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SECOND-ORDER FORMULATION FOR THE QUASI-STATIC FIELD FROM A VERTICAL ELECTRIC DIPOLE ON A LOSSY HALF-SPACE

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Abstract—Improved quasi-static expressions are derived for the time-harmonic electromagnetic (EM) field components excited by a vertical electric dipole (VED) lying on the surface of a flat and homogeneous lossy half-space. An analytical procedure is developed that allows to evaluate the complete integral representations for the fields, once the non-oscillating part of the integrand in the expression of the magnetic vector potential is replaced with its quadratic approximation for small values of the ratio between the wavenumbers in free-space and in the conducting medium. The advantage of the proposed second-order quasi-static approximations resides in the possibility of relaxing the assumption of highly conducting half-space. This makes it possible to overcome the limitations implied by the previously published zeroth-order formulation, whose validity is restricted to extremely low frequencies for poorly conducting media. Numerical results are presented to illustrate the reduction of relative percent error arising from using the improved quasi-static field expressions.

1. INTRODUCTION

The problem of evaluating the electromagnetic field produced by an elementary dipole source located above a plane conducting half-space has attracted the attention of many researchers beginning with Sommerfeld [1–4]. Yet, to date the derivation of closed-form expressions from the complete integral representations for the field components has proven to be a prohibitive task, except for certain special cases which, fortunately, cover many practical applications (EM sounding to detect buried objects, radio propagation, therapeutic

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heating of tissues) [3–11]. Useful simple expressions have been obtained for the case of the vertical electric dipole, under the assumptions that both the source and field points are located on the surface of the medium, and the operating frequency is low enough that $k_0\rho \ll 1$, being k_0 the free-space wavenumber and ρ the source-receiver distance. Such quasi-static field approximations, the summary of which is given by Bannister [8], are subject to the further condition that the half-space is much more dense than air ($|k_1| \gg k_0$), and this is the reason why they may be referred to as zeroth-order approximations [12, 13], whose frequency range of validity strongly depends upon the electromagnetic properties of the material half-space. Due to this feature, the zeroth-order formulation fails at extremely low frequencies if the electrical conductivity of the material half-space is very poor, like, for instance, in the case of permafrost, igneous and metamorphic rocks [5, 6].

The scope of the present paper is two-fold. First, to develop a rigorous analytical procedure that allows to derive second-order quasi-static expressions for the field components produced by a VED lying on the surface of a lossy medium, so to overcome the described limitation implied by the previously published formulation. Second, to quantify through numerical simulations the error that arises from applying the zeroth-order quasi-static approximations in the case of poorly conducting media. In Section 2 the non-oscillating part of the integrand in the expression for the vertical component of the magnetic vector potential is replaced with its quadratic approximation for small values of $k_0/|k_1|$. Next, the field integrals are cast into forms involving only known tabulated Sommerfeld Integrals. In Section 3, the relative percent error resulting from applying the zeroth- and second-order formulas instead of performing numerical evaluation of the field integrals is computed as a function of frequency. The obtained results demonstrate that, up to $k_0\rho = 0.36$, second-order expressions exhibit a good level of accuracy for all the values assumed for the electromagnetic parameters of the half-space, while the zeroth-order approximations begin to breakdown far before entering the non-quasi-static frequency region if the conductivity of the half-space is sufficiently small. It is seen that the zeroth-order formulation may produce, even in the quasi-static frequency region, a relative percent error beyond 30%, while the relative error arising from applying the improved expressions for the fields does not exceed 7% up to $k_0\rho = 0.36$.

2. THEORY

Consider a VED of moment $pe^{j\omega t}$ lying on the surface of a flat, homogeneous, isotropic and linear lossy medium. The EM parameters

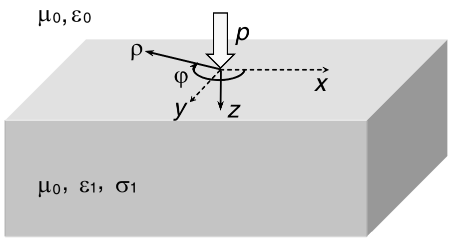

Figure 1. Sketch of a vertical electric dipole on a homogeneous lossy medium.

of the medium are as depicted in Fig. 1, and a cylindrical coordinate system (ρ, φ, z) is introduced. The frequency-domain integral representations for the EM field components E_ρ, E_z, and H_φ generated in the air region may be written as [14–18]

$$E_\rho = -\frac{j\omega}{k_0^2}\frac{\partial}{\partial\rho}\frac{\partial A}{\partial z}, \tag{1}$$

$$E_z = \frac{j\omega}{k_0^2}\nabla_t^2 A, \tag{2}$$

$$H_\varphi = -\frac{1}{\mu_0}\frac{\partial A}{\partial\rho}, \tag{3}$$

where

$$\nabla_t^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} = \left(\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right)\frac{\partial}{\partial\rho} \tag{4}$$

is the transverse Laplacian operator, and

$$A = \frac{\mu_0\, p}{2\pi}\int_0^\infty e^{u_0 z}\frac{1}{u_0 + \tau^2 u_1}J_0(\lambda\rho)\lambda d\lambda \tag{5}$$

is the z-component of the magnetic vector potential, being $J_0(\xi)$ the zeroth-order Bessel function, and

$$u_n = \left(\lambda^2 - k_n^2\right)^{1/2}, \tag{6}$$

$$k_n = \left(\omega^2\mu_n\epsilon_n + j\omega\mu_n\sigma_n\right)^{1/2}, \tag{7}$$

$$\tau = \frac{k_0}{k_1}. \tag{8}$$

The scope of this paper is to find explicit expressions for the components E_ρ, E_z, and H_φ at $z = 0^-$ for the quasi-static case $k_0\rho \ll 1$, which are valid regardless of the order of magnitude of the half-space conductivity. A glance to (1)–(3) suggests that this problem reduces to that of evaluating the derivatives

$$\left.\frac{\partial A}{\partial z}\right|_{z=0}, \qquad \left.\frac{\partial A}{\partial \rho}\right|_{z=0}. \tag{9}$$

To this goal, at first the identity

$$\frac{1}{u_0+\tau^2 u_1} = \frac{1}{u_0} - \frac{\tau^2 u_1}{(u_0+\tau^2 u_1)u_0} \tag{10}$$

and the well known result [19, p. 9, No. 24]

$$\int_0^\infty \frac{e^{u_n z}}{u_n} J_0(\lambda\rho)\lambda d\lambda = \frac{e^{-jk_n r}}{r}, \tag{11}$$

with $r=\sqrt{\rho^2+z^2}$, are used to cast (5) in the form

$$A = \frac{\mu_0\, p}{2\pi}\left[\frac{e^{-jk_0 r}}{r} - \tau^2 \int_0^\infty e^{u_0 z}\frac{u_1}{(u_0+\tau^2 u_1)u_0} J_0(\lambda\rho)\lambda d\lambda\right]. \tag{12}$$

Next, since $|\tau|$ is always less than unity, one can introduce the power series expansion

$$\frac{u_1}{(u_0+\tau^2 u_1)\, u_0} = \frac{u_1}{u_0^2} - \tau^2\frac{u_1^2}{u_0^3} + \mathcal{O}\left(\tau^4\right) \tag{13}$$

and write the z- and ρ-derivatives of A as

$$\begin{aligned}\frac{\partial A}{\partial z} &= \frac{\mu_0\, p}{2\pi}\left[-(1+jk_0 r)z\frac{e^{-jk_0 r}}{r^3} - \tau^2\int_0^\infty e^{u_0 z}\frac{u_1}{u_0}J_0(\lambda\rho)\lambda d\lambda\right.\\ &\quad \left.+\tau^4\int_0^\infty e^{u_0 z}\frac{u_1^2}{u_0^2}J_0(\lambda\rho)\lambda d\lambda + \mathcal{O}(\tau^6)\right],\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial A}{\partial \rho} &= \frac{\mu_0\, p}{2\pi}\left[\frac{\partial}{\partial\rho}\left(\frac{e^{-jk_0 r}}{r}\right) + \tau^2\int_0^\infty e^{u_0 z}\frac{u_1}{u_0^2}J_1(\lambda\rho)\lambda^2 d\lambda\right.\\ &\quad \left.-\tau^4\int_0^\infty e^{u_0 z}\frac{u_1^2}{u_0^3}J_1(\lambda\rho)\lambda^2 d\lambda + \mathcal{O}(\tau^6)\right].\end{aligned} \tag{15}$$

Setting $z=0$ and taking account of the relation

$$\left[\frac{\partial}{\partial\rho}\left(\frac{e^{-jk_0 r}}{r}\right)\right]_{z=0} = \frac{d}{d\rho}\left(\frac{e^{-jk_0\rho}}{\rho}\right) \cong \frac{d}{d\rho}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}, \tag{16}$$

that holds since $k_0\rho \ll 1$, provides the expressions

$$\begin{aligned}\left.\frac{\partial A}{\partial z}\right|_{z=0} &= \frac{\mu_0\, p\,\tau^2}{2\pi}\left[-\lim_{z\to 0}\int_0^\infty e^{u_0 z}\frac{u_1}{u_0}J_0(\lambda\rho)\lambda d\lambda\right.\\ &\quad \left.+\tau^2\lim_{z\to 0}\int_0^\infty e^{u_0 z}\frac{u_1^2}{u_0^2}J_0(\lambda\rho)\lambda d\lambda + \mathcal{O}(\tau^4)\right],\end{aligned} \tag{17}$$

$$\left.\frac{\partial A}{\partial \rho}\right|_{z=0} = \frac{\mu_0\, p}{2\pi}\left[-\frac{1}{\rho^2}+\tau^2\lim_{z\to 0}\int_0^\infty e^{u_0 z}\frac{u_1}{u_0^2}J_1(\lambda\rho)\lambda^2 d\lambda\right.$$
$$\left.-\tau^4\lim_{z\to 0}\int_0^\infty e^{u_0 z}\frac{u_1^2}{u_0^3}J_1(\lambda\rho)\lambda^2 d\lambda+\mathcal{O}(\tau^6)\right], \quad (18)$$

in the former of which the factor τ^2, which was common to all the terms in the square brackets, has been moved outside. Notice that in the mathematical development that follows this factor will disappear as a result of algebraic simplifications.

It should be also noted that, since the aim of the present study is to derive second-order accurate expressions for the quasi-static fields, the term on the order of τ^4 in (18) is a higher-order contribution and may be included in the remainder of the power series expansion. It reads

$$\left.\frac{\partial A}{\partial \rho}\right|_{z=0} = \frac{\mu_0\, p}{2\pi}\left[-\frac{1}{\rho^2}+\tau^2\lim_{z\to 0}\int_0^\infty e^{u_0 z}\frac{u_1}{u_0^2}J_1(\lambda\rho)\lambda^2 d\lambda+\mathcal{O}\left(\tau^4\right)\right]. \quad (19)$$

Then, use of (6) in conjunction with the Bessel differential equation [20]

$$\left(\nabla_t^2+\lambda^2-\frac{\nu^2}{\rho^2}\right)J_\nu(\lambda\rho)=0 \quad (20)$$

leads to the identity

$$u_n^2 J_\nu(\lambda\rho) = -\left(\nabla_t^2-\frac{\nu^2}{\rho^2}+k_n^2\right)J_\nu(\lambda\rho), \quad (21)$$

which makes it possible to express (17) and (19) as

$$\left.\frac{\partial A}{\partial z}\right|_{z=0} = \frac{\mu_0\, p\,\tau^2}{2\pi}\left[\left(\nabla_t^2+k_1^2\right)\left(S_{1,1}-\tau^2 S_{2,0}\right)\right], \quad (22)$$

$$\left.\frac{\partial A}{\partial \rho}\right|_{z=0} = -\frac{\mu_0\, p}{2\pi}\left[\frac{1}{\rho^2}+\tau^2\left(\nabla_t^2-\frac{1}{\rho^2}+k_1^2\right)S_{2,1}\right], \quad (23)$$

with

$$S_{2,1} = \int_0^\infty \frac{1}{u_0^2 u_1}J_1(\lambda\rho)\lambda^2 d\lambda, \quad (24)$$

$$S_{1,1} = \int_0^\infty \frac{1}{u_0 u_1}J_0(\lambda\rho)\lambda d\lambda, \quad (25)$$

$$S_{2,0} = \int_0^\infty \frac{1}{u_0^2}J_0(\lambda\rho)\lambda d\lambda, \quad (26)$$

and where only the zeroth- and second-order terms in the square brackets have been retained. Notice that casting (17) and (19) into the forms (22) and (23) is allowed to the extent that a derivative of arbitrary order with respect to ρ can be moved outside the integral sign, as follows

$$\int_0^\infty e^{u_0 z} f(\lambda) \frac{\partial^l J_\nu(\lambda\rho)}{\partial\rho^l}\, d\lambda = \frac{\partial^l}{\partial\rho^l} \int_0^\infty e^{u_0 z} f(\lambda) J_\nu(\lambda\rho) d\lambda, \tag{27}$$

where $f(\lambda)$ is the non-oscillating and non-exponential part of the generic integrand. The property (27) holds in virtue of the continuous dominated convergence theorem (CDCT) [21, Chap. 19], because the function

$$\left| e^{u_0 z} f(\lambda) \frac{\partial^l J_\nu(\lambda\rho)}{\partial\rho^l} \right| = e^{\Re\{u_0\} z} \left| f(\lambda) \frac{\partial^l J_\nu(\lambda\rho)}{\partial\rho^l} \right| \tag{28}$$

is integrable over $[0,\infty)$ for all $\rho > 0$ and for all $l \geq 0$, as it exponentially decays with increasing λ.

All of the integrals in (24)–(26) can be reduced to well-known tabulated Sommerfeld Integrals. Under the assumption that $u_0 \cong \lambda$, which holds since $k_0 \to 0$, one can apply formulas [19, p. 18, No. 3] and [19, p. 8, No. 16] respectively to (24) and (25) and obtain

$$S_{2,1} = \int_0^\infty \frac{1}{u_1} J_1(\lambda\rho) d\lambda = \frac{1 - e^{-jk_1\rho}}{jk_1\rho}, \tag{29}$$

$$S_{1,1} = \int_0^\infty \frac{1}{u_1} J_0(\lambda\rho)\, d\lambda = K_0\left(\frac{jk_1\rho}{2}\right) I_0\left(\frac{jk_1\rho}{2}\right), \tag{30}$$

being $I_0(\xi)$ and $K_0(\xi)$ the zeroth-order modified Bessel functions of the first and second kind. On the other hand, application of [9, p. 11, No. 45] to (26) yields the expression

$$S_{2,0} = K_0\left(jk_0\rho\right), \tag{31}$$

which, since $k_0\rho \ll 1$, can be simplified by introducing the asymptotic form of K_0 for small arguments [20, 9.6.11]. It is found that

$$S_{2,0} \cong -\gamma - \ln\frac{jk_0\rho}{2}, \tag{32}$$

being $\gamma = 0.5772\ldots$, the Euler's constant [20, p. 255]. Substituting the results of the differentiations

$$\left(\nabla_t^2 - \frac{1}{\rho^2} + k_1^2\right) S_{2,1} = -\frac{e^{-jk_1\rho} + jk_1\rho}{\rho^2}, \tag{33}$$

$$\begin{aligned}\left(\nabla_t^2 + k_1^2\right) S_{1,1} = \frac{k_1^2}{2}\Bigg[&K_0\left(\frac{jk_1\rho}{2}\right) I_0\left(\frac{jk_1\rho}{2}\right) \\ &+K_1\left(\frac{jk_1\rho}{2}\right) I_1\left(\frac{jk_1\rho}{2}\right)\Bigg],\end{aligned} \tag{34}$$

$$\left(\nabla_t^2 + k_1^2\right) S_{2,0} = -k_1^2\left(\gamma + \ln\frac{jk_0\rho}{2}\right) \tag{35}$$

into (22) and (23) provides

$$\left.\frac{\partial A}{\partial z}\right|_{z=0} = \frac{\mu_0 k_0^2 p}{2\pi}\left[\frac{1}{2}\left(K_0 I_0 + K_1 I_1\right) + \tau^2\left(\gamma + \ln\frac{jk_0\rho}{2}\right)\right], \quad (36)$$

$$\left.\frac{\partial A}{\partial \rho}\right|_{z=0} = -\frac{\mu_0 p}{2\pi\rho^2}\left[1 - \tau^2\left(e^{-jk_1\rho} + jk_1\rho\right)\right], \quad (37)$$

where the argument of the modified Bessel functions has been omitted for notational simplicity. With the use of (36) and (37), we are now able to perform the differentiations in (1)–(3) and obtain second-order quasi-static expressions for the EM field components at the air-medium interface. After some algebra, it is found that

$$E_{\rho 0} = \frac{j\omega\mu_0 p}{2\pi\rho}\left(K_1 I_1 - \tau^2\right), \quad (38)$$

$$E_{z0} = -\frac{p}{2\pi j\omega\epsilon_0\rho^3}\left[1 - \tau^2\left(1 + jk_1\rho\right)e^{-jk_1\rho}\right], \quad (39)$$

$$H_{\varphi 0} = \frac{p}{2\pi\rho^2}\left[1 - \tau^2\left(e^{-jk_1\rho} + jk_1\rho\right)\right], \quad (40)$$

where the subscript "0" denotes calculation at $z = 0$. The zeroth-order quasi-static approximations for the fields given in [8] may be obtained directly from (38), (39), and (40) by setting $\tau = 0$.

3. RESULTS AND DISCUSSION

The zeroth- and second-order quasi-static approximations are applied to the computation of the amplitudes of the EM field components generated on the top surface of a medium with $\epsilon_1 = 10\,\epsilon_0$ at distance $\rho = 90/\pi$ m from a unit-moment VED. Figs. 2–4 show the relative percent error that results from comparing the obtained results with those arising from the numerical evaluation of the field integrals, plotted versus frequency. To highlight the difference in accuracy between the two formulations when the ratio

$$\frac{k_0}{|k_1|} = \left[\epsilon_{r_1}^2 + \left(\frac{\sigma_1}{\omega\epsilon_0}\right)^2\right]^{-\frac{1}{4}} \quad (41)$$

is not negligible with respect to unity, poorly conducting media are considered. Thus, the electrical conductivity σ_1, taken as a parameter, is assumed to be equal to 0.01, 0.1, and 1 mS/m. Common examples of materials with conductivity on these orders of magnitude are permafrost, igneous and metamorphic rocks [5, 6].

The numerical evaluation of the field integrals has been performed through the rigorous quasi-analytical procedure described in [7, 22].

The procedure consists of applying the Cauchy's residue theorem to the z- and the ρ-derivatives of (5) at $z = 0$, which are Hankel transforms of the form

$$g(\rho) = c \int_0^\infty f(\lambda) J_\nu(\lambda\rho) \lambda^{\nu+1} d\lambda. \tag{42}$$

Then, proceeding as discussed in [22] turns (42) into

$$g(\rho) = -jc \sum_{m=1}^{M} r_m \lambda_m^\nu K_\nu(\lambda_m \rho), \tag{43}$$

where the λ_m's and r_m's are the coefficients of the rational approximation

$$f(\lambda) \cong \sum_{m=1}^{M} \frac{r_m}{\lambda_m} \left[\frac{1}{\lambda - j\lambda_m} - \frac{1}{\lambda + j\lambda_m} \right], \tag{44}$$

to be determined using the least squares-based fitting algorithm described in [23]. The method has been shown [7] to ensure at least 13 digits of precision, and this happens because the accuracy of the result of the computation depends only on the quality of the fitting process.

Figure 2 depicts percent errors arising from the calculation of the amplitude of E_ρ, and zeroth-order curves are marked with points to be distinguished from the second-order ones. It should be observed that for $\sigma_1 = 0.01\,\text{mS/m}$ the error produced by the zeroth-order approximation exceeds 30% almost everywhere in the $0 \leq k_0\rho \leq$

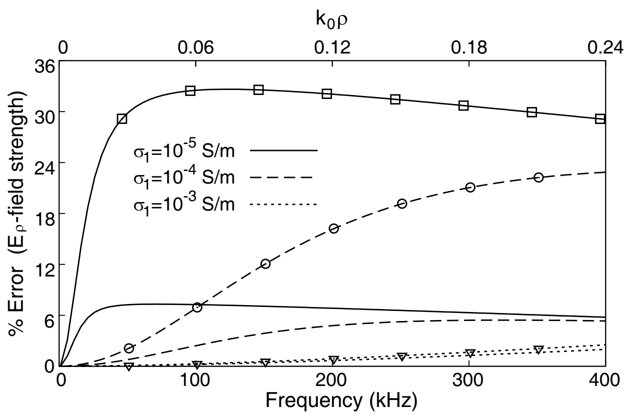

Figure 2. Relative errors of the zeroth-order (lines and points) and second-order (lines) quasi-static approximations of E_ρ as compared to the exact results. Errors are plotted versus frequency with varying σ_1.

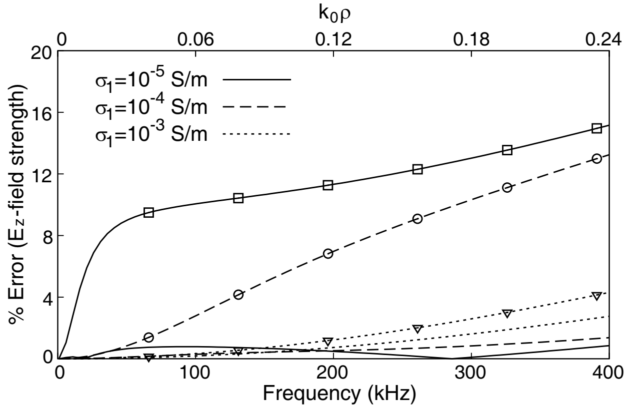

Figure 3. Relative errors of the zeroth-order (lines and points) and second-order (lines) quasi-static approximations of E_z as compared to the exact results. Errors are plotted versus frequency with varying σ_1.

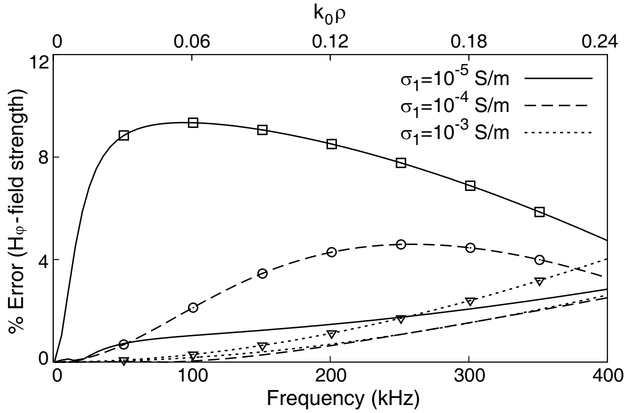

Figure 4. Relative errors of the zeroth-order (lines and points) and second-order (lines) quasi-static approximations of H_φ as compared to the exact results. Errors are plotted versus frequency with varying σ_1.

0.24 range, while the error occurring when applying the improved formula (38) is always less than 7%.

The discrepancy between the relative errors produced by the two formulations becomes smaller and smaller as conductivity increases, and the two trends are about to overlap for $\sigma_1 = 1\,\text{mS/m}$. The same can be argued from the analysis of Figs. 3 and 4, which show, respectively, the relative percent error occurring when computing E_z and H_φ under the quasi-static field assumption. Improved formulas for these field components allow to keep the relative error in the $0 \leq k_0\rho \leq 0.24$ range below the threshold of 3%, while application of the zeroth-order ones in the same range would lead to higher errors, up to 15% for the E_z-field and 9% for the H_φ-field.

It is also of interest to study the effect of varying the electrical permittivity of the half-space on the relative percent errors that result from applying the quasi-static approximations. This aspect is

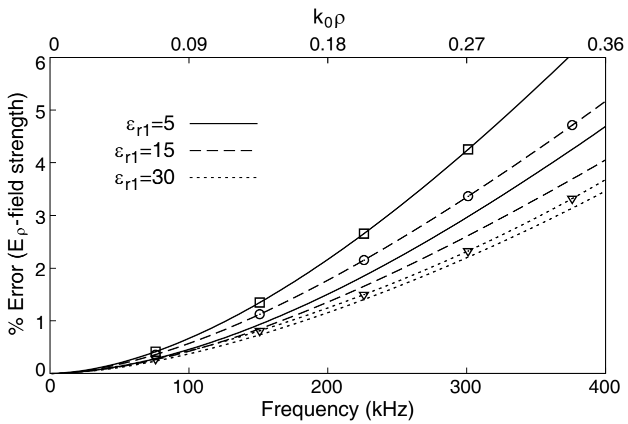

Figure 5. Relative errors of the zeroth-order (lines and points) and second-order (lines) quasi-static approximations of E_ρ as compared to the exact results. Errors are plotted versus frequency with varying ϵ_{r_1}.

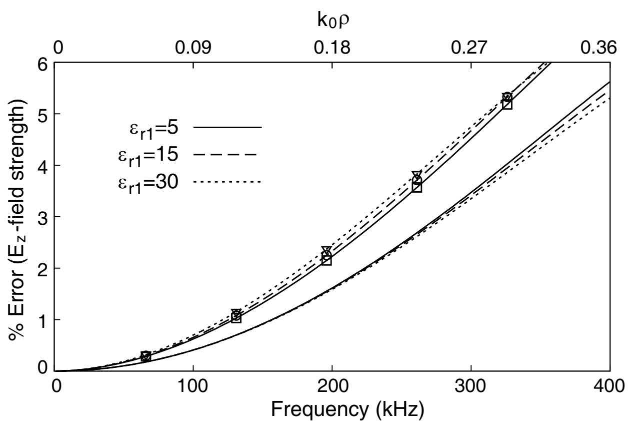

Figure 6. Relative errors of the zeroth-order (lines and points) and second-order (lines) quasi-static approximations of E_z as compared to the exact results. Errors are plotted versus frequency with varying ϵ_{r_1}.

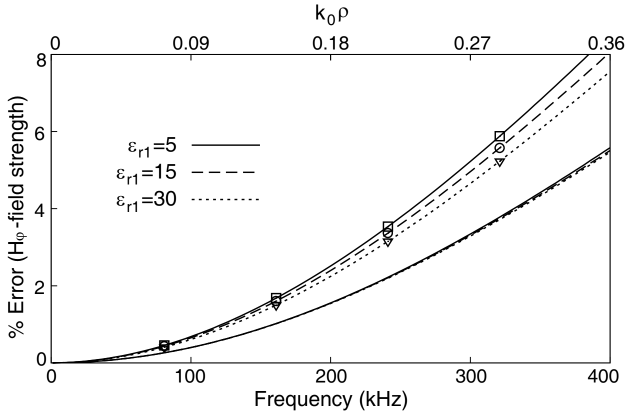

Figure 7. Relative errors of the zeroth-order (lines and points) and second-order (lines) quasi-static approximations of H_φ as compared to the exact results. Errors are plotted versus frequency with varying ϵ_{r_1}.

pointed out in Figs. 5–7, which illustrate the behavior of the relative errors against frequency, assuming $\rho = 135/\pi\,\mathrm{m}$ and $\sigma_1 = 1\,\mathrm{mS/m}$, and taking the relative permittivity ϵ_{r_1} as a parameter. Significant conclusions can be drawn from the analysis of the plotted curves. First, it is confirmed that using the second-order formulation in place of the zeroth-order one permits to reduce the relative percent error. Second, what emerges is that in the quasi-static frequency range the relative error is only weakly affected by a permittivity change, and this is to be attributed to the fact that the conduction currents are large enough to predominate over the displacement currents. In fact, as long as σ_1 is sufficiently larger than $\omega\epsilon_1$, k_1 suffers only from a mild alteration while varying ϵ_1, and so do both the quasi-static approximations and the errors produced by them, which depend on k_1.

4. CONCLUSION

Second-order quasi-static approximations for the radial distributions of the EM field components excited by a VED on the surface of a material half-space have been derived in this paper. The expressions are in terms of modified Bessel functions (E_ρ-field) and exponential functions (E_z- and H_φ-field), and have been obtained through a rigorous analytical procedure after replacing the non-oscillating part of the integrand in the integral representation for the z-directed magnetic vector potential with its quadratic approximation for small values of $k_0/|k_1|$. Numerical results are presented to show the advantages of the proposed formulation over the zeroth-order one in terms of accuracy. In the quasi-static frequency range and beyond, use of the derived improved expressions makes it possible to reduce the maximum relative error in the calculation of the fields from about 30% down to less than 7%. The effect of varying the electrical conductivity and permittivity of the half-space on the accuracy of both the approximations is also investigated. It is deduced that the variation of permittivity does not significantly alter the accuracy of the quasi-static formulations, and this is due to the fact that in the quasi-static frequency range the conduction currents predominate over the displacement currents.

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