

THE ADI-FDTD METHOD INCLUDING LUMPED NETWORKS USING PIECEWISE LINEAR RECURSIVE CONVOLUTION TECHNIQUE

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Abstract—The lumped network alternating direction implicit finite difference time domain (LN-ADI-FDTD) technique is proposed as an extension of the conventional ADI-FDTD method in this paper, which allows the lumped networks to be inserted into some ADI-FDTD cells. Based on the piecewise linear recursive convolution (PLRC) technique, the current expression of the loaded place can be obtained. Then, substituting the expression into the ADI-FDTD formulas, the difference equations including an arbitrary linear network are derived. For the sake of showing the validity of the proposed scheme, lumped networks are placed on the microstrip and the voltage across the road is computed by the lumped network finite difference time domain (LN-FDTD) method and LN-ADI-FDTD method, respectively. Moreover, the results are compared with those of obtained by using the circuitual simulator ADS. The agreement among all the simulated results is achieved, and the extended ADI-FDTD method has been shown to overcome the Courant-Friedrichs-Lewy (CFL) condition.

1. INTRODUCTION

The traditional finite-difference time-domain (FDTD) method is constrained by the Courant-Friedrichs-Lewy (CFL) condition [1], which affects its computational efficiency for electrically fine and high Q structures. An alternative to overcome this limitation is the alternating-direction implicit (ADI)-FDTD method [2, 3]. It is based on the time split scheme where time marching over one full time step

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is broken up into two sub-time step computations. Such time splits lead to computational expenditures that are higher than those of the conventional FDTD method.

In recent years, a number of techniques have been proposed to incorporate lumped elements into the standard FDTD algorithm. The so called lumped network finite difference time domain (LN-FDTD) method allows complex microwave circuits to be successfully analyzed; bridging the gap between electromagnetic-field and circuit-based simulators [4]. The key point of researching the LN-FDTD method is to import the current expression of lumped network into the Maxwell's equations. Generally, there are three common methods to derive the current expression at the loaded place: directly deducing by the volt-ampere characteristic [5], basing on the piecewise linear recursive convolution (PLRC) technique [6–8], using Z-transform approach [9–12]. Furthermore, the PLRC technique and Z-transform can be applied to model arbitrary linear lumped network, whereas arbitrary linear lumped network is difficult to be modeled by the volt-ampere characteristic. In addition, Z-transform should save $2M - 1$ additional variables, where M represents the order of the impedance function [4]. Compared to Z-transform, PLRC technique only needs to obtain the current expression in the time domain, and the update value of electric field can be computed immediately which can save fewer field variables. Recently, the Z-transform has been applied into the LN-ADI-FDTD method in [13]. Nevertheless, to the best of our knowledge, the PLRC technique has not been applied to the LN-ADI-FDTD method, and the study on the PLRC technique applying to the LN-ADI-FDTD method is very useful.

The ADI-FDTD method is extended to include arbitrary higher-order lumped networks based on PLRC technique in this paper. Firstly, the one step ($n \rightarrow n + 1$) of conventional PLRC technique is split into two sub-steps ($n \rightarrow n + 1/2$, $n + 1/2 \rightarrow n + 1$). Secondly, the admittance parameters in Laplace domain of the lumped network are abstracted. Then, the time domain admittance parameters can be obtained by inverse Fourier transform. Fourthly, the time domain current formula is substituted into the ADI-FDTD equations based on the PLRC technique. At last, to validate the above analysis, a microstrip including lumped networks is considered and the voltage across the lumped networks has been computed. The result is compared with those obtained by ADS simulator and the agreement among all the simulated results is achieved. Furthermore, the LN-ADI-FDTD method has been shown to overcome the Courant-Friedrichs-Lewy (CFL) condition.

2. EXTENDED ADI-FDTD INCLUDING LUMPED NETWORKS BASED ON THE PLRC TECHNIQUE

Assuming that the lumped networks are replaced along the $+z$ direction, the contribution of the lumped networks is presented by \vec{J}_{Lz} , and the current to the current density as $I_L = \Delta x \Delta y J_{Lz}$, only the update formulations of E_z are different from the traditional ADI-FDTD method. Therefore, the formulations of E_z in sub-steps can be derived as follows

sub-step 1:

$$E_z^{n+1/2} = E_z^n + \frac{\Delta t}{2\varepsilon} \left(\frac{\partial H_y^{n+1/2}}{\partial x} - \frac{\partial H_x^n}{\partial y} \right) - \frac{\Delta t}{2\varepsilon \Delta x \Delta y} I_L^{n+1/4} \quad (1)$$

$$H_y^{n+1/2} = H_y^n + \frac{\Delta t}{2\mu} \left(\frac{\partial E_z^{n+1/2}}{\partial x} - \frac{\partial E_x^n}{\partial z} \right) \quad (2)$$

sub-step 2:

$$E_z^{n+1} = E_z^{n+1/2} - \frac{\Delta t}{2\varepsilon} \left(\frac{\partial H_y^{n+1/2}}{\partial x} - \frac{\partial H_x^{n+1}}{\partial y} \right) - \frac{\Delta t}{2\varepsilon \Delta x \Delta y} I_L^{n+3/4} \quad (3)$$

$$H_x^{n+1} = H_x^{n+1/2} + \frac{\Delta t}{2\mu} \left(\frac{\partial E_y^{n+1/2}}{\partial z} - \frac{\partial E_z^{n+1}}{\partial y} \right) \quad (4)$$

For sub-step 1, at the time step $t = (n + 1/4)\Delta t$, the update formulation of E_z can be deduced by combining (1) and (2),

$$\begin{aligned} & -\frac{(\Delta t)^2}{4\varepsilon\mu(\Delta x)^2} E_z^{n+1/2}(i-1, j, k) + \left(1 + \frac{(\Delta t)^2}{2\varepsilon\mu(\Delta x)^2} \right) E_z^{n+1/2}(i, j, k) \\ & -\frac{(\Delta t)^2}{4\varepsilon\mu(\Delta x)^2} E_z^{n+1/2}(i+1, j, k) = E_z^n(i, j, k) \\ & + \frac{\Delta t}{2\varepsilon\Delta x} \left(H_y^n \left(i + \frac{1}{2}, j, k + \frac{1}{2} \right) - H_y^n \left(i - \frac{1}{2}, j, k + \frac{1}{2} \right) \right) \\ & - \frac{\Delta t}{2\varepsilon\Delta y} \left(H_x^n \left(i, j + \frac{1}{2}, k + \frac{1}{2} \right) - H_x^n \left(i, j - \frac{1}{2}, k + \frac{1}{2} \right) \right) \\ & - \frac{(\Delta t)^2}{4\varepsilon\mu\Delta x\Delta z} \left(E_x^n \left(i + \frac{1}{2}, j, k + 1 \right) - E_x^n \left(i + \frac{1}{2}, j, k \right) \right) - \frac{\Delta t}{2\varepsilon\Delta x\Delta y} I_L^{n+1/4} \quad (5) \end{aligned}$$

For a lumped network consisting of arbitrary connecting of several linear elements, $Y(s)$ can be expressed as

$$Y(s) = \sum_{i=1}^N \frac{c_i}{s - a_i} + g + sh = \sum_{i=1}^N Y_i(s) + Y_o(s) \quad (6)$$

where the coefficients g and h are real numbers, while a_i and c_i are real or conjugate complex numbers.

In the Laplace domain, $I(s)$ is defined as:

$$I(s) = V(s)Y(s) \quad (7)$$

Furthermore, the voltage and current characteristic equations of lumped networks in the time domain can be derived by PLRC technique [7], and then the update expression of E_z can be obtained.

The PLRC technique is divided from one time step ($n \rightarrow n+1$) into two sub-steps ($n \rightarrow n+1/2$, $n+1/2 \rightarrow n+1$). Supposing $I(t) = \int_0^t V(t-\tau)Y(\tau)d\tau$, discrete the time of $I(t)$,

$$I^n = \sum_{m=0}^{2n-1} \int_{\frac{m}{2}\Delta t}^{\frac{m+1}{2}\Delta t} V(n\Delta t - \tau)Y(\tau)d\tau \quad (8)$$

During the region of $(m/2)\Delta t \sim (m/2 + 1/2)\Delta t$, $V(n\Delta t - \tau)$ changes linearly, and it can be expressed as,

$$V(n\Delta t - \tau) = V^{n-\frac{m}{2}} + \frac{V^{n-\frac{m}{2}-\frac{1}{2}} - V^{n-\frac{m}{2}}}{\Delta t/2} \left(\tau - \frac{m}{2}\Delta t \right) \quad (9)$$

By substituting (9) into (8), the expression of I^n can be deduced as:

$$\begin{aligned} I^n &= \sum_{m=0}^{2n-1} \left[V^{n-\frac{m}{2}} \cdot Y^{\frac{m}{2}} + \left(V^{n-\frac{m}{2}-\frac{1}{2}} - Y^{n-\frac{m}{2}} \right) \xi^{\frac{m}{2}} \right] \\ &= (y^0 - \xi^0) V^n + \xi^0 V^{n-\frac{1}{2}} + \sum_{m=1}^{2n-1} \left[V^{n-\frac{m}{2}} \cdot Y^{\frac{m}{2}} + \left(V^{n-\frac{m}{2}-\frac{1}{2}} - Y^{n-\frac{m}{2}} \right) \xi^{\frac{m}{2}} \right] \\ &= (y^0 - \xi^0) V^n + \xi^0 V^{n-\frac{1}{2}} + \sum_{m=0}^{2n-2} \left[V^{n-\frac{m}{2}-\frac{1}{2}} \cdot Y^{\frac{m}{2}+\frac{1}{2}} + \left(V^{n-\frac{m}{2}-1} \right. \right. \\ &\quad \left. \left. - V^{n-\frac{m}{2}-\frac{1}{2}} \right) \xi^{\frac{m}{2}+\frac{1}{2}} \right] \end{aligned} \quad (10)$$

where

$$y^{\frac{m}{2}} = \int_{\frac{m}{2}\Delta t}^{\frac{m+1}{2}\Delta t} Y(\tau)d\tau \quad (11)$$

$$\xi^{\frac{m}{2}} = \frac{1}{\Delta t/2} \int_{\frac{m}{2}\Delta t}^{\frac{m+1}{2}\Delta t} \left(\tau - \frac{m}{2}\Delta t\right) Y(\tau) d\tau \tag{12}$$

If the following relationship can be established,

$$\rho^{\frac{m}{2}} = \frac{y^{\frac{m}{2} + \frac{1}{2}}}{y^{\frac{m}{2}}} = \frac{\xi^{\frac{m}{2} + \frac{1}{2}}}{\xi^{\frac{m}{2}}} \tag{13}$$

I^n can be generated as:

$$I^n = (y^0 - \xi^0) V^n + \xi^0 V^{n-\frac{1}{2}} + \rho^0 I^{n-\frac{1}{2}} \tag{14}$$

Regarding (6), the total electricity can be expressed as,

$$I_L^{n+1/4} = \sum_{i=0}^N I_{Li}^{n+1/4} + I_{Lo}^{n+1/4} \tag{15}$$

Next, $I_L^{n+1/4}$ can be derived in the following procedures.

2.1. a_i Is Zero

The expression of $Y_i(s) = \frac{c_i}{s}$ is transformed into the time domain,

$$Y(t) = f^{-1} \left(\frac{c_i}{s} \right) = c_i u(t) \tag{16}$$

Then substituting (16) into (11)–(13), we can obtain

$$y_i^{\frac{m}{2}} = \int_{\frac{m}{2}\Delta t}^{\frac{m+1}{2}\Delta t} c_i d\tau = \frac{c_i \Delta t}{2} \tag{17}$$

$$\xi_i^{\frac{m}{2}} = \frac{1}{\Delta t/2} \int_{\frac{m}{2}\Delta t}^{\frac{m+1}{2}\Delta t} \left(\tau - \frac{m}{2}\Delta t\right) c_i d\tau = \frac{c_i \Delta t}{4} \tag{18}$$

$$\rho_i = 1 \tag{19}$$

In this case, (15) can be expressed as follows.

$$I_{Li}^{n+1/4} = \left(I_{Li}^{n+1/2} + I_{Li}^n \right) / 2 = \left[(y_i^0 - \xi_i^0) V_{Zi}^{n+1/2} + \xi_i^0 V_{Zi}^n + (\rho_i^0 + 1) I_{Li}^n \right] / 2 \tag{20}$$

2.2. a_i and c_i Are Real Numbers, Moreover a_i Is Nonzero

The inversed Fourier transform of $Y_i(s) = \frac{c_i}{s-a_i}$ is:

$$Y_i(t) = f^{-1} \left(\frac{c_i}{s-a_i} \right) = c_i e^{a_i t} \tag{21}$$

According to (11)–(13), the following expressions can be obtained:

$$y_i^{\frac{m}{2}} = -\frac{c_i}{a_i} \left(1 - e^{\frac{a_i \Delta t}{2}}\right) e^{\frac{ma_i \Delta t}{2}} \quad (22)$$

$$\xi_i^{\frac{m}{2}} = -\frac{2c_i}{a_i^2 \Delta t} \left[\left(1 - \frac{a_i \Delta t}{2}\right) e^{\frac{a_i \Delta t}{2}} - 1 \right] e^{\frac{ma_i \Delta t}{2}} \quad (23)$$

$$\rho_i = e^{\frac{a_i \Delta t}{2}} \quad (24)$$

In this way, the current expressions of (15) and (20) are similar.

2.3. a_i and a_{i+1} , c_i and c_{i+1} Are Conjugate Complex Pairs

Based on the method from [8], (20) can be simplified as

$$I_{Li}^{n+1/4} + I_{Li(i+1)}^{n+1/4} = \text{Re} \left[(y_i^0 - \xi_i^0) V_{Zi}^{n+1/2} + \xi_i^0 V_{Zi}^n + (\rho_i^0 + 1) I_{Li}^n \right] \quad (25)$$

The expression of $Y_0(s) = g + sh$ is transformed into the time domain

$$I_{Lo}^{n+1/4} = \left(\frac{g}{2} + \frac{h}{\Delta t/2} \right) V_z^{n+1/2} + \left(\frac{g}{2} - \frac{h}{\Delta t/2} \right) V_z^n \quad (26)$$

Finally, considering the discussion above, if c_i and a_i are consist of N_r real quantities, N_g pairs of conjugate numbers, the total electricity is

$$I_L^{n+1/4} = \left(y_t^0 - \xi_t^0 + \frac{g}{2} + \frac{h}{\Delta t/2} \right) V_z^{n+1/2} + \left(\xi_t^0 + \frac{g}{2} - \frac{h}{\Delta t/2} \right) V_z^n + I_{Lt}^n \quad (27)$$

where

$$y_t^0 = \frac{1}{2} \sum_{i=1}^{N_r} y_i^0 + \sum_{i=N_r+1}^{N_r+N_g} \text{Re} (y_i^0) \quad (28)$$

$$\xi_t^0 = \frac{1}{2} \sum_{i=1}^{N_r} \xi_i^0 + \sum_{i=N_r+1}^{N_r+N_g} \text{Re} (\xi_i^0) \quad (29)$$

$$I_t^n = \frac{1}{2} \sum_{i=1}^{N_r} (\rho_i + 1) I_{Li}^n + \sum_{i=N_r+1}^{N_r+N_g} \text{Re} [(\rho_i + 1) I_{Li}^n] \quad (30)$$

For sub-step 2, the formulations are similar to the above equations, which are not shown here.

Equations (5) and (27) are the governing difference equations of the LN-ADI-FDTD method written in operating form. Thus the extraction of current expression constitutes the fatal point for researching the ADI-FDTD method including lumped network.

3. NUMERICAL RESULTS

In order to verify the theoretical analysis above, a microstrip structure with lumped networks is simulated as shown in Fig. 1. The computational domain is $12 \times 18 \times 2$ mm, $\Delta x = \Delta y = 0.2$ mm, $\Delta z = 0.1$ mm, leading to a mesh number of $60 \times 90 \times 20$. In addition, the thickness of the dielectric plane is $3\Delta z$, the dielectric constant is $\epsilon_r = 2.2$, and the dimension of the metal strip is $6\Delta x \times 80\Delta y$. Mur's first-order absorbing boundary condition is applied on the truncated boundary to absorb out-going waves except for the $z = 0$ plane. Moreover, a sinusoidal voltage source with an internal impedance of 100Ω is connected to one termination of the metal strip, which is 20 GHz frequency and 1 V amplitude. Furthermore, at the node $(30\Delta x, 80\Delta y)$, a plane of 7×3 identical lumped elements are used between the metal strip and the infinite ground plane. Fig. 2 shows the circuit diagram of one lumped network at the loaded place. CFLN is defined as the ratio between the time step taken and the maximum CFL limit of the FDTD method, which can be expressed as $CFLN = \Delta t / \Delta t_{max}$.

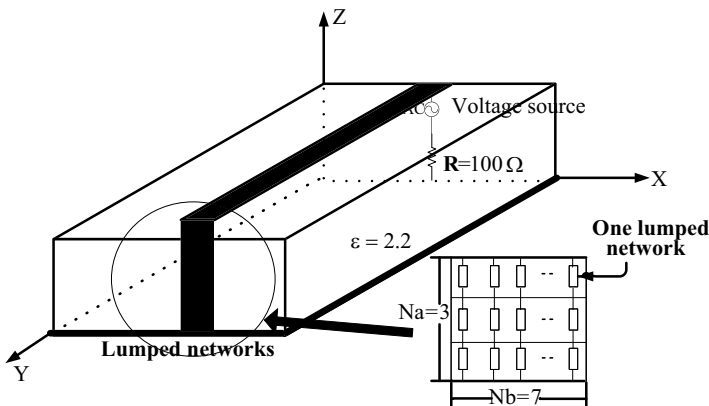


Figure 1. Configuration of a microstrip structure with lumped networks.

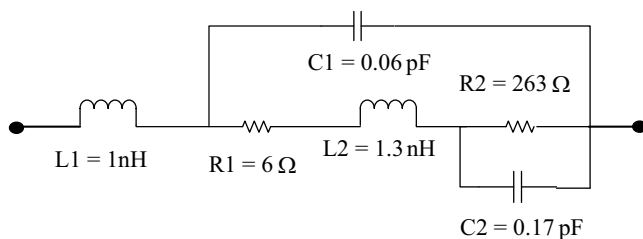


Figure 2. Circuit diagram of one lumped network.

Here, $\Delta t_{\max} = 0.2667$ ps is the maximum time step size to satisfy the limitation of the 3D CFL condition in the conventional FDTD method. For the conventional FDTD method, $CFLN = 0.75$ and the step number is 2000, and the total simulation time is selected to be 400 ps. The simulations are performed on a computer of Pentium IV with 2GB RAM, and the computer program is developed with C++.

Figure 3 presents the voltages across lumped networks computed by the LN-ADI-FDTD method and the LN-FDTD method, and simulated by the ADS approach. As can be seen from Fig. 3, the results of the LN-FDTD method and LN-ADI-FDTD method are both close to that of given by ADS. Generally speaking, good agreement is observed among these three techniques.

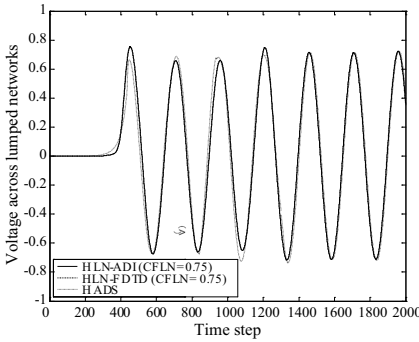


Figure 3. Voltage across the lumped network by different methods.

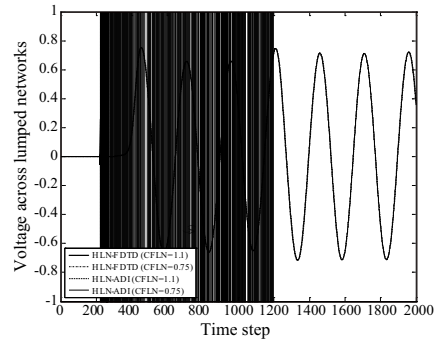


Figure 4. Voltage across the lumped networks with different time steps.

Figures 4 and 5 show the time domain response of the voltage across lumped networks obtained by the proposed LN-ADI-FDTD formulations and the conventional LN-FDTD formulations for different CFLN values. As can be clearly seen from Fig. 4, the LN-FDTD method is divergent for $CFLN = 1.1$, but stable for $CFLN = 0.75$; while for the LN-ADI-FDTD method, the result is stable for $CFLN = 1.1$ and is consistent with the result for $CFLN = 0.75$. It also can be observed that the LN-ADI-FDTD method is unconditionally stable, whereas, it is conditionally stable in the LN-FDTD method.

The computed results computed by the LN-ADI-FDTD method as $CFLN$ varying from 0.75 to 4.5 are shown in Fig. 5 and good agreement with those obtained by the LN-FDTD method when $CFLN = 0.75$ is achieved. As $CFLN$ increases, the results of the LN-ADI-FDTD method deviate from these obtained by the LN-FDTD method. In order to further illustrate the relationship between $CFLN$ and

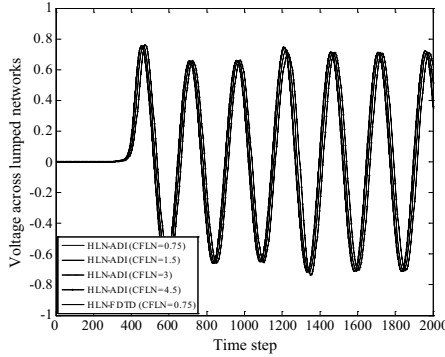


Figure 5. Voltage across the lumped networks with different time steps.

efficiency, Table 1 shows the comparison of the results of the LN-ADI-FDTD method and the LN-FDTD method. For clarity, the definition of the average relative errors for the LN-ADI-FDTD method and the LN-FDTD method are shown as follows:

$$\text{Error-ave}|_{\text{LN-ADI}} = \sum_{i=1}^N \text{abs} [V_{\text{LN-ADI}}(i \times n) - V_{\text{LN-FDTD}}(i)] / (N/n) \quad (31)$$

where N is the step number of the LN-FDTD method and n the ratio of N to the step number of the LN-ADI-FDTD method.

From Table 1, when CFLN increases, the average relative error becomes larger; however the CPU time is reduced. Moreover, the CPU time of the LN-ADI-FDTD method when CFLN = 3 and CFLN = 4.5 is less than the LN-FDTD method when CFLN = 0.75.

Table 1. The comparison of results computed by the LN-ADI-FDTD method with the LN-FDTD method.

Scheme	CFLN	Step number	Average relative error	CPU time (s)
LN-FDTD	0.75	2000		35
LN-ADI-FDTD	0.75	2000	0.0048	100
	1.5	1000	0.0248	49
	3	500	0.0907	25
	4.5	333	0.1796	17

For instance, the LN-ADI-FDTD method requires the CPU time of 25s when $CFLN = 3$, while the LN-FDTD method requires the CPU time of 35s when $CFLN = 0.75$. The saving in CPU time of the extended LN-ADI-FDTD method is 28% in comparison with the LN-FDTD method. Meanwhile, the average relative error is 0.0907, and to some extent, the result can be accepted.

Consequently, the unconditional stability of the proposed method has been demonstrated by the numerical experiments in the above. Following the Z transform, theoretical proof of the unconditional stability of the proposed method can be shown. Due to space limitations, this is not done here and is left to a future publication.

4. CONCLUSION

The PLRC technique has been split into two steps ($n \rightarrow n + 1/2$, $n + 1/2 \rightarrow n + 1$) from one step ($n \rightarrow n + 1$). Then the LN-ADI-FDTD method has been proposed based on the PLRC technique. Finally, a microstrip with lumped networks has been computed by the LN-ADI-FDTD method and the LN-FDTD method, simulated by the ADS and HFSS+ADS approach. From the results, good agreement is observed among these four techniques. Furthermore, the LN-ADI-FDTD method can overcome the CFL condition which the LN-FDTD method should be constrained. In addition, the PLRC technique has only applied to the ADI-FDTD method including one-port lumped network in this paper, and it will be extended to the ADI-FDTD method including two-port lumped networks. Generalizing these extensions will be our future work.

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