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TIME-DOMAIN *REAL-VALUED TM*-MODAL WAVES IN LOSSY WAVEGUIDES

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Abstract—The waveguide has a perfectly conducting surface. Its cross section domain is bounded by a singly-connected contour of a rather arbitrary but enough smooth form. Possible waveguide losses are modeled by a homogeneous conductive medium in the waveguide. The boundary-value problem for the system of Maxwell's equations with time derivative is solved in the time domain. The real-valued solutions are obtained in Hilbert space L_2 in a form of transverse-longitudinal decompositions. Every field component is a product of the vector element of the modal basis dependent on transverse coordinates, and the modal amplitudes dependent on time and axial coordinate. Three examples are included. The dynamic properties of the modal waves and concomitant energetic waves are studied and their dependence on time illustrated graphically.

1. INTRODUCTION

A layout of any analytical study of the waveguide time-domain problems consists of two principal parts. The first one applies to derivation of the waveguide modal fields, the amplitudes of which depend on time, t, and an axial waveguide coordinate, z. At this step, the one-dimensional Klein-Gordon Equation (KGE) appears, eventually, which specifies the modal amplitudes. The second part relates to solving the KGE and has directly to do with the physical analysis of the *dynamic* processes, in which the modal amplitudes participate.

In the first part of our layout, we apply the simple mathematical technique of the vector analysis for a *straightforward* derivation† of a

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† Thereby, this article is fed to the young researchers of the PhD and graduate student levels. Our former publications (see, for example, [1–5]) addressed, mostly, to an advanced

complete set of the TM-time-domain modal waves from the system of Maxwell's equations with the time derivative, ∂_t. Klein-Gordon equation appears in this way naturally enough. Our main goal in the time-domain studies is a fresh look at the *dynamic* physical properties of the waveguide waves.

All the electromagnetic quantities in Maxwell's equations are measurable. Therefore, the solutions to Maxwell's equations, with their pertinent physical content, should be found in a class of the *real-valued* functions.

The classical time-harmonic field approach operates in a space of complex-valued solutions. The real-valued fields, which are needed for the physical analysis, can be obtained then via superposition of *two* solutions as

$$\mathcal{E}\left(\mathbf{R},t\right) = \mathbf{E}\left(\mathbf{R},\omega\right)\exp\left(-i\omega t\right) + \mathbf{E}^*\left(\mathbf{R},\omega\right)\exp\left(i\omega t\right) \tag{1}$$

where $\mathcal{E}(\mathbf{R},\, t)$ and $\mathbf{E}(\mathbf{R},\, \omega)$ are the electric field strength and its phasor, respectively; $\mathbf{R}$ is a position vector at a point of observation, t an observation time, $-\infty < t < \infty$, and ω a frequency parameter, $-\infty < \omega < \infty$.

Just the *superposition* in (1) presupposes *linearity* of Maxwell's equations.

We find the solutions to this problem within the framework of a waveguide version of the evolutionary approach to electromagnetics[‡], see [2–5]. The TM-wave solutions are obtained directly in the Hilbert space, L_2, of the *real-valued* functions in a form, which can be exhibited symbolically as follows:

$$\mathcal{H}(\mathbf{R},t) = I(z,t)\mathbb{H}(\mathbf{r}) \quad \text{and} \quad \mathcal{E}(\mathbf{R},t) = V(z,t)\mathbb{E}(\mathbf{r}) + e(z,t)\mathbb{Z}(\mathbf{r}), \tag{2}$$

where $\mathbb{H}(\mathbf{r})$ and $\mathbb{E}(\mathbf{r})$ are the two-component basis vectors in the waveguide cross section; $\mathbb{Z}(\mathbf{r})$ is a one-component basis vector with the unit vector $\mathbf{z}$, z the axial variable, and $\mathbf{r}$ a projection of $\mathbf{R}$ onto the waveguide cross section. The scalar factors, $I(z,\, t)$, $V(z,\, t)$ and $e(z,\, t)$, are the amplitudes, physically.

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reader who are familiar with the linear functional spaces, in general, and the Hilbert space, L_2, especially. Besides, that reader should have a notion about a topology of the Hilbert space in the sense of Weyl theorem from the functional analysis.

[‡] Postulation of the *harmonic in time* varying the fields (like in Eq. (1)) "kills" the time derivative, ∂_t, in Maxwell's equations at the very beginning. We keep ∂_t from the beginning and up to the end in our analysis. In this way, we obtain the *KGE* where ∂_t^2 participates. Mathematicians call all the differential equations with time derivatives as *evolution* (or *evolutionary*) equations, see, for example, *Journal of Evolution Equations* [6]. Apparently, this term was induced by Cauchy theorem for dynamic systems. Any solution, obtained via the Cauchy theorem, exhibits how a process *progresses in time* (i.e., *evolves*, shortly) starting from a given initial state and up to the state at a time of observation. That is why we name this method as *Evolutionary* Approach to Electromagnetics (*EAE*).

We had an opportunity to discuss a background history of the time-domain electromagnetics in our previous publications, e.g., [7–10]. A reader can find additional information on this topic in other publications, e.g., [11–17].

The article is organized as follows:

In Section 2, the time-domain problem is stated for the transverse-longitudinal decompositions of Maxwell's equations with time derivative, ∂_t. Hilbert space, L_2, of the real-valued functions is defined as the space of solutions.

In Section 3, this problem is solved, and main results are listed. A complete set of the basis elements, $\{\mathbb{H}_m, \mathbb{E}_m, \mathbb{Z}_m\}_{m=1}^{\infty}$, is derived with needed their physical dimensions. The "magnetic" basis elements, $\mathbb{H}_m(\mathbf{r})$, are obtained with dimension *ampere per meter*, whereas the "electric" elements, $\mathbb{E}_m(\mathbf{r})$ and $\mathbb{Z}_m(\mathbf{r})$, are derived with dimension *volt per meter*, both. The modal amplitudes, $I_m(z, t)$, $V_m(z, t)$ and $e_m(z, t)$, which are attached to the basis elements (like in Eq. (2)), have dimensionless quantities. The modal amplitude problem is obtained in a general form. The modal basis problem and modal amplitude problem are obtained as the *autonomous* ones.

In Section 4, the modal amplitude problem is analyzed. Eventually, this problem comes to solving the *Klein-Gordon Equation* for the modal amplitude $e_m(z, t)$. As soon as $e_m(z, t)$ is obtained, the other amplitudes, $I_m(z, t)$ and $V_m(z, t)$, can be found as t- and z-derivatives of $e_m(z, t)$, respectively. Implementation of this scheme is illustrated by two examples where the variables (z, t) are separated and exact explicit solutions are obtained.

In Section 5, a new version of separation of the variables in the *KGE* is considered. This version is based on Miller's concept about existence of *ten* hidden *"orbits of symmetry"* in the *KGE* [20]. One of the orbits is taken for analysis herein. A final physical result can be shortly announced as follows. The symmetry of *KGE* on the chosen orbit discloses existence of a *new* countable set of the modal amplitudes oscillating with the same cut-off frequency. Besides, these elements have some polynomial factors, which involve z and t variables. The lowest element coincides with that one what yields separation of variables on the orbit (z, t). All other elements belong to another orbit of symmetry.

In Section 6, the time-domain energetic quantities are listed. The conservation of energy law is presented as a continuity equation with z- and t-derivatives. The modal energetic waves, which propagate accompany with the field waves, are discovered and illustrated graphically.

In Section 7, a short summary of new research findings is given.

2. THE NOTATIONS AND THE GOVERNING EQUATIONS

2.1. A Description of the Waveguide and the Notations

The waveguide is geometrically homogeneous along its axis, Oz. Its invariable cross-section domain, S, is bounded by a closed singly-connected contour, L. The shape of L may be rather arbitrary provided that none of the possible its *inner* angles (i.e., measured *within* S) exceed π. The standard waveguides with the rectangular cross section satisfy this requirement. A right-handed triplet $\{\mathbf{z}, \mathbf{l}, \mathbf{n}\}$ (where $\mathbf{z} \times \mathbf{l} = \mathbf{n}$ and so on) of the mutually orthogonal unit vectors is introduced. The vector $\mathbf{z}$ is oriented along the waveguide axis, Oz; the vector $\mathbf{l}$ is tangential to the contour, L; and $\mathbf{n}$ is the outer normal to the domain S. The waveguide surface has the properties of the perfect electric conductor. The waveguide is filled with a lossy medium specified by its conductivity, σ. The relative permittivity and permeability of the medium within the waveguide are taken as $\varepsilon = \mu = 1$.

2.2. The Governing Equations

The electromagnetic field quantities are the functions of *four* independent variables, namely: two transverse coordinates, which are accumulated in the vector $\mathbf{r}$, and the variables $(z,\, t)$. In the form (2) of the expected solution, every field component is presented as a *product* of two functions. In every product, one function depends on $(z,\, t)$, only, and the other one depends on $\mathbf{r}$, solely. This trick is known in the partial differential equation theory as an *incomplete* separation of the variables [18]. In order to apply this method effectively, we should first rearrange Maxwells equations. Projecting these equations onto the waveguide cross section and the waveguide axis results in

$$\partial_z \mathbf{E} + \mu_0 \partial_t \left[\mathbf{H} \times \mathbf{z}\right] = \nabla_\perp E_z \tag{3a}$$

$$\nabla_\perp \cdot \left[\mathbf{z} \times \mathbf{E}\right] = 0 \tag{3b}$$

$$\partial_{\mathbf{z}} \left[\mathbf{z} \times \mathbf{H}\right] = \epsilon_0 \partial_t \mathbf{E} + \sigma \mathbf{E} \tag{3c}$$

$$\nabla_\perp \cdot \left[\mathbf{H} \times \mathbf{z}\right] = \epsilon_0 \partial_t E_z + \sigma E_z \tag{3d}$$

$$\nabla_\perp \cdot \mathbf{H} = 0 \tag{3e}$$

$$\nabla_\perp \cdot \mathbf{E} + \partial_z E_z = -\frac{\sigma}{\epsilon_0} \int_0^t \left(\nabla_\perp \cdot \mathbf{E} + \partial_z E_z\right) dt' \tag{3f}$$

where $\nabla_\perp$ is the transverse part of the operator nabla, ∇. This operator, $\nabla_\perp$, acts on the transverse variables, $(\mathbf{r})$, *only*. Eqs. (3a) and (3b) are the projections of the vector equation, $\nabla \times \mathcal{E} = -\mu_0 \partial_t \mathcal{H}$, onto the waveguide cross-section, S, and axis Oz, respectively. Eqs. (3c) and (3d) are the projections of equation $\nabla\times\mathcal{H}=\epsilon_0\partial_t\mathcal{E}+\sigma\mathcal{E}$ onto the domain S and the axis Oz, respectively. Eq. (3e) is the divergent equation, $\mu_0\nabla\cdot\mathbf{H}=0$. Combination of the equation $\epsilon_0\nabla\cdot\mathcal{E}=\rho$ and the continuity equation, $\nabla\cdot\mathcal{J}=-\partial_t\rho$, for the induced current and charge densities, $\mathcal{J}=\sigma\mathcal{E}$ and ρ, results in Eq. (3f). A complete set of the Maxwell's differential Eqs. (3a)–(3f) should be supplemented with the boundary conditions as

$$E_z|_L=0,\quad \mathbf{l}\cdot\mathbf{E}|_L=0,\quad \mathbf{n}\cdot\mathbf{H}|_L=0. \tag{4}$$

2.3. The Space of Solutions

A Hilbert space, L_2, of the *real-valued* functions is chosen as a space of solutions and defined by an inner product as

$$\langle\mathcal{X}_1,\mathcal{X}_2\rangle=\int_{z_1}^{z_2}\int_{t_1}^{t_2}\int_S\left[\mu_0\mathbf{H}_1\cdot\mathbf{H}_2+\epsilon_0\left(\mathbf{E}_1\cdot\mathbf{E}_2+E_{z1}\cdot E_{z2}\right)\right]dsdtdz \tag{5}$$

where $\mathcal{X}_1=\mathrm{col}(\mathcal{E}_1,\mathcal{H}_1)$ and $\mathcal{X}_2=\mathrm{col}(\mathcal{E}_2,\mathcal{H}_2)$ is a pair of the *column*-vectors from the space of solutions, $z_1\le z\le z_2$ and $t_1\le t\le t_2$. The free-space constants, ϵ_0 and μ_0, play role of the weighting coefficients herein. Notice that the procedure of complex conjugation is absent in the integrand in Eq. (5).

3. SOLVING THE PROBLEM (3)–(4)

Equations (3b) and (3e) suggest to search out the vectors $\mathbf{E}$ and $\mathbf{H}$ as

$$\begin{aligned}\mathbf{E}(\mathbf{r},z,t)&=V(z,t)\,\epsilon_0^{-\frac{1}{2}}\nabla_\perp\phi(\mathbf{r})\\ \mathbf{H}(\mathbf{r},z,t)&=I(z,t)\,\mu_0^{-\frac{1}{2}}\left[\mathbf{z}\times\nabla_\perp\phi(\mathbf{r})\right]\end{aligned} \tag{6}$$

where the scalar functions, $V(z,t)$, $I(z,t)$ and $\phi(\mathbf{r})$, should be found afterwards. The free-space constants, ϵ_0 and μ_0, are installed in (6) specially in order to provide later the fields $\mathbf{E}$ and $\mathbf{H}$ with their physical dimensions Vm^{-1} (*volt per meter*) and Am^{-1} (*ampere per meter*), respectively. The longitudinal field component, E_z, can be presented similarly as

$$E_z(\mathbf{r},z,t)=E(z,t)\,\epsilon_0^{-\frac{1}{2}}\,\phi(\mathbf{r}) \tag{7}$$

where $E(z,t)$ is one more unknown scalar function, as yet.

Substitution of $\mathbf{H}$ and E_z from Eqs. (6) and (7) to Eq. (3d) results in

$$I(z,t)\ \nabla_\perp^2\phi(\mathbf{r})=\left[c^{-1}\partial_tE(z,t)+2\gamma c^{-1}E(z,t)\right]\ \phi(\mathbf{r}) \tag{8}$$

where $c^{-1} = \sqrt{\epsilon_0\mu_0}$ is the light speed, and $2\gamma = \sigma/\epsilon_0$ is a lossy parameter.

Mention here that the boundary condition $E_z|_L = 0$ results in $\phi(\mathbf{r})|_L = 0$. This fact, jointly with presence of the factor $\nabla_\perp^2\phi(\mathbf{r})$ in Eq. (8), suggests to introduce into consideration the well-studied Dirichlet *boundary-eigenvalue problem* for the transverse Laplacian, $\nabla_\perp^2$, as

$$\nabla_\perp^2\phi_m(\mathbf{r}) + \kappa_m^2\phi_m(\mathbf{r}) = 0 \quad \text{and} \quad \phi_m(\mathbf{r})|_L = 0 \tag{9}$$

where $\{\kappa_m^2\}_{m=1}^\infty$ is a set of the real-valued eigenvalues; $\kappa_0^2 = 0$ is also the eigenvalue, and an eigensolution $\phi_0(\mathbf{r})$, corresponding to κ_0^2, is a solution to the problem as $\{\nabla_\perp^2\phi_0 = 0,\ \phi_0|_L = 0\}$. This solution is equal to *zero* in accordance with the minimum-maximum principle for the harmonic functions[§]. The subscript $m = 1, 2, \ldots$ regulates positions of the other eigenvalues, $\kappa_m^2 > 0$, on a real axis in order of increasing their numerical values. The set of eigenfunctions, $\{\phi_m\}_{m=1}^\infty$, corresponding to all the eigenvalues, κ_m^2, is complete. This set originates a *basis* in Hilbert space $L_2(S)$ provided that the problem (9) is supplemented with the appropriate normalization conditions for the functions $\phi_m(\mathbf{r})$. Hence, any twice-differentiable function, $\phi(\mathbf{r})$, satisfying the same boundary condition, $\phi(\mathbf{r})|_L = 0$, can be expanded in terms of the basis elements.

Let us take the eigenfunctions $\phi_m(\mathbf{r})$ as the potential $\phi(\mathbf{r})$ in formulas (6) and (7), and scale by κ_m the factor $E(z,t)$ as follows: $E(z,t) = \kappa_m e_m(z,t)$. Then Eqs. (7) and 8) take the form as

$$\begin{aligned} E_{z\,m}(\mathbf{r},z,t) &= e_m(z,t)\,\epsilon_0^{-\frac{1}{2}}\,\kappa_m\phi_m(\mathbf{r}) \\ I_m(z,t) &= -(\kappa_m c)^{-1}\left[\partial_t e_m(z,t) + 2\gamma\, e_m(z,t)\right] \end{aligned} \tag{10}$$

where $m = 1, 2, \ldots$. Eq. (3f) yields a relationship between V_m and e_m as

$$V_m(z,t) - \kappa_m^{-1}\partial_z e_m(z,t) + 2\gamma\int_0^t \left[V_m(z,t') - \kappa_m^{-1}\partial_z e_m(z,t')\right] dt' = 0. \tag{11}$$

Factually, this relationship is much simpler. If we introduce a function as

$$f(z,t) = \int_0^t \left[V_m(z,t') - \kappa_m^{-1}\partial_z e_m(z,t')\right] dt' \tag{12}$$

and notice that the first pair in Eq. (11) is $\partial_t f(z,\, t)$, then Eq. (11) yields an initial-value problem for the function $f(z,\, t)$ as

$$\partial_t f(z,t) + 2\gamma f(z,t) = 0 \quad \text{and} \quad f(z,t)|_{t=0} = 0 \tag{13}$$

[§] It is so because the cross section S is chosen as a singly-connected domain.

where the initial condition follows from the mean-value theorem applied to the integral in Eq. (12). Solving this problem results in $f(z,t) \equiv 0$. Hence,

$$V_m(z,t) = \kappa_m^{-1} \partial_z e_m(z,t). \tag{14}$$

We shall operate henceforward with the field components taken as

$$\begin{aligned} \mathbf{H}_m(\mathbf{r},z,t) &= I_m(z,t)\,\mu_0^{-\frac{1}{2}}\,\left[\mathbf{z} \times \nabla_\perp \phi_m(\mathbf{r})\right] \\ \mathbf{E}_m(\mathbf{r},z,t) &= V_m(z,t)\,\epsilon_0^{-\frac{1}{2}}\,\nabla_\perp \phi_m(\mathbf{r}) \\ E_{z\,m}(\mathbf{r},z,t) &= e_m(z,t)\,\epsilon_0^{-\frac{1}{2}}\,\kappa_m \phi_m(\mathbf{r}). \end{aligned} \tag{15}$$

Substitution of the formulas (15) to Eq. (3a) yields

$$\partial_z V_m(z,t) + c^{-1} \partial_t I_m(z,t) - \kappa_m\, e_m(z,t) = 0. \tag{16}$$

Finally, the formulas (10) and (14), being substituted to Eq. (16), results in the well known *Telegraph equation* [19], i.e.,

$$(\kappa_m c)^{-2}\, \partial_t^2 e_m + 2\gamma\,(\kappa_m c)^{-2}\, \partial_t e_m - \kappa_m^{-2}\, \partial_z^2 e_m + e_m = 0. \tag{17}$$

Substitution of the fields (15) to Eq. (3c) yields identity, $0 = 0$.

3.1. Main Results

3.1.1. Normalization of the Solutions to Dirichlet Problem

The *TM*-modal wave problem starts with solving the Dirichlet boundary-value problem. Supplement that with the normalization condition, which we choose as

$$\nabla_\perp^2 \phi_m(\mathbf{r}) + \kappa_m^2 \phi_m(\mathbf{r}) = 0, \quad \phi_m(\mathbf{r})|_{\mathbf{r}\in L} = 0, \quad \frac{\mathcal{N}_m^2}{S} \int_S \phi_m^2(\mathbf{r}) ds = N, \tag{18}$$

where $\mathcal{N}_m$ is the normalization constant, and N is the "nominated" number 1. That is, we take this number, 1, and assign the force unit N = kgms^{-2} (*newton*). Evidently, that the constant, $\mathcal{N}_m$, has the physical dimension $N^{\frac{1}{2}}$ provided that the potential $\phi_m(\mathbf{r})$ is a dimensionless quantity, see **Example 1.**

Example 1 *Solve the problem (18) for a standard waveguide with the rectangular cross section specified as* $0 \le x \le a$ *and* $0 \le y \le b$. *Separation of the variables in Helmholtz Eq. (18) yields the potential,* $\phi_m(\mathbf{r}) \equiv \phi_m(x,y)$, *as*

$$\phi_m(x,y) \equiv \phi_{p,\,q}(x,y) = \sin(\pi p\, x/a) \sin(\pi q\, y/b), \tag{19}$$

which corresponds to the eigenvalues distinct from zero,

$$\kappa_m^2 \equiv \kappa_{p,\,q}^2 = \pi^2 \left[(p/a)^2 + (q/b)^2\right]. \tag{20}$$

The subscript parameter, m, *is a doublet,* (p,q), *composed of the integers,* $p = 1, 2, \ldots$ *and* $q = 1, 2, \ldots$. *The normalization condition yields* $\mathcal{N}_m = 2N^{\frac{1}{2}}$.

3.1.2. The Waveguide Modal Basis

Complete set of the normalized solutions to the problem (18) generates a complete set of elements of a *modal basis* as

$$\begin{aligned} \mathbb{H}_m(\mathbf{r}) &= \mathbf{z} \times \nabla_\perp \phi_m(\mathbf{r}) \left[\mu_0^{-\frac{1}{2}} \mathcal{N}_m \right] \\ \mathbb{E}_m(\mathbf{r}) &= \nabla_\perp \phi_m(\mathbf{r}) \left[\epsilon_0^{-\frac{1}{2}} \mathcal{N}_m \right] \\ \mathbb{Z}_m(\mathbf{r}) &= \mathbf{z} \kappa_m \phi_m(\mathbf{r}) \left[\epsilon_0^{-\frac{1}{2}} \mathcal{N}_m \right] \end{aligned} \tag{21}$$

where $\mathcal{N}_m$ is proportional to $N^{\frac{1}{2}}$. Products $[\mu_0^{-\frac{1}{2}} N^{\frac{1}{2}}]$ and $[\epsilon_0^{-\frac{1}{2}} N^{\frac{1}{2}}]$ have physical dimensions A (*ampere*) and $V = \mathrm{kgm^2A^{-1}s^{-3}}$ (*volt*), respectively. Square root of the eigenvalues, $\sqrt{\kappa_m^2} = \kappa_m$, and the operator $\nabla_\perp$, have dimension m^{-1} (*inverse meter*). Thus, the elements $\mathbb{H}_m$ have dimension Am^{-1} (*ampere per meter*), and $\mathbb{E}_m$ and $\mathbb{Z}_m$, both, have dimension Vm^{-1} (*volt per meter*). Hence, the modal amplitudes in Eq. (15) are the dimensionless quantities.

3.1.3. The Waveguide Evolutionary Equations

It is convenient to operate with the dimensionless variables, which we introduce by scaling z and t as

$$\boxed{\xi = \kappa_m z \quad \text{and} \quad \tau = \omega_m t} \tag{22}$$

where κ_m and $\omega_m = \kappa_m c$ are the cut-off wave number and the cut-off frequency, physically. Then the modal *TM*-waves are presentable as

$$\boxed{\mathcal{H}_m = I_m(\xi,\tau)\mathbb{H}_m(\mathbf{r}) \quad \text{and} \quad \mathcal{E}_m = V_m(\xi,\tau)\mathbb{E}_m(\mathbf{r}) + e_m(\xi,\tau)\mathbb{Z}_m(\mathbf{r}).} \tag{23}$$

The modal amplitudes, dependent on (ξ, τ), are the dimensionless quantities. The amplitude $e_m(\xi, \tau)$ should be found by solving equation

$$\partial_\tau^2 e_m(\xi,\tau) + 2\varrho \partial_\tau e_m(\xi,\tau) - \partial_\xi^2 e_m(\xi,\tau) + e_m(\xi,\tau) = 0 \tag{24}$$

where $\varrho = \gamma/\omega_m$ is a dimensionless lossy parameter. The other amplitudes are

$$V_m(\xi,\tau) = \partial_\xi e_m(\xi,\tau) \quad \text{and} \quad I_m(\xi,\tau) = -\partial_\tau e_m(\xi,\tau) - 2\varrho e_m(\xi,\tau). \tag{25}$$

3.1.4. A Conclusion

The modal amplitude problem (24), (25) and the modal basis problem (21) are *autonomous.* Tracing back analysis given in this Section, a reader can be certain that the problems (21) and (24), (25) were extracted from Maxwell's equations with ∂_t *rigorously*, in the mathematical sense.

4. EXAMPLES OF SOLVING KLEIN-GORDON EQUATION

Time derivative of the first order, ∂_τ, can be eliminated from Eq. (24) by applying a substitution for the expected solution, $e_m(\xi,\,\tau)$, as follows:

$$\boxed{e_m\left(\xi,\tau\right)=e^{-\varrho\tau}\tilde{e}_m\left(\xi,\tau\right)} \tag{26}$$

where $\tilde{e}_m(\xi,\,\tau)$ is a new unknown function. Simple manipulations with Eqs. (26) and (24) result in canonical *Klein-Gordon Equation* (*KGE*) as

$$\boxed{\partial_\tau^2\tilde{e}_m\left(\xi,\tau\right)-\partial_\xi^2\tilde{e}_m\left(\xi,\tau\right)+\eta^2\tilde{e}_m\left(\xi,\tau\right)=0} \tag{27}$$

and slightly changed formulas (25) to the form of

$$\boxed{V_m=e^{-\varrho\tau}\partial_\xi\tilde{e}_m(\xi,\tau)\quad\text{and}\quad I_m=-e^{-\varrho\tau}[\partial_\tau\tilde{e}_m(\xi,\tau)+\varrho\tilde{e}_m(\xi,\tau)]} \tag{28}$$

where $\eta=\sqrt{1-\varrho^2}\geq 0$, $\varrho=\gamma/\omega_m$, and $\gamma=\sigma/\left(2\epsilon_0\right)$ is the lossy parameter.

The modal basis problem, (18) and (21), is common for the *time-domain* fields and also for the *time-harmonic* fields presentable in the form (23). The modal amplitude problem rests upon solving the *KGE* (27). This solution yields then the amplitude of the longitudinal field component by formula (26). The amplitudes of the transverse components can be calculated by formulas (28).

As an implementation of this scheme, let us look for the amplitudes of the time-harmonic fields. The method of separation of ξ- and τ-variables can be applied for solving the *KGE* (27). Exact explicit solutions for the *real-valued* time-harmonic modal amplitudes are obtained in two examples below.

Example 2 Let an expected solution to Eq. (27) be in the form of

$$\tilde{e}_m\left(\xi,\tau\right)=A_m\sin\left[\left(\varpi\tau-\xi\,\Gamma_m\right)+\varphi_m\right] \tag{29}$$

where $\varpi=\omega/\omega_m$ is a dimensionless frequency, ω is a frequency parameter, $\omega_m=\kappa_m c$, κ_m is square root of an eigenvalue from

Dirichlet problem (9), c is the light speed. The constants, A_m and φ_m, are real-valued free numerical parameters[‖]. Substitution of the function $\tilde{e}_m(\xi, \tau)$ to Eq. (27) yields $\Gamma_m = \pm\sqrt{\varpi^2 - \eta^2}$ provided that $\varpi^2 - \eta^2 \geq 0$. In the sign doublet, $(\pm)$, the upper sign corresponds to the wave propagation lengthwise Oz-axis, and the lower sign corresponds to the opposite direction. In order to simplify notations henceforth, we now introduce two "phase functions" as

$$\Phi_m(\xi,\tau) = \varpi\tau - \xi\Gamma_m + \varphi_m \equiv \omega t - (z/c)\sqrt{\omega^2 - \omega_m^2 + \gamma^2} + \varphi_m \quad (30a)$$

$$\mathring{\Phi}_m(\xi,\tau) = \Phi_m(\xi,\tau) - \vartheta_m \quad (30b)$$

where $\vartheta_m = \sin^{-1}(\varrho/\sqrt{\varpi^2 + \varrho^2})$ is a "lossy" phase shift. If $\varrho = \gamma/\omega_m = 0$, (i.e., $\gamma = 0$, the waveguide is lossless), then $\vartheta_m = 0$ and $\mathring{\Phi}_m(\xi,\tau) = \Phi_m(\xi,\tau)$.

Let us take the upper sign in $(\pm\Gamma_m)$ and substitute solution (29) to Eqs. (26) and (28). It results in the modal amplitudes of the *TM*-waves as

$$e_m^{\varpi}(\xi,\tau) = A_m e^{-\varrho\tau} \sin[\Phi_m(\xi,\tau)] \quad (31a)$$

$$V_m^{\varpi}(\xi,\tau) = -A_m\sqrt{\varpi^2 - \eta^2}\, e^{-\varrho\tau} \cos[\Phi_m(\xi,\tau)] \quad (31b)$$

$$I_m^{\varpi}(\xi,\tau) = -A_m\sqrt{\varpi^2 + \varrho^2}\, e^{-\varrho\tau} \cos\left[\mathring{\Phi}_m(\xi,\tau)\right], \quad m = 1, 2, \ldots. \quad (31c)$$

Condition $\Gamma_m = 0$ yields an equation for the frequency parameter, $\varpi_{\textit{cut-off}}$. Solving this equation specifies so-called "cut-off" frequencies as

$$\varpi_{\textit{cut-off}} = \eta = \sqrt{1 - \varrho^2} \quad (32)$$

where $\varrho = \gamma/\omega_m \leq 1$, $\omega_m = \kappa_m c$. Nominally admissible value of the parameter $\gamma = \kappa_m c$ yields $\omega_{\textit{cut-off}} = 0$. The values of κ_m, $m = 1, 2 \ldots$, are called as the "cut-off wave numbers" of the lossless waveguides.

Notice that all information about the shape and size of the contour, L, of the waveguide cross section, S, is accumulated just in the numbers κ_m. For the cavity with a rectangular cross section, the values of κ_m are obtained in Example 1. In a similar way, one can easily find (analytically or numerically) these parameters, κ_m, for the other singly-connected contours.

It should be noticed, as well, that the graphical presentations of the amplitude in terms of ξ- and τ-variables are preferable. All specificity of the waveguide contour L is already put away in κ_m, see Eq. (22). Thus, the amplitudes for the rectangular and circular waveguides look graphically *equally*.

‖ Involvement of *two* parameters, A_m and φ_m, in the solution (29) is equivalent to presentation that via two linearly independent *sine* and *cosine* functions as $\tilde{e}_m(\xi, \tau) = a_m \sin(\cdot) + b_m \cos(\cdot)$ where $(\cdot)$ is $(\tau\omega/\omega_m - \xi\,\Gamma_m)$, $a_m = A_m \cos\varphi_m$, $b_m = A_m \sin\varphi_m$.

Example 3 Let a **cavity** be a piece of the short-circuited waveguide where $0 \le z \le \ell$. Apply the boundary condition, $\mathbf{z} \times \mathcal{E}(\mathbf{r}, z, t) = 0$, at $z = 0$ and $z = \ell$; $\mathbf{r} \in S$. This condition is equivalent to $\partial_\xi \tilde{e}_m|_{z=0,\,\ell} = 0$. Separation of the variables (ξ, τ) by factorization of an expected solution to (27) as $\tilde{e}_m = T_m(\tau) X_m(\xi)$ yields

$$T_m^{-1}(\tau) \tfrac{d^2}{d\tau^2} T_m(\tau) + \eta^2 = X_m^{-1}(\xi) \tfrac{d^2}{d\xi^2} X_m(\xi) = -\Lambda_m^2 \tag{33}$$

where $\Lambda_m^2 > 0$ is a real-valued constant of separation of the variables. The first problem, $\{X_m(\xi) : \frac{d^2}{d\xi^2} X_m + \Lambda_m^2 X_m = 0,\ \partial_\xi X_m|_{z=0,\,\ell} = 0\}$, yields $X_m(\xi) = \cos(\Lambda_m \xi)$ where $\Lambda_m = \pi s/(\kappa_m \ell)$, $s = 1, 2, \ldots$, and κ_m is the waveguide cut-off wave number. The other problem, $\{T_m(\tau) : \frac{d^2}{d\tau^2} T_m + (\Lambda_m^2 + \eta^2) T_m = 0\}$, yields $T_m(\tau) = B_m \sin(\tau \sqrt{\Lambda_m^2 + \eta^2} + \check{\varphi}_m)$ provided that $\sqrt{\Lambda_m^2 + \eta^2} \ge 0$. The parameters, B_m and $\check{\varphi}_m$, can be specified by applying the appropriate initial conditions.

Introduce one more pair of the "phase functions" as

$$\Psi_{ms}(\tau) = \tau \Omega_{ms} + \check{\varphi}_m \equiv t\,\omega_m \Omega_{ms} + \check{\varphi}_m \tag{34a}$$

$$\mathring{\Psi}_{ms}(\tau) = \Psi_{ms}(\xi, \tau) - \delta_s \tag{34b}$$

where a dimensionless frequency, Ω_{ms}, and a "lossy" phase shift, δ_s, are

$$\Omega_{ms} = \sqrt{1 + [\pi s/(\kappa_m \ell)]^2 - \varrho^2}, \quad \delta_s = \sin^{-1}\left[\varrho/\sqrt{1 + [\pi s/(\kappa_m \ell)]^2}\right]. \tag{35}$$

Then the amplitudes of the TM-cavity modes can be shortly written as

$$I_{ms}^{(c)}(z, \tau) = -B_m \sqrt{1 + [\pi s/(\kappa_m \ell)]^2} e^{-\varrho\tau} \cos\left[\mathring{\Psi}_{ms}(\tau)\right] \cos(\pi s z/\ell) \tag{36a}$$

$$V_{ms}^{(c)}(z, \tau) = -B_m [\pi s/(\kappa_m \ell)]\, e^{-\varrho\tau} \sin[\Psi_{ms}(\tau)] \sin(\pi s z/\ell) \tag{36b}$$

$$e_{ms}^{(c)}(z, \tau) = B_m e^{-\varrho\tau} \sin[\Psi_{ms}(\tau)] \cos(\pi s z/\ell), \quad s = 1, 2, \ldots \tag{36c}$$

where the superscript, $^{(c)}$, implies "cavity". Nominally admissible values of the lossy parameter $\varrho = \gamma/\omega_m$ are within the interval $0 \le \varrho \le \sqrt{1 + [\pi s/(\kappa_m \ell)]^2}$.

5. THE *KGE* AND PARABOLIC CYLINDER FUNCTIONS

Klein-Gordon Eq. (27) is relativistic. Hence, that must maintain its form under action of the relativistic Lorentz transformations in any *inertial* reference frame. Therefore, the solutions to *KGE* obey specific properties of symmetry, as well. These symmetries were studied by Miller within the framework of the Group Theory in [20]. Some Miller's ideas were used in our previous publications [5, 9]. In this article, we continue expansion of the group-theoretical results on development of the time-domain electromagnetics.

The Group Theory operates with a so-called *point transformation.* This implies, technically, that in addition to the *independent* variables in Eq. (27), i.e., ξ and τ, one should introduce a pair of *dependent* variables as $u \equiv u(\xi, \tau)$ and $v \equiv v(\xi, \tau)$. Then solution to Eq. (27), $\tilde{e}_m(\xi, \tau)$, can be interpreted as $\tilde{e}_m[u(\xi, \tau), v(\xi, \tau)]$. Formal substitution of the latter to Eq. (27) yields

$$\left\{\begin{matrix} [(\partial_\tau u)^2 - (\partial_\xi u)^2]\partial_u^2 + \left[(\partial_\tau v)^2 - (\partial_\xi v)^2\right]\partial_v^2 \\ + \left[\partial_\tau^2 u - \partial_\xi^2 u\right]\partial_u + \left[\partial_\tau^2 v - \partial_\xi^2 v\right]\partial_v \\ +2\left[(\partial_\tau u)(\partial_\tau v) - (\partial_\xi u)(\partial_\xi v)\right]\partial_{uv}^2 + \eta^2 \end{matrix}\right\}\tilde{e}_m(u, v) = 0 \qquad (37)$$

where the dependent variables are undefined as yet. Right now, one can say only that $(u,\ u_\xi,\ u_\tau,\ u_{\xi\tau},\ u_{\xi\xi},\ u_{\tau\tau})$ and $(v,\ v_\xi,\ v_\tau,\ v_{\xi\tau},\ v_{\xi\xi},\ v_{\tau\tau})$ should exist.

Just the Group Theory proposes a way for definition of $u(\xi, \tau)$ and $v(\xi, \tau)$ proceeding from the symmetry¶ of (27). In this article, we consider one of the possible other 10 cases, see Appendix in [5]. Specifically, that looks as

$$\eta\tau = \left(u^2 + v^2\right)/2 \quad \text{and} \quad \eta\xi = uv \qquad (38)$$

where $-\infty < u < \infty$ and $0 \le v < \infty$. Inversion of Eq. (38) yields

$$u(\xi,\tau) = \sqrt{\frac{\eta}{2}}\left(\sqrt{\tau+\xi} + \sqrt{\tau-\xi}\right) \text{ and } v(\xi,\tau) = \sqrt{\frac{\eta}{2}}\left(\sqrt{\tau+\xi} - \sqrt{\tau-\xi}\right) \quad (39)$$

where we suppose initially that $\tau \ge 0$ and $0 \le \xi \le \tau$.

We can calculate now the variable coefficients, which are placed in the square brackets in Eq. (37). The first pair of the coefficients, which stands in front of the derivatives, ∂_u^2 and ∂_v^2, are distinct from *zero* and, respectively, are

$$\left[(\partial_\tau u)^2 - (\partial_\xi u)^2\right] = -\left[(\partial_\tau v)^2 - (\partial_\xi v)^2\right] = \eta^2/\left(u^2 - v^2\right). \qquad (40)$$

All the other coefficients are equal to *zero.* Finally, Eq. (37) looks as

$$\frac{1}{u^2 - v^2}\partial_u^2\tilde{e}_m(u, v) - \frac{1}{u^2 - v^2}\partial_v^2\tilde{e}_m(u, v) + \tilde{e}_m(u, v) = 0. \qquad (41)$$

¶ Miller has established *ten* of so-called "orbits of symmetry." These result in definition for the *eleven* pairs of the dependent variables, $\{u(\xi, \tau), v(\xi, \tau)\}$. The first pair as $u(\xi, \tau) = \xi$ and $v(\xi, \tau) = \tau$ is trivial. This case directs electromagnetics to the time-harmonic field approach. The other ten pairs are capable of elucidation of new aspects in the time-domain electromagnetics. One can find a complete list of these pairs in Appendix [5].

Virtually, the chosen symmetry transformation of Eq. (27) rearranges that to its *equivalent* form (41). In other words, (41) holds whenever (27) holds.

It is evident that Eq. (41) can be solved by separation of the variables, u and v. To this aim, the solution should be factorized as follows:

$$\tilde{e}_m(u,v) = \mathbb{U}_m(u)\,\mathbb{V}_m(v). \tag{42}$$

Substitution of the product (42) to Eq. (41) yields

$$\tfrac{d^2}{du^2}\mathbb{U}_m + \left(u^2+\lambda\right)\mathbb{U}_m = 0 \quad \text{and} \quad \tfrac{d^2}{dv^2}\mathbb{V}_m + \left(v^2+\lambda\right)\mathbb{V}_m = 0 \tag{43}$$

where λ is a constant (possibly, complex-valued) of separation of the variables.

Both Eq. (43) are pertained to the type of ordinary differential equations for the *parabolic cylinder functions.* However, it is necessary to rearrange their form to a canonical one (see [21]) which looks as

$$\tfrac{d^2}{dx^2}y(x) - \left(\tfrac{1}{4}x^2 + a\right)y(x) = 0 \tag{44}$$

In order to put in order Eq. (43), we slightly change the variables, u and v, and the parameter λ, appropriately, what yields

$$\tfrac{d^2}{d\mathring{u}^2}\mathbb{U}(\mathring{u}) - \left(\tfrac{1}{4}\mathring{u}^2+\alpha\right)\mathbb{U}(\mathring{u}) = 0 \quad \text{and} \quad \tfrac{d^2}{d\mathring{v}^2}\mathbb{V}(\mathring{v}) - \left(\tfrac{1}{4}\mathring{v}^2+\alpha\right)\mathbb{V}(\mathring{v}) = 0 \tag{45}$$

where α is a new parameter, subscript $({}_m)$ is omitted. The new variables are

$$\mathring{u} = u\sqrt{i2} = \sqrt{i\eta}\left(\sqrt{\tau+\xi} + \sqrt{\tau-\xi}\right) \tag{46a}$$

$$\mathring{v} = v\sqrt{i2} = \sqrt{i\eta}\left(\sqrt{\tau+\xi} - \sqrt{\tau-\xi}\right) \tag{46b}$$

where $i = \sqrt{-1}$ is the imaginary unit.

Both equations in Eq. (45) coincide (with accuracy to notations). Therefore, we can take their solution as

$$\mathbb{U}(\mathring{u}) = U(\alpha, \mathring{u}) \quad \text{and} \quad \mathbb{V}(\mathring{v}) = U(\alpha, \mathring{v}) \tag{47}$$

where $U(\alpha, \cdot)$ is one of the possible form of the solutions[+] to Eq. (44), parameter α has a half-integer value, namely: $\alpha = -(2n+1)/2$, $n = 0, 1, 2, \ldots$.

Finally, we are ready to write down two linearly independent solutions to the equations in (45). They are expressible via products of

[+] The solutions to Eq. (44) can be expressed via confluent hypergeometric series (Kummer's and Whittaker's functions). We use standard solutions denoted in [21] as $U(\alpha, x)$.

the parabolic cylinder functions (47). Therefore, they are parametrized by superscript $^{(\alpha)}$ as

$$\hat{e}_m^{(\alpha)}(\xi,\tau)=e^{-\varrho\tau}\left[U(\alpha,\mathring{u})\,U(\alpha,\mathring{v})+U(\alpha,\mathring{u}^*)\,U(\alpha,\mathring{v}^*)\right]/2 \quad (48a)$$

$$\breve{e}_m^{(\alpha)}(\xi,\tau)=e^{-\varrho\tau}\left[U(\alpha,\mathring{u})\,U(\alpha,\mathring{v})-U(\alpha,\mathring{u}^*)\,U(\alpha,\mathring{v}^*)\right]/2i \quad (48b)$$

where $\mathring{u}^*$ is $\mathring{u}$ complex conjugated and $\mathring{v}^*$ is $\mathring{v}$ complex conjugated. These solutions satisfy Eq. (27). So long as the parabolic cylinder functions with

$$\alpha=-(2n+1)/2 \quad \text{where} \quad n=0,1,2,\ldots \quad (48c)$$

are convertible to the Hermite polynomials, calculations by formulas (48a)–(48c) result in the simple explicit expressions. In particular, if $n=0$ then

$$\hat{e}_m^{\left(-\frac{1}{2}\right)}=e^{-\varrho\tau}\cos(\eta\tau) \quad \text{and} \quad \breve{e}_m^{\left(-\frac{1}{2}\right)}=-e^{-\varrho\tau}\sin(\eta\tau). \quad (49)$$

One can form a linear combination of these solutions as

$$e_m^{\eta}=a_m\hat{e}_m^{\left(-\frac{1}{2}\right)}-b_m\breve{e}_m^{\left(-\frac{1}{2}\right)}\equiv e^{-\varrho\tau}A_m\sin(\eta\tau+\varphi_m) \quad (50)$$

where a_m and b_m are numerical parameters which specify A_m and φ_m as

$$A_m=\sqrt{a_m^2+b_m^2} \quad \text{and} \quad \varphi_m=\sin^{-1}(a_m/A_m)=\cos^{-1}(b_m/A_m). \quad (51)$$

This linear combination one-to-one corresponds to Eq. (31a) provided that the frequency parameter, ϖ, coincides with the cut-off frequency, η, see Eq. (32).

The modal amplitudes V_m^{η} and I_m^{η}, which are generated by e_m^{η}, can be found by formulas (25). The results of calculations of V_m^{η} and I_m^{η} one-to-one correspond to Eqs. (31b), (31c) provided that $\varpi=\eta$. The lossy phase shift, ϑ_m, from Eq. (30b) is found as $\vartheta_m=\sin^{-1}\varrho=\cos^{-1}\eta$ for this case.

As it follows from Eqs. (48a)–(48c), a *countable set* of other solutions exists for the frequency $\varpi=\eta$. If $n=1$ in (48c), the formulas (48a), (48b) result in

$$\hat{e}_m^{(-3/2)}(\xi,\tau) = e^{-\varrho\tau}\eta\xi\sin(\eta\tau) \quad (52a)$$

$$\breve{e}_m^{(-3/2)}(\xi,\tau) = e^{-\varrho\tau}\eta\xi\cos(\eta\tau). \quad (52b)$$

The case $n=2$ yields one more pair of the solutions as

$$\hat{e}_m^{(-5/2)}(\xi,\tau) = e^{-\varrho\tau}\left[\left(1/4-\eta^2\xi^2\right)\cos(\eta\tau)-\eta\tau\sin(\eta\tau)\right] \quad (53a)$$

$$\breve{e}_m^{(-5/2)}(\xi,\tau) = -e^{-\varrho\tau}\left[\eta\tau\cos(\eta\tau)+\left(1/4-\eta^2\xi^2\right)\sin(\eta\tau)\right]. \quad (53b)$$

And so on. Thus, in the large, we obtained a new countable set of *exact* solutions to the relativistic Maxwell's equations with time derivative.

Remark The orbit of symmetry of KGE, where $u(\xi,\,\tau)=\xi$ and $v(\xi,\,\tau)=\tau$ hold, absorbs the time-harmonic solutions like (31a)–(31c) and (36a)–(36c). In these solutions, the frequency parameter does not suffer restrictions. In the orbit of symmetry, where Eqs. (46a), (46b) hold, the frequency parameter is restricted by condition (32). The solutions, oscillating with the cut-off frequency, appear to be arranged in a *new* countable set of the fields generated by Eqs. (48a)–(48c).

6. THE INSTANT ENERGETIC CHARACTERISTICS

6.1. The General Energetic Relationships

Introduce a control volume, δV, limited by two consecutive waveguide cross sections located at coordinates z and $z+\delta z$. Apply Poynting's theorem to the TM-modal field (23). Take the limit when $\delta z\to 0$. On this way*, two energetic characteristics appear as

$$P_{m\,z}\left(\xi,\tau\right) = \left[\frac{1}{S}\int_S \mathbf{z}\cdot\mathbb{E}_m\times\mathbb{H}_m ds\right]V_m I_m = \left[c\kappa_m^2 N\right]\mathcal{P}_{m\,z}\left(\xi,\tau\right) \tag{54a}$$

$$W_m\left(\xi,\tau\right) = \left[\kappa_m^2 N\right]\mathcal{W}_m\left(\xi,\tau\right) \tag{54b}$$

where factor $[c\kappa_m^2 N]$ has physical dimension Wm^{-2} (*watt per meter*^{2}) and factor $[\kappa_m^2 N]$ has physical dimension Jm^{-3} (*joule per meter*^{3}). If $\delta V|_{\delta z\to 0}\ \to\ 0$, *dimensional* quantity $\mathcal{P}_{m\,z}$ specifies the modal power flow through a fixed waveguide cross section. The *dimensional* quantity $\mathcal{W}_m$ specifies the modal field energy density stored at the same cross section. *Dimensionless* energetic quantities are presentable via the dimensionless modal amplitudes as

$$\mathcal{P}_{m\,z}\left(\xi,\tau\right) = I_m\left(\xi,\tau\right)V_m\left(\xi,\tau\right) \tag{55a}$$

$$\mathcal{W}_m\left(\xi,\tau\right) = \mathcal{W}_m^{\mathbf{m}}\left(\xi,\tau\right)+\mathcal{W}_m^{\mathbf{e}}\left(\xi,\tau\right) \tag{55b}$$

$$\mathcal{W}_m^{\mathbf{m}}\left(\xi,\tau\right) = I_m^2\left(\xi,\tau\right)/2 \tag{55c}$$

$$\mathcal{W}_m^{\mathbf{e}}\left(\xi,\tau\right) = \left[V_m^2\left(\xi,\tau\right)+e_m^2\left(\xi,\tau\right)\right]/2 \tag{55d}$$

where $\mathcal{W}_m^{\mathbf{m}}$ and $\mathcal{W}_m^{\mathbf{e}}$ are the energy densities stored in the magnetic and electric parts of the modal field, respectively. Mathematically, characteristics (55a)–(55d) specify the *global* field properties in the space of solutions.

Poynting's theorem yields an *energetic* continuity equation as

$$\partial_\xi\mathcal{P}_{m\,z}\left(\xi,\tau\right)+\partial_\tau\mathcal{W}_m\left(\xi,\tau\right)+2\varrho\left[V_m^2\left(\xi,\tau\right)+e_m^2\left(\xi,\tau\right)\right]=0\,. \tag{56}$$

This is the time-domain law of conservation of the modal field energy, physically. Mathematically, that specifies the *local* field properties in the space solutions.

* One can find details of similar procedure in article [10].

Notice in passing that the solutions, which were obtained in *Examples* 2 and 3, satisfy the energetic continuity Eq. (56). Besides, all the solutions expressible via the *parabolic cylinder functions* satisfy the conservation of energy law (56), as well, for arbitrary $\alpha = -(2n+1)/2$, $n \geq 0$.

Combination of Umov theorem [22] and Poynting's theorem [23] yields the definition for *instant velocity* of transportation of the modal field energy as

$$\mathrm{v}_m(\xi, \tau) = c\,\frac{\mathcal{P}_{m\,z}(\xi, \tau)}{\mathcal{W}_m(\xi, \tau)} \tag{57}$$

where c is the light speed. One can find discussion on this topic in [10].

6.2. The Time-harmonic *Energetic* Waveguide Waves

Exact explicit solutions (31a)–(31c) make possible for a fresh look at the *energetic* field properties in the *time-domain.* For the time being, we ignore the phase shift, ϑ_m, in Eq. (30b). This results in $\mathring{\Phi}_m = \Phi_m$ in Eq. (31a). Observation of Eqs. (31b), (31c) under this supposition suggests to introduce a new energetic quantity as

$$\mathring{S}_m^{\varpi}(\xi, \tau) = (1/2)\left(I_m^{\varpi 2} - V_m^{\varpi 2}\right) = (1/2)A_m^2 e^{-2\varrho\tau}\cos^2[\Phi_m(\xi, \tau)]. \tag{58}$$

Observation of Eq. (31a) suggests to introduce one more energetic quantity as

$$w_m^{\varpi}(\xi, \tau) = (1/2)\, e_m^{\varpi\,2} = (1/2)\; A_m^2\, e^{-2\varrho\tau}\sin^2[\Phi_m(\xi, \tau)]. \tag{59}$$

Notice that the quantity $\mathring{S}_m^{\varpi}(\xi,\, \tau)$ has the exact physical sense provided that $\varrho = 0$ because $\vartheta_m|_{\varrho=0} = 0$. That quantity species *difference* of the energy density stored in the *transverse* components of the magnetic and electric fields, see Eqs. (55c), (55d). In the meanwhile, $w_m^{\varpi}(\xi,\, \tau)$ is the energy density stored in the *longitudinal* component of the electric field. So, propagation of the TM-modal electromagnetic wave with its amplitudes (31a)–(31c) is accompanied with propagation of two antiphase *energetic* waves, (58) and (59). This means, physically, that a periodical *exchange by energy* occurs between $\mathring{S}_m^{\varpi}(z,\, t)$ and $w_m^{\varpi}(z,\, t)$.

Averaging these quantities over the period of oscillations, $T = 2\pi/\omega$, yields

$$\tfrac{1}{T}\int_{t_0}^{t_0+T} \mathring{S}_m^{\varpi}(\xi, \tau)\; dt = \tfrac{1}{T}\int_{t_0}^{t_0+T} w_m^{\varpi}(\xi, \tau)\, dt = (1/4)\; A_m^2. \tag{60}$$

So, the averaged values depend neither on time t_0 nor on coordinate ξ.

Drop that supposition, $\mathring{\Phi}_m = \Phi_m$, and for the general case introduce a new energetic quantity as

$$S_m^{\varpi}(\xi, \tau) = (1/2)\left(I_m^{\varpi 2} - V_m^{\varpi 2}\right) \tag{61}$$

where V_m^{ϖ} and I_m^{ϖ} are specified in Eqs. (31b), (31c). It seems natural to name this quantity, $S_m^{\varpi}(\xi,\,\tau)$, as a "*surplus*" of the energy density stored in the transverse field components of a modal wave.

Let us put $A_m = 1$ in formula (59) (for the sake of simplicity) and do the same in Eqs. (31a)–(31c) and (61). One can then study the variations in time dependence of the energetic quantities $S_m^{\varpi}(\xi,\,\tau)$ and $w_m^{\varpi}(\xi,\,\tau)$ in any waveguide cross section by specifying a chosen coordinate ξ.

In Fig. 1, time dependence of the quantities $S_m^{\varpi}(0,\,\tau)$ and $w_m^{\varpi}(0,\,\tau)$ are presented. It is evident that these quantities accomplish *antiphased* oscillations. Hence, propagation of the *TM*-modal wave with its amplitudes (31a)–(31c) is accompanied by the *energetic wave process* where exchange by energy between $S_m^{\varpi}(\xi,\,\tau)$ and $w_m^{\varpi}(\xi,\,\tau)$ occurs.

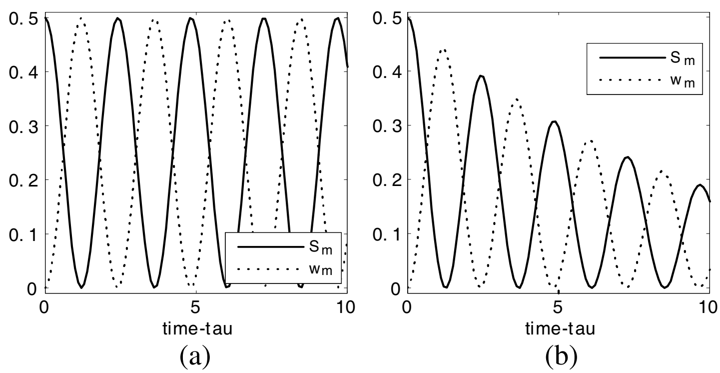


Figure 1. Exchange by energy between the surplus of energy, $\mathcal{S}_m\,(\xi,\,\tau)$; stored in the transverse field components and the energy, $w_m\,(\xi,\,\tau)$; stored in the longitudinal field component; $\varpi = 1.3$, $\xi = 0$, and (a) $\varrho = 0$, (b) $\varrho = 0.05$.

In Fig. 2, time dependence of the power flow, $\mathcal{P}_{m\,z}^{\varpi}(\xi,\,\tau)$, and the energy density, $\mathcal{W}_m^{\varpi}(\xi,\,\tau)$, are presented for the same values $\xi = 0$ and $A_m = 1$. Case (a) corresponds to the lossless waveguide, and case (b) corresponds to the value $\varrho = 0.05$ of the dimensionless lossy parameter $\varrho = \gamma/\omega_m$, where $\gamma = \sigma/(2\epsilon_0)$ and ω_m is the cut-off frequency.

Formula for the normalized by c velocity of transportation of the time-harmonic modal field energy was derived by definition (57) with making usage of formulas (55a)–(55d) and (31a)–(31c). The final result is

$$\frac{\mathrm{v}_m^{\varpi}(\xi,\tau)}{c} = \sqrt{(\varpi^2-\eta^2)\,(\varpi^2+\varrho^2)}\,\frac{2\cos[\Phi_m(\xi,\tau)]\cos[\Phi_m(\xi,\tau)-\vartheta_m]}{I_m^{\varpi 2}(\xi,\tau)+V_m^{\varpi 2}\,(\xi,\tau)+e_m^{\varpi 2}(\xi,\tau)} \quad (62)$$

where $\vartheta_m = \sin^{-1}(\varrho/\sqrt{\varpi^2+\varrho^2})$ is the lossy phase shift.

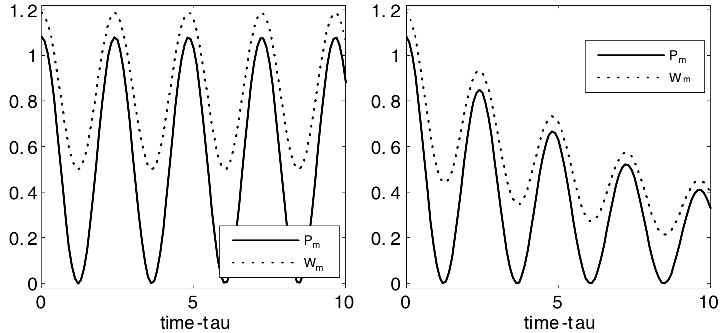


Figure 2. Time dependence of the energy flow density, $\mathcal{P}_m(\xi,\,\tau)$, and the energy density, $\mathcal{W}_m(\xi,\,\tau)$ for $\varpi = 1.3$, $\xi = 0$ and (a) $\varrho = 0$, (b) $\varrho = 0.05$.

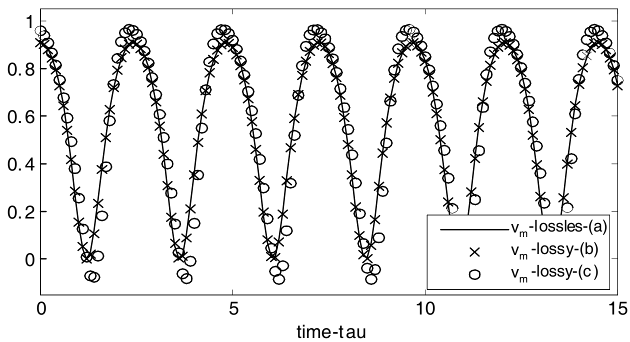


Figure 3. Normalized by c instant velocity of transportation energy, v_m/c, for $\varpi = 1.3$, $\xi = 0$ and (a) $\varrho = 0$, (b) $\varrho = 0.05$, (c) $\varrho = 0.5$.

In Fig. 3, variations of $\mathrm{v}_m^{\varpi}(0,\,\tau)/c$ are presented. The maximal values of v_m^{ϖ}/c never exceed 1. The minimal values of v_m^{ϖ}/c are actually slightly *negative* if $\varrho \neq 0$. This is caused by presence of the lossy phase shift, ϑ_m, in (62).

6.3. The Time-harmonic *Energetic* Cavity Oscillations

Calculations by formulas (55c), (55d) and (36a)–(36c) of the energy densities stored in the electric and magnetic parts, individually, for the cavity field yield

$$\mathcal{W}_m^{\mathbf{e}} = \frac{1}{2} B_m^2 \left\{ \cos^2\left(\pi s \frac{z}{\ell}\right) + \sin^2\left(\pi s \frac{z}{\ell}\right) \left(\frac{\pi s}{\kappa_m \ell}\right)^2 \right\} e^{-2\varrho\tau} \sin^2 \Psi_{ms}(\tau) \quad (63a)$$

$$\mathcal{W}_m^{\mathbf{m}} = \frac{1}{2} B_m^2 \left\{ 1 + \left(\frac{\pi s}{\kappa_m \ell}\right)^2 \right\} \cos^2\left(\pi s \frac{z}{\ell}\right) e^{-2\varrho\tau} \cos^2 \mathring{\Psi}_{ms}(\tau). \quad (63b)$$

Perform averaging Eqs. (63a), (63b) as $(1/\ell) \int_0^\ell (\cdot)\, dz$. It is evident that

$$\tfrac{1}{\ell} \int_0^\ell \cos^2(\pi s z/\ell)\, dz = \tfrac{1}{\ell} \int_0^\ell \sin^2(\pi s z/\ell)\, dz = \tfrac{1}{2}, \quad s = 1, 2, \ldots. \quad (64)$$

Thus, the averaged values, $\bar{\mathcal{W}}_m^{\mathbf{e}}$ and $\bar{\mathcal{W}}_m^{\mathbf{m}}$, depend on time as

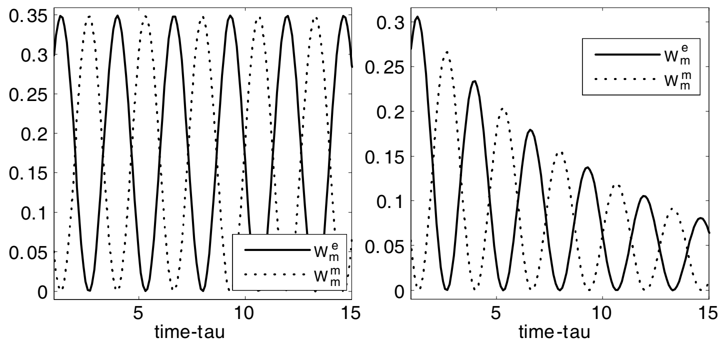
$$\bar{\mathcal{W}}_m^{\mathbf{e}}(\tau) = \mathcal{B}_{ms}\, e^{-2\varrho\tau} \sin^2 \Psi_{ms}(\tau) \quad (65a)$$

$$\bar{\mathcal{W}}_m^{\mathbf{m}}(\tau) = \mathcal{B}_{ms}\, e^{-2\varrho\tau} \cos^2 \mathring{\Psi}_{ms}(\tau) \quad (65b)$$

where $\mathcal{B}_{ms} = \frac{1}{4} B_m^2 \{1 + [\pi s / (\kappa_m \ell)]^2\}$.

If the cavity is lossless (i.e., if $\varrho = 0$), then $\delta_s = 0$, and hence, $\mathring{\Psi}_{ms} = \Psi_{ms}$. In this case, Eqs. (65a) and (65b) describe *antiphased* oscillations of the electric and magnetic field energy in the cavity. This process is analogous to the oscillations of the kinetic and potential energies of a pendulum.

Figure 4 exhibits just these oscillations of the averaged quantities (65a), (65b).



(a) (b)

Figure 4. Oscilations of the electric and magnetic field energy stored in the cavity $\bar{\mathcal{W}}_m^e(\tau)$, $\bar{\mathcal{W}}_m^e(\tau)$, for $s = 2$, $k_m \ell = 10$, $\xi = 1$ and (a) $\varrho = 0$, (b) $\varrho = 0.05$.

In Fig. 5, the instant velocity of transportation of energy by the cavity field is presented. Calculations were performed by formula

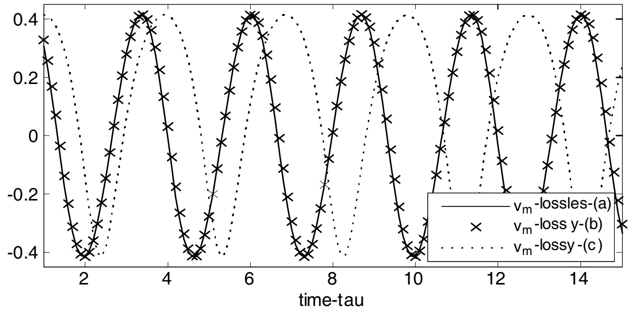


Figure 5. Normalized by c instant velocity of transportation energy, v_m/c, for $s = 2$, $k_m\ell = 10$, $\xi = 1$ and (a) $\varrho = 0$, (b) $\varrho = 0.05$, (c) $\varrho = 0.5$.

(57). The needed amplitudes were taken from Eqs. (36a)–(36c). The waveguide cross section is fixed by specifying $\xi = 1$. The normalized by c velocity, v_m/c, *oscillates* with respect to its averaged in time value $\bar{v}_m/c = 0$. This implies, physically, that we observe a *standing energetic wave* in the cavity.

7. SUMMARY OF THE RESEARCH FINDINGS

The system of Maxwell's equations with ∂_t is solved explicitly by a simple method via the straightforward calculations. The solution is obtained in Hilbert space L_2 of the real-valued functions. The time-domain modal fields are found in a form of the transverse-longitudinal decompositions. Each of the field components is a product of two factors. One factor is a vectorial element of the modal basis, dependent only on transverse waveguide coordinates. The basis elements are obtained with the required physical dimensions. The other factors are the appropriate modal amplitudes (dimensionless), each of which dependent solely on time, t, and axial waveguide coordinate, z. The modal basis and modal amplitude problems are autonomous.

Besides the time-domain fields, their instant energetic characteristics are obtained explicitly. The energetic waves, which propagate accompanying the field waves, are established and analyzed. The velocity of transportation of the field energy is obtained explicitly as the function of the variables (t, z).

Existence of a new countable set of the waveguide modes is established. Every modal field from this set oscillates with the same cut-off frequency, but different modes have different amplitudes as the polynomials in t and z.

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