NEW METHOD FOR ULTRA WIDE BAND AND HIGH GAIN RECTANGULAR DIELECTRIC ROD ANTENNA DESIGN

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Abstract—A novel method for tapered rectangular dielectric rod antenna design is presented. This method can be used to design from millimetre to terahertz antennas. The work modes of antenna have been analysed. The input modes are fundamental mode and second mode, and the end-fire mode is only fundamental mode. The calculation formulas for bottom diameter and top diameter are given according to the work modes. In order to avoid standing wave in the antenna, the wave will not reflect on the boundary surface of antenna. The calculation formula of antenna length is given based on the radial theory. Different shaped 300 GHz antennas have been designed based on the method. The results indicate that this method is suitable for different shapes of rectangular rod antennas. We also give the other two 1 THz and 32 GHz antenna design to demonstrate this method. The antenna gain will increase with the length expanding based on our design. The bottom and top diameters can be tuned slightly because the work mode formulas of rectangular dielectric waveguide are derived approximately. As an example different tuning designs of 100 GHz indicate that the tuning region is based on the calculation results. The design results have ultra wide bandwidth which is almost 50% of centre frequency and high gain.

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1. INTRODUCTION

Dielectric rod antennas have been researched for many years [1–6]. Dielectric resonator antennas have also been studied for many years. Resonator antenna has lower gain comparing with tapered rod antenna because most of the wave inside the antenna is standing wave. Normally dielectric rod antennas are tapered at the end of the antenna in order to get high radiant direction.

There are two shapes of dielectric rod antenna. One is the circular cross section antenna, and the other is the rectangle cross section antenna. We will design rectangle cross section antenna in this paper. In [1], the calculation formulas for bottom and top diameter had been given. Generally speaking, the gain will increase with the antenna length expanding and the same standpoint in [7]. In [1], 14.5 dB for the $L = 5\lambda_0$ case to approximately 16.2 dB for $L = 8\lambda_0$. But the $L = 15\lambda_0$ antenna did not reveal any increase in gain as compared to that of $L = 8\lambda_0$. Actually the gain will increase with the length increasing only for certain condition which will be analyzed in Section 3.

In [8] a two-dimensional dielectric slab-wedge configuration was created to analyze the antenna, but the assuming modes are TE and TM. In fact, the modes in rectangular dielectric waveguide are quasi-TEM modes. In [9], the input and end-fire modes are all multi-modes, and the gain is lower (about 2 dB) even though the expected gain is about 20 dB. Other structure rectangular antennas have been designed in [10], and some cylindrical dielectric antenna have been studied and designed [11–15]. Theory analysis for dielectric antenna is studied in [16]. In [17–20] only the design results for a certain frequency are presented without presenting a detailed design method.

In this paper a new method for rectangular dielectric rod antenna is presented. This method can be used to design rectangular dielectric rod antennas with any dielectric materials at any frequency. The proposed design method is based on work modes of rectangular dielectric waveguide, which is described in Section 2. The design method is presented in Section 3. Design examples and their simulated performance are discussed in Section 4, which will be followed by brief concluding remarks in Section 5.

2. THE WORK MODES OF RECTANGULAR DIELECTRIC ANTENNA

The dielectric rod antennas can be considered as travelling wave structures. As the wave travels along the antenna, the reflected wave should be minimized in order to increase the radiation at the end of the

antenna. In this paper we will use the theory of rectangular dielectric waveguide to analyze the work modes of antenna.

It is not possible to derive a precise analytical solution for a rectangular dielectric waveguide. However, for rectangular dielectric waveguide there are two well-known approximate solutions. One is the method proposed by Marcatili using approximate boundary conditions for an analytical modal analysis [21]. The other, proposed by Goell [22], is based on the circular harmonics series. In Marcatili' solution most of the energy is located at the center of the dielectric waveguide. The modes E_z and H_z are both small and the electromagnetic field behaves like a quasi-TEM mode over the central region. Such modal waves can be classified into two groups. One is the E_{mn}^x mode consisted of E_x and H_y, the other is E_{mn}^y mode with dominant E_y and H_x [21].

For E_{mn}^y mode, the approximate solutions for wave numbers are [22],

$$k_y = \frac{m\pi}{a}\left(1 + \frac{2\varepsilon_2 A}{\pi\varepsilon_1 b}\right)^{-1} \tag{1}$$

$$k_x = \frac{n\pi}{a}\left(1 + \frac{2A}{\pi a}\right)^{-1} \tag{2}$$

For E_{mn}^x mode, the approximate solutions for wave numbers are [22],

$$k_y = \frac{m\pi}{ba}\left(1 + \frac{2A}{\pi b}\right)^{-1} \tag{3}$$

$$k_x = \frac{n\pi}{a}\left(1 + \frac{2\varepsilon_2 A}{\pi\varepsilon_1 a}\right)^{-1} \tag{4}$$

$$m, n = 0, 1, 2, 3 \ldots$$

where a is the long diameter of the rectangular dielectric waveguide, b is the short diameter of the rectangular dielectric waveguide. If the dielectric waveguide is surrounded by the air, the A is by,

$$A = \frac{\lambda}{2\sqrt{\varepsilon_r - 1}} \tag{5}$$

The wave numbers k_x, k_y and k_z should satisfy the separation equation for the guided wave

$$k_z = \sqrt{\varepsilon_r k_0^2 - k_x^2 - k_y^2} > 0 \tag{6}$$

where k_0 is the free-space wave number.

$$k_0 = \omega\sqrt{\mu_0\varepsilon_0} \tag{7}$$

2.1. The Cutoff Frequencies of the Mode E_{mn}^y

For E_{mn}^y mode, combining formulas (1), (2), (6) and (7), we obtain,

$$\omega^2 \mu_0 \varepsilon_0 \varepsilon_r > \left(\frac{m\pi}{b}\right)^2 \left(1 + \frac{2A}{\pi \varepsilon_r b}\right)^{-2} + \left(\frac{n\pi}{a}\right)^2 \left(1 + \frac{2A}{\pi a}\right)^{-2} \quad (8)$$

From formula (8), E_{01}^y ($m = 0$, $n = 1$) has the lowest cutoff frequency. The cutoff frequency for the lowest order mode is given by,

$$f_{cEy01} = \frac{\omega}{2\pi} = \frac{c}{2\pi\sqrt{\varepsilon_r}} \frac{\pi}{a} \left(1 + \frac{2A}{\pi a}\right)^{-1} \quad (9)$$

From (5) and (9), we obtain

$$f_{cEy01} = \frac{c}{a\sqrt{\varepsilon_r}} \left(\frac{1}{2} - \frac{\sqrt{\varepsilon_r}}{\pi\sqrt{\varepsilon_r - 1}}\right) \quad (10)$$

2.2. The Cutoff Frequency of the Mode E_{mn}^x

For E_{mn}^x mode, we have,

$$\omega^2 \mu_0 \varepsilon_0 \varepsilon_r > \left(\frac{m\pi}{b}\right)^2 \left(1 + \frac{2A}{\pi b}\right)^{-2} + \left(\frac{n\pi}{a}\right)^2 \left(1 + \frac{2A}{\pi \varepsilon_r a}\right)^{-2} \quad (11)$$

The mode E_{01}^x ($m = 0$, $n = 1$) has the lowest cutoff frequency, which is given by,

$$f_{cEx01} = \frac{c}{a\sqrt{\varepsilon_r}} \left(\frac{1}{2} - \frac{1}{\pi\sqrt{\varepsilon_r}\sqrt{\varepsilon_r - 1}}\right) \quad (12)$$

So for rectangular dielectric waveguide, the first mode cutoff frequency is

$$f_1 = \frac{c}{a\sqrt{\varepsilon_r}} \left(\frac{1}{2} - \frac{\sqrt{\varepsilon_r}}{\pi\sqrt{\varepsilon_r - 1}}\right) \quad (13)$$

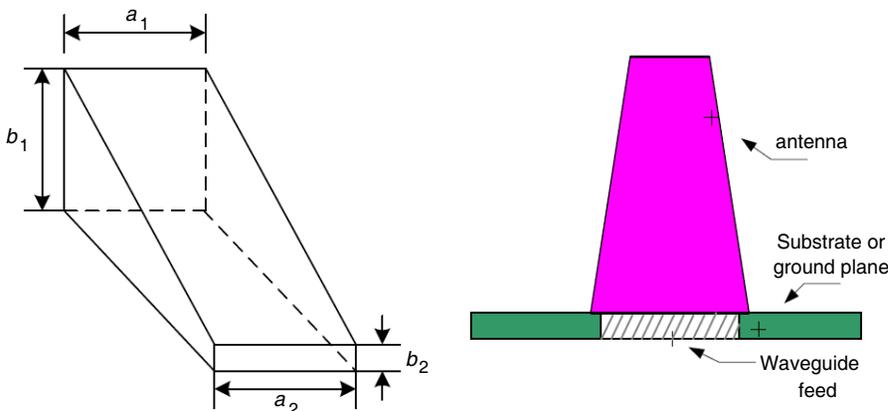The second mode cutoff frequency is

$$f_2 = \frac{c}{a\sqrt{\varepsilon_r}} \left(\frac{1}{2} - \frac{1}{\pi\sqrt{\varepsilon_r}\sqrt{\varepsilon_r - 1}}\right) \quad (14)$$

The third mode cutoff frequency is

$$f_3 = \frac{c}{a\sqrt{\varepsilon_r}} \left(1 - \frac{\sqrt{\varepsilon_r}}{\pi\sqrt{\varepsilon_r - 1}}\right) \quad (15)$$

3. THE DIMENSION DESIGN OF THE ANTENNA

The structure of the dielectric rod antenna is shown in Figure 1, $a_1 \geq b_1 \geq a_2 \geq b_2$. The tapered rectangular dielectric rod is fed by waveguide, and the feed structure is shown in Figure 2.

Figure 1. The structure of antenna.

Figure 2. Feed structure.

3.1. The Feed Structure of the Antenna

The antenna can be fed by a waveguide, and is simply placed on the top of rectangular waveguide. The waveguide dimension can be the same as the bottom plane, or matched with the frequency. In our method, the waveguide dimension which corresponds to the frequency is bigger than the bottom plane dimension, there should be substrate made of the same dielectric material under the antennas. In this situation, the antenna can be designed as antenna array, but every rod is the same as the single antenna design.

3.2. The Diameter Calculation

In order to get high gain and bandwidth, the work mode at the column bottom which is called input work mode has two modes, i.e., the fundamental mode and the second mode. The work mode at the column top which is called radiation work mode has only the fundamental mode. The modes can convert smoothly inside the antenna with the dimension of column section [6].

According to the formulas (14) and (15), the bottom diameters should satisfy the following condition,

$$\frac{c}{f\sqrt{\varepsilon_r}}\left(\frac{1}{2}-\frac{1}{\pi\sqrt{\varepsilon_r}\sqrt{\varepsilon_r-1}}\right) < a_1(b_1) < \frac{c}{f\sqrt{\varepsilon_r}}\left(1-\frac{\sqrt{\varepsilon_r}}{\pi\sqrt{\varepsilon_r-1}}\right) \quad (16)$$

According to formulas (13) and (14), the top diameters should

satisfy the following condition,

$$\frac{c}{f\sqrt{\varepsilon_r}}\left(\frac{1}{2}-\frac{\sqrt{\varepsilon_r}}{\pi\sqrt{\varepsilon_r-1}}\right) < a_2(b_2) < \frac{c}{f\sqrt{\varepsilon_r}}\left(\frac{1}{2}-\frac{1}{\pi\sqrt{\varepsilon_r}\sqrt{\varepsilon_r-1}}\right) \quad (17)$$

where f is the center frequency of the antennas.

When the bottom plane and top plane are rectangle, the small side diameter is calculated by formulas (16) and (17).

Decreasing the top diameters a_2 or b_2 can increase the bandwidth and shift it to right due to cutoff frequency reduction [see (10)]. Increasing the bottom diameters a_1 or b_1 will make the frequency band shift to left, inversely reducing the bottom diameters will make the frequency band shift to right.

Because the cutoff frequencies are derived from the approximate boundary conditions, it is possible to tune the antenna dimensions based on the formula (16) and (17). We will give some design examples in Section 4.

3.3. The Dimension of the Antenna Length

The cross-section graph of the antenna is shown in Figure 3. Assuming the incident wave angle θ_i parallels to radial direction, the tapered angle of the antenna is $\delta/2$ as shown in Figure 3. The total reflection during the interface of the dielectric will happen for $\theta_i > \theta_c$.

From Figure 3 we can get

$$\tan\frac{\delta}{2} = \frac{a_1/2 - a_2/2}{h} \quad \text{or} \quad \tan\frac{\delta}{2} = \frac{b_1/2 - b_2/2}{h} \quad (18)$$

In order to avoid the resonance in the antenna, and make these wave move to the top plane and radiate, θ_r should be smaller than
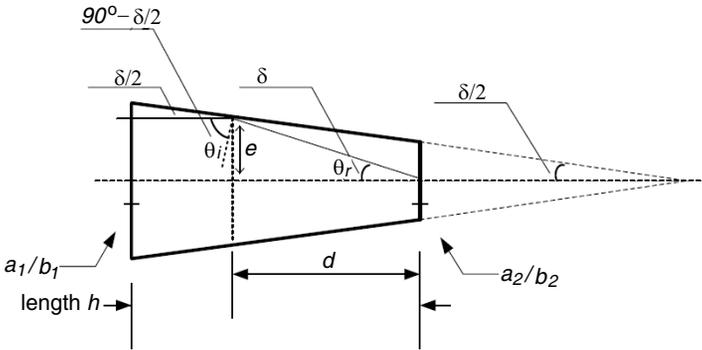

Figure 3. Cross-section of dielectric rod antenna.

critical angle θ_c, and the following inequalities should be satisfied.

$$d = e \tan(90^0 - \delta) < \frac{a_1}{2} \tan(90^0 - \delta) > h$$

i.e.,

$$\frac{a_1}{2} \tan(90^0 - \delta) > h \qquad (19)$$

As a quasi-travelling antenna, the length of the rod should normally be larger than one wavelength in dielectric.

From (18) and (19), the relationship of bottom diameter, top diameter and the length should be

$$a_1(b_1) < a_2(b_2) + h \tan\left(\frac{90^0 - \arctan(h/a_1(b_1))}{2}\right) \qquad (20)$$

4. THE EXAMPLES OF ANTENNA DESIGN AND DISCUSSION

Some examples 32 GHz, 100 GHz, 300 GHz, and 1 THz antennas design according to the critical approach introduced above have been presented and discussed.

4.1. $f = 300\,\text{GHz}$

If the frequency is 300 GHz, and the dielectric is high resistance silicon which the dielectric constant is $\varepsilon_r = 11.9$, the design results for different dimension antennas are shown in Figure 4.
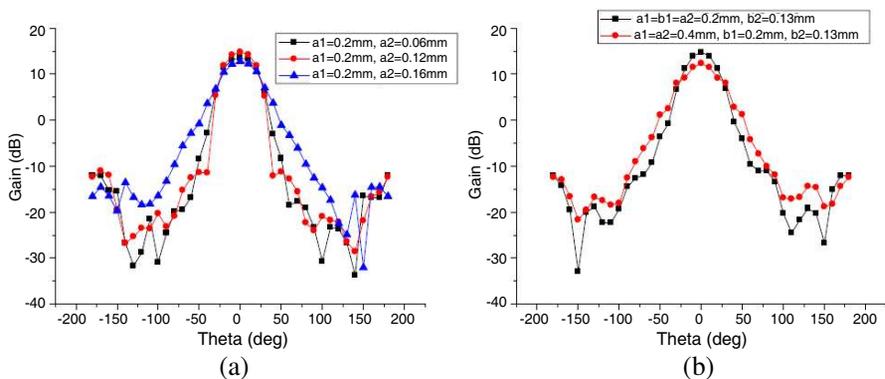
Figure 4. 300 GHz antenna far field pattern for different diameters with length $h = 5\,\text{mm}$.

In Figure 4, the length of antenna is all 5λ. In Figure 4(a), the bottom plane and top plane are all square. From formula (16), the bottom diameter is $0.136\,\text{mm} < a_1 < 0.193\,\text{mm}$.

From (17), we obtain $0.048\,\text{mm} < a_2 < 0.136\,\text{mm}$.

Because the work modes formulas are derived from the approximate method, the antenna dimension region derived from the formulas (16) and (17) can be tuned slightly. In Figure 4(a), the bottom diameter $a_1 = 0.2\,\text{mm}$ is closed to the value calculated by the formula.

We have designed more than 50 different dimension antennas and received the conclusion that in order to get high gain the bottom diameter a_1 should be as big as possible.

In Figure 4(a), the top diameter a_2 are either satisfied the calculated value or tuned slightly. The $a_2 = 0.12\,\text{mm}$ is in the region of calculation, and the corresponding gain is biggest.

In Figure 4(b), there are two antennas design. The bottom plane of one antenna is square with $a_1 = b_1 = 0.2\,\text{mm}$, the top plane is rectangle with $a_2 = 0.2\,\text{mm}$, $b_2 = 0.13\,\text{mm}$. Another antenna with the bottom plane and top plane are all rectangle with $a_1 = 0.4\,\text{mm}$, $b_1 = 0.2\,\text{mm}$, $a_2 = 0.4\,\text{mm}$, $b_2 = 0.13\,\text{mm}$. The gain of these two antennas is similar. The maximum gain is $14\,\text{dB}$ and the bandwidth is $170\,\text{GHz}$. The results for different shape antennas are similar.

Increasing the bottom diameters a_1 or b_1 will make the frequency band shift to low frequency region; inversely reducing the bottom diameters will make the frequency band shift to high frequency region.

4.2. $f = 32\,\text{GHz}$, and $f = 1\,\text{THz}$, $\varepsilon_r = 11.9$

For the millimeter wave $32\,\text{GHz}$ antenna, the design method is the same as introduction above. According to the above method, the bottom diameter and the top diameter should be

$$1.275\,\text{mm} < a_1(b_1) < 1.809\,\text{mm}, \quad 0.45\,\text{mm} < a_2(b_2) < 1.275\,\text{mm}$$

We choose $a_1 = b_1 = 2\,\text{mm}$, $a_2 = b_2 = 1.1\,\text{mm}$. From formula (20), we choose $h = 7\,\text{mm} = 0.75\lambda$. The radiation pattern and the S_{11} are shown in Figure 5(a). The antenna gain is $7.3\,\text{dB}$, the bandwidth is $7\,\text{GHz}$.

When $f = 1\,\text{THz}$, $\varepsilon_r = 11.9$, if $a_1 = b_1 = a_2 > b_2$, from formula (16) and (17), the diameters should be

$$0.0408\,\text{mm} < a_1 < 0.0579\,\text{mm}, \quad 0.0144\,\text{mm} < b_2 < 0.0408\,\text{mm}$$

We choose $a_1 = a_2 = b_1 = 0.07\,\text{mm}$, $b_2 = 0.036\,\text{mm}$. From formula (20), we choose $h = 0.9\,\text{mm} = 3\lambda$. The radiation pattern and the S_{11} are shown in Figure 5(b). The antenna gain is $12.5\,\text{dB}$.

(a) $f = 32$ GHz, $h = 7$ mm $= 0.75\lambda$

(b) $f = 1$ THz, $h = 0.9$ mm $= 3\lambda$
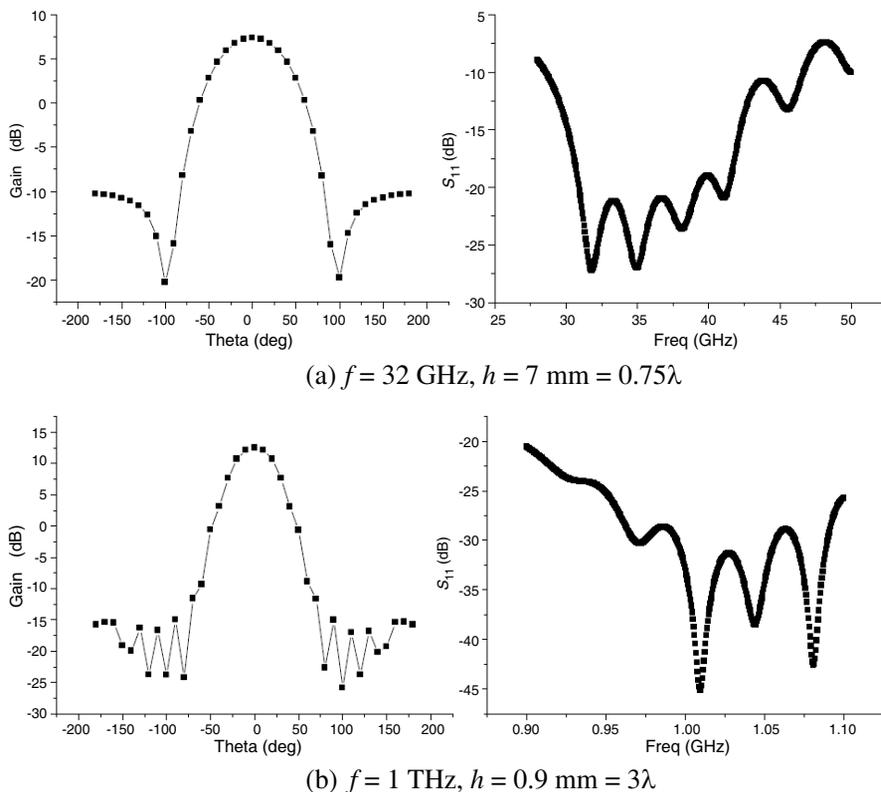
Figure 5. The gain and S_{11} for antenna. (a) $f = 32$ GHz, (b) $f = 1$ THz.

4.3. Discussion

If the diameters and length are satisfied the condition given in this paper, the gain will increase with the length expanding. For 300 GHz, if the bottom plane and top plane are all square which the dimension are $a_1 = b_1 = 0.2$ mm, $a_2 = b_2 = 0.13$ mm, the far field pattern with different length are shown in Figure 6. In Figure 6, When the length $h = 2.5$ mm, the antenna gain is 11.2 dB. The gain increases to 16.6 dB for $h = 10$ mm, 20.6 dB for $h = 40$ mm. All of the three antennas have wide bandwidth.

The diameters can tune beyond the calculated value slightly, but if the diameter is too big or too small, the antenna gain and the bandwidth will be small or worse. As an example, the different design results for $f = 100$ GHz are shown in Figure 7.

For 100 GHz antenna, according to (16) and (17), the diameters
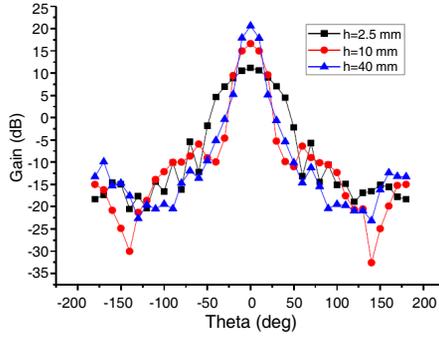
Figure 6. The 300 GHz antenna far field pattern for different length.
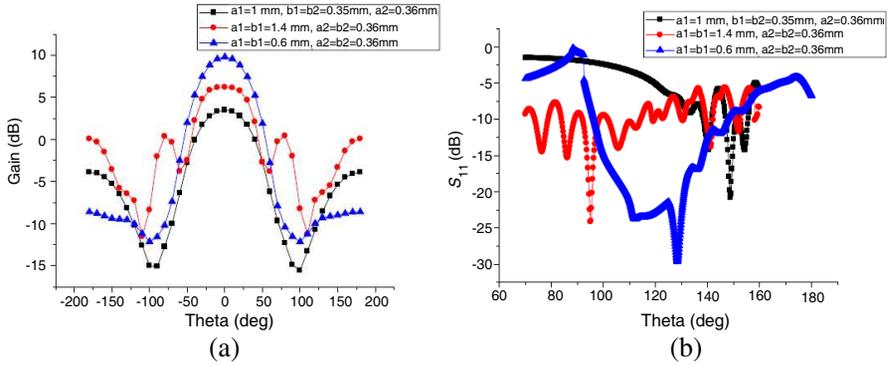
Figure 7. The gain and S_{11} for antenna $f = 100\,\text{GHz}$.

should be

$$0.408\,\text{mm} < a_1(b_1) < 0.579\,\text{mm}, \quad 0.144\,\text{mm} < a_2(b_2) < 0.408\,\text{mm}$$

When we choose the bottom diameters $a_1 = b_1 = 0.6\,\text{mm}$, top diameters $a_2 = b_2 = 0.36\,\text{mm}$, and the length $h = 4\,\text{mm} = 1.3\lambda$, the gain is 9.8 dB, and the band width is 90 GHz.

With the same top diameters and the same length, if the bottom diameters are $a_1 = b_1 = 1.4\,\text{mm}$, which is too big comparing to the calculation value 0.579 mm, the antenna gain in only 6.2 dB, the band width is very narrow. The same situation happens for too small diameters antenna. If the short side bottom diameter is $b_1 = 0.35\,\text{mm}$ even with big long side diameter $a_1 = 1\,\text{mm}$, the antenna gain is only 3.5 dB with worse band width shown in Figure 7(b).

5. CONCLUSION

A new method for rectangular dielectric rod antennas design was presented. With this new method we can design very wide bandwidth and high gain antennas. For our design method, the input work modes are fundamental mode and second mode, the end-fire mode is only fundamental mode. Most of the single reflected ray fields will be transmitted through the upper plane of the antenna and radiate. According to the calculation formulas for bottom diameter, top diameter and length, we have designed different shape $300\,\mathrm{GHz}$ antennas. For the same length $h = 5\,\mathrm{mm} = 5\lambda$, the gain is $14\,\mathrm{dB}$, and the bandwidth is $170\,\mathrm{GHz}$. The gain increases with the length expanding. When the length is $h = 2.5\,\mathrm{mm}$, the antenna gain is $11.2\,\mathrm{dB}$. The gain increases to $16.6\,\mathrm{dB}$ for $h = 10\,\mathrm{mm}$, $20.6\,\mathrm{dB}$ for $h = 40\,\mathrm{mm}$.

This method is also suitable for the terahertz and other millimeter antenna design. The antenna gain is $7.3\,\mathrm{dB}$, the bandwidth is $7\,\mathrm{GHz}$ for $32\,\mathrm{GHz}$ with $h = 0.75\lambda$. The antenna gain is $12.5\,\mathrm{dB}$ with wide band width for $1\,\mathrm{THz}$ with $h = 3\lambda$.

The diameters can tune beyond the calculated value slightly, but if the diameter is too big or too small, the antenna gain and the bandwidth will be small or worse. As an example, the different dimension results for $f = 100\,\mathrm{GHz}$ are designed. Increasing the bottom diameters a_1 or b_1 will make the frequency band shift to low frequency; inversely reducing the bottom diameters will make the frequency band shift to high frequency.

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