to verify the validity and highlight the advantages of the proposed approximate formulas, in this section the zeroth- and second-order quasi-static approximations for the fields are used to calculate the magnitudes of the transverse EM field components produced by a unit-moment HED at a point 30 m away from it. The source and observation points are located on the surface of a medium with  $\epsilon_1=10\epsilon_0$ , while the radial position of the observation point and the dipole axis are 45 degrees apart from each other ( $\varphi = \pi/4$ ). As in the previous work focused on the field of a VED [21], the conductivity  $\sigma_1$  of the half-space is taken as a parameter and assumed to be equal to 0.01, 0.1,

and 1 mS/m. Permafrost, igneous and metamorphic rocks [18] are examples of materials with conductivity on these orders of magnitude.

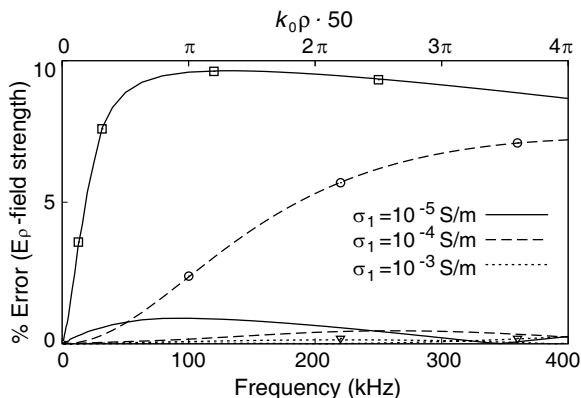
Figures 2–5 show the relative percent error that originates from comparing the results provided by the quasi-static expressions for the transverse fields with those arising from the numerical evaluation of the integrals in (13), plotted versus frequency. The highly accurate numerical evaluation of the field integrals is obtained through the quasi-analytical procedure described in [28, 29], which has been proven to ensure at least 13 digits of precision. According to this procedure, application of the Cauchy’s residue theorem allows to write

$$\int_0^\infty f(\lambda)J_\nu(\lambda\rho)\lambda^{\nu+1}d\lambda \cong j(-1)^n \sum_{l=1}^L r_l \lambda_l^\nu K_\nu(\lambda_l \rho), \quad (72)$$

the left-hand side of which is any of the Hankel transforms in (13). In the above equation  $f(\lambda)$  is an even function, while the  $\lambda_l$ ’s and  $r_l$ ’s are the coefficients of the rational approximation

$$f(\lambda) \cong \sum_{l=1}^L \frac{r_l}{2\lambda_l} \left[ \frac{1}{\lambda + j\lambda_l} - \frac{1}{\lambda - j\lambda_l} \right], \quad \Re[\lambda_l] > 0. \quad (73)$$

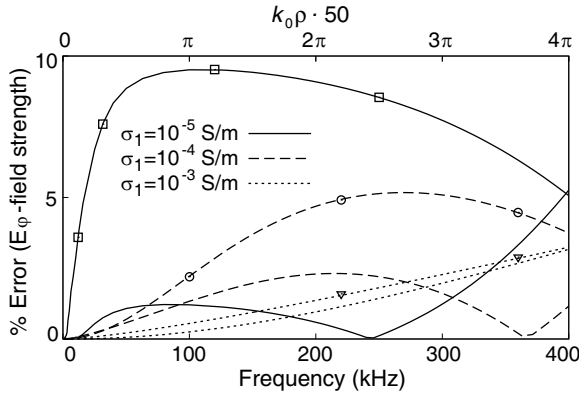
Figure 2 depicts percent errors resulting from the computation of the magnitude of  $E_\rho$ , and zeroth-order curves are marked with points to be discerned from the second-order ones. It can be noticed that for  $\sigma_1=0.01$  mS/m the error generated by the zeroth-order approximation approaches 10% for most of the frequencies in the  $0 \leq k_0\rho \leq 0.25$



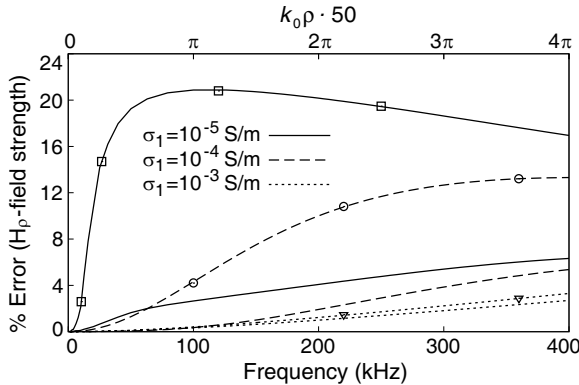
**Figure 2.** Relative errors of the zeroth-order (lines and points) and second-order (lines) quasi-static expressions for  $E_\rho$  as compared to the exact results. Errors are plotted versus frequency with varying  $\sigma_1$ .

range, while the error due to the improved formula (66) is always less than 1%.

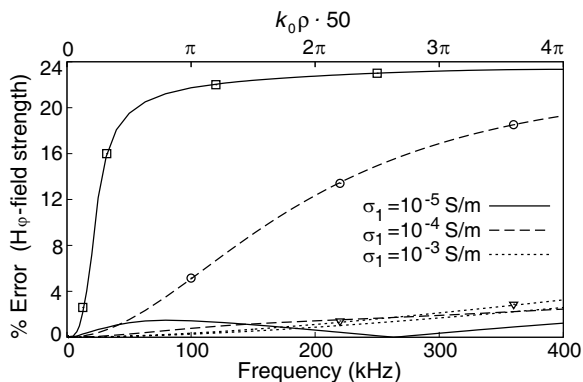
As  $\sigma_1$  increases, the gap between the relative errors generated by the two formulations becomes narrower and narrower, up to cancel out for  $\sigma_1 \geq 1$  mS/m. Analogous considerations can be drawn from the analysis of Figs. 3, 4, and 5, which illustrate, respectively, the relative percent error that originates from using the quasi-static expressions for the  $E_\varphi^-$ ,  $H_\rho^-$ , and  $H_\varphi^-$ -fields. Improved expressions for these field



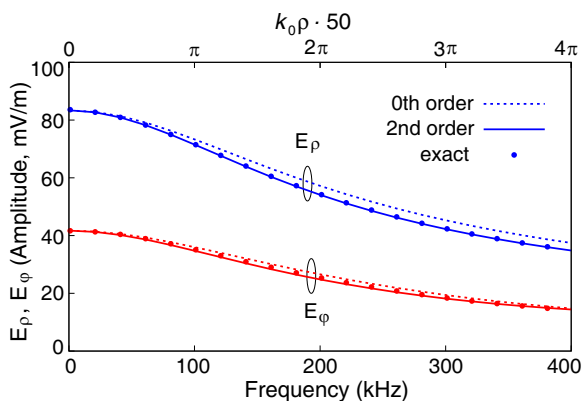
**Figure 3.** Relative errors of the zeroth-order (lines and points) and second-order (lines) quasi-static expressions for  $E_\varphi$  as compared to the exact results. Errors are plotted versus frequency with varying  $\sigma_1$ .



**Figure 4.** Relative errors of the zeroth-order (lines and points) and second-order (lines) quasi-static expressions for  $H_\rho$  as compared to the exact results. Errors are plotted versus frequency with varying  $\sigma_1$ .



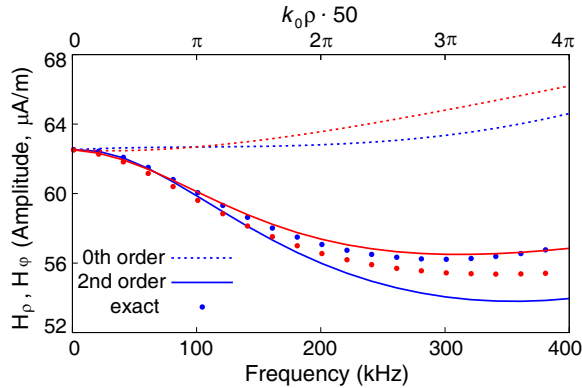
**Figure 5.** Relative errors of the zeroth-order (lines and points) and second-order (lines) quasi-static expressions for  $H_\varphi$  as compared to the exact results. Errors are plotted versus frequency with varying  $\sigma_1$ .



**Figure 6.** Amplitude-frequency spectra of the zeroth- and second-order quasi-static expressions for  $E_\rho$  (blue color) and  $E_\varphi$  (red color), compared to the exact solutions.

components permit to maintain the relative error in the  $0 \leq k_0\rho \leq 0.25$  range well below the threshold of 7%, while previously published formulas would produce, in the same interval, errors up to more than 9% ( $E_\varphi$ ), 20% ( $H_\rho$ ), and 23% ( $H_\varphi$ ).

To better appreciate the difference in accuracy exhibited by the zeroth- and second-order quasi-static approximations, plots of the magnitudes of the transverse EM field components, resulting from either using the two approaches or numerically evaluating the field



**Figure 7.** Amplitude-frequency spectra of the zeroth- and second-order quasi-static expressions for  $H_\rho$  (blue color) and  $H_\varphi$  (red color), compared to the exact solutions.

integrals, are presented in Figs. 6 and 7. Fig. 6 shows the amplitude-frequency spectra of  $E_\rho$  (blue color) and  $E_\varphi$  (red color), assuming  $\rho = 30$  m and  $\sigma_1 = 0.1$  mS/m. What is pointed out is the absence of discrepancy between the second-order accurate and the numerical outcomes related to the  $E_\rho$ -field. As to the  $E_\varphi$ -field, the second-order quasi-static approximation is seen to be slightly more accurate than the zeroth-order one. On the other hand, with the same assumptions for  $\rho$  and  $\sigma_1$  as in the previous example, Fig. 7 depicts the trends of the amplitudes of  $H_\rho$  and  $H_\varphi$  against frequency. As can be seen from the plotted curves, only the second-order accurate trends reproduce with acceptable accuracy the exact solutions (the maximum percent error is, for the two components, less than 6% and 3%), while the zeroth-order spectra diverge from the exact ones, starting to diverge at very low frequencies.

#### 4. CONCLUSION

This paper presents second-order quasi-static approximations for the radial distributions of the EM field components produced by a HED on the surface of a homogeneous lossy half-space. The complete integral expressions for the fields are expanded into power series of the ratio between the wavenumbers in free-space and in the material half-space. Then, the lower-order terms of the expansions are reduced to known tabulated Sommerfeld Integrals. Numerical simulations show that, in the quasi-static frequency range and beyond, the derived improved

expressions for the fields exhibit better accuracy than the previously published zeroth-order approximations, to a greater extent for smaller values of the medium conductivity.

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