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MAGNETIC ENERGY OF SURFACE CURRENTS ON A TORUS

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Abstract—The magnetic energy and inductance of current distributions on the surface of a torus are considered. Specifically, we investigate the influence of the aspect ratio of the torus, and of the pitch angle for helical current densities, on the energy. We show that, for a fixed surface area of the torus, the energy experiences a minimum for a certain pitch angle. New analytical relationships are presented as well as a review of results scattered in the literature. Results for the ideally conducting torus, as well as for thin rings are given.

1. INTRODUCTION

Here we investigate the inductance and magnetic energy of surface currents on a torus, i.e., a toroid of circular cross section, also called an anchor ring or doughnut. Since the torus is the most symmetric not simply connected body, toroidal currents and their magnetic energy are of great theoretical interest. A toroidal magnetic field has been used in an experimental verification of the Aharonov-Bohm effect [1, 2]. However, the problem has also technological import in, e.g., plasma fusion research and astrophysics. In the past, the work on toroids has focused on the vector potential and magnetic fields of such currents [3–15], on their inductance and energy [16–22], as well as on force free configurations of such currents [23–27]. In particular the problem of surface currents, either due to skin effect, or due to superconductivity or perfect conductivity of the tori, has been investigated [28–34].

This paper is organised as follows. We first introduce the general notation needed to describe a torus. The surface current density can be seen as a superposition of toroidal and poloidal currents resulting

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in a helical current. The total energy is shown to be the sum of a toroidal and a poloidal contribution. A specific class of helical curves on a torus is proposed and the resulting current density is discussed. The energies of such current densities are then given as integrals over the torus surface, as functions of the aspect ratio. The case of a purely poloidal current, the toroidal solenoid, is unique and easily solved analytically. There is a discussion of different purely toroidal current densities and some relevant results for them. We discuss how the energy varies as toroidal and poloidal currents are superposed. We focus on a few types of such toroidal surface current densities and their energies. We then review results that have been obtained in the literature using expansion in terms of toroidal functions. In particular we discuss the energy minimising toroidal current distribution on an ideally conducting torus, and in connection with this we present the necessary background relating to toroidal coordinates. Finally the limiting case of a thin ring is reviewed. An appendix presents general formulas for magnetic energy and another appendix gives some torus formulas. At the very end an appendix presents a method for removing the Coulomb singularity in the integrations.

We aim at some completeness when it comes to presenting mathematical expressions relevant to surface currents on a torus and for a thin ring. While we treat the integral form of torus magnetic energy in some detail, the results relating to expansions in terms of toroidal functions and thin rings are presented only very briefly; for actual derivations we refer to the quoted literature.

2. THE TORUS — GEOMETRY AND NOTATION

A *toroid* is a solid of revolution obtained by rotating a closed plane curve about an axis in the plane of the curve. A *torus* (or anchor ring or “doughnut”) results when the curve is a circle. We denote the radius of the rotated circle, the *minor radius*, by b. The distance between the center of the circle and the rotation axis, which we take to be the z-axis, is the *major radius* c, of the torus. We put the origin at the point on the z-axis closest to the circle. The equation for the surface of the torus is then given by,

$$\left(c - \sqrt{x^2 + y^2}\right)^2 + z^2 = b^2. \tag{1}$$

The points of a torus can also be given on parametric form as,

$$\begin{aligned} x &= (c + \beta \cos\chi)\cos\varphi, \\ y &= (c + \beta \cos\chi)\sin\varphi, \\ z &= \beta \sin\chi, \end{aligned} \tag{2}$$

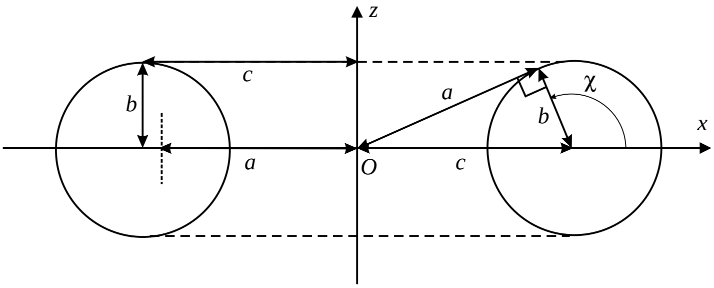

Figure 1. Section of torus in the xz-plane. The major radius is denoted c, the minor radius b. The length parameter a of the toroidal coordinates then obeys, $a^2 + b^2 = c^2$, as illustrated by the right angled triangle.

where $0 \leq \beta \leq b$, $-\pi < \chi \leq \pi$, and $0 \leq \varphi < 2\pi$, so that $\beta = b$ for the points on the surface. These quantities are illustrated in Fig. 1. The same figure shows that the line from the origin that is tangent to the torus touches it at at a point for which the angle χ obeys, $\cos(\pi - \chi) = b/c$. The distance a to this point obeys $a^2 + b^2 = c^2$. We call a the *length parameter* of the torus.

Below we will often use *cylindrical coordinates* (ρ, φ, z) given by,

$$x = \rho \cos\varphi, \quad y = \rho \sin\varphi, \quad z = z, \tag{3}$$

$$\rho = \sqrt{x^2 + y^2}, \quad \varphi = \arctan(y/x), \quad z = z, \tag{4}$$

in terms of the Cartesian (x, y, z). Here the angle φ is the same angle of rotation about the z-axis as in Eq. (2). The distance element is $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2$, and the metric coefficients are thus, $g_{\rho\rho} = 1$, $g_{\varphi\varphi} = \rho^2$, $g_{zz} = 1$. Unit vectors in the direction of increasing ρ, φ, z, β, and χ are,

$$\mathbf{u}_\rho = \cos\varphi\, \mathbf{u}_x + \sin\varphi \mathbf{u}_y, \quad \mathbf{u}_\varphi = -\sin\varphi \mathbf{u}_x + \cos\varphi \mathbf{u}_y, \quad \mathbf{u}_z = \mathbf{u}_z, \tag{5}$$

$$\mathbf{u}_\beta = \cos\chi \mathbf{u}_\rho + \sin\chi\, \mathbf{u}_z, \quad \mathbf{u}_\chi = -\sin\chi \mathbf{u}_\rho + \cos\chi \mathbf{u}_z, \tag{6}$$

respectively, in terms of the Cartesian basis vectors. The Eqs. (1) and (2) then give,

$$(c - \rho)^2 + z^2 = b^2, \quad \text{and} \quad \rho = c + b\cos\chi, \quad z = b\sin\chi, \tag{7}$$

respectively, for a torus. The parameter form (2), with $\beta = b$, gives,

$$\mathbf{r}(\varphi, \chi) = c\mathbf{u}_\rho(\varphi) + b\, \mathbf{u}_\beta(\varphi, \chi) \tag{8}$$

using (5) and (6), for the surface of the torus.

3. MAGNETIC ENERGY FOR SURFACE CURRENT DENSITY ON A TORUS

A surface current density on a torus is spanned by unit tangent vectors $\mathbf{u}_\varphi(\varphi)$ in the toroidal direction, and $\mathbf{u}_\chi\,(\varphi,\chi)$ in the poloidal direction. The general surface current density, with toroidal symmetry, on the torus can thus be written

$$\mathbf{J}(\chi,\varphi) = \mathbf{J}_\varphi + \mathbf{J}_\chi = J_\varphi(\chi)\mathbf{u}_\varphi(\varphi) + J_\chi(\chi)\mathbf{u}_\chi(\varphi,\chi). \tag{9}$$

The magnetic energy of Eq. (A7) is then

$$W = W_\varphi + W_{\varphi\chi} + W_\chi \tag{10}$$

$$= \frac{\mu_0}{8\pi}\int_{\partial V}\int_{\partial V} \frac{\mathbf{J}_\varphi(\mathbf{r})\cdot\mathbf{J}_\varphi(\mathbf{r}') + 2\mathbf{J}_\varphi(\mathbf{r})\cdot\mathbf{J}_\chi(\mathbf{r}') + \mathbf{J}_\chi(\mathbf{r})\cdot\mathbf{J}_\chi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}S'\mathrm{d}S. \tag{11}$$

Here the cross term $W_{\varphi\chi}$ is necessarily zero since it changes sign when one of the current densities is reversed, and this would mean that helical currents of different handedness had different energy. We can therefore write,

$$W = W_\varphi + W_\chi = \frac{\mu_0}{8\pi}\int_{\partial V}\int_{\partial V} \frac{\mathbf{J}_\varphi(\mathbf{r})\cdot\mathbf{J}_\varphi(\mathbf{r}') + \mathbf{J}_\chi(\mathbf{r})\cdot\mathbf{J}_\chi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}S'\,\mathrm{d}S, \tag{12}$$

and discuss the two terms separately.

Using that,

$$\mathbf{u}_\rho(\varphi)\cdot\mathbf{u}_\rho(\varphi') = \mathbf{u}_\varphi(\varphi)\cdot\mathbf{u}_\varphi(\varphi') = \cos(\varphi-\varphi') \tag{13}$$

$$\mathbf{u}_\beta(\varphi,\chi)\cdot\mathbf{u}_\beta(\varphi',\chi') = \cos\chi\cos\chi'\,\cos(\varphi-\varphi') + \sin\chi\sin\chi' \tag{14}$$

$$\mathbf{u}_\chi(\varphi,\chi)\cdot\mathbf{u}_\chi(\varphi',\chi') = \sin\chi\sin\chi'\cos(\varphi-\varphi') + \cos\chi\cos\chi' \tag{15}$$

$$\mathbf{u}_\rho(\varphi)\cdot\mathbf{u}_\beta(\varphi',\chi') = \cos\chi\cos(\varphi-\varphi') \tag{16}$$

$$\mathbf{u}_\varphi(\varphi)\cdot\mathbf{u}_\chi(\varphi',\chi') = \sin\chi'\sin(\varphi-\varphi'), \tag{17}$$

and (8), we find the expression

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'|^2 &= |\mathbf{r}(\varphi,\chi)-\mathbf{r}(\varphi',\chi')|^2\\ &= 2\left\{[c^2 + cb(\cos\chi+\cos\chi')][1-\cos(\varphi-\varphi')]\right.\\ &\quad \left.+\, b^2[1-\cos\chi\cos\chi'\cos(\varphi-\varphi') - \sin\chi\sin\chi']\right\}\end{aligned} \tag{18}$$

for the distance between two points on the torus. Using (9) and the scalar products above one also finds

$$\mathbf{J}_\varphi(\mathbf{r})\cdot\mathbf{J}_\varphi(\mathbf{r}') = J_\varphi(\chi)J_\varphi(\chi')\cos(\varphi-\varphi') \tag{19}$$

$$\mathbf{J}_\chi(\mathbf{r})\cdot\mathbf{J}_\chi(\mathbf{r}') = J_\chi(\chi)J_\chi(\chi')[\sin\chi\sin\chi'\,\cos(\varphi-\varphi') + \cos\chi\cos\chi'] \tag{20}$$

Using these results one obtains definite integrals for W_φ and W_χ of (12) provided the functions $J_\varphi(\chi)$ and $J_\chi(\chi)$ are known. We next address this question.

4. CURRENT DENSITY FOR A CLASS OF HELICAL TRAJECTORIES ON THE TORUS

The requirement that the poloidal current $\mathbf{J}_\chi$ is divergence free leads to the constraint,

$$\mathbf{J}_\chi(\chi) = \frac{j_\chi c}{\rho(\chi)} = \frac{j_\chi}{(1+\delta\cos\chi)} \tag{21}$$

where j_χ is constant and where we have put,

$$\rho(\chi) = c + b\cos\chi = c(1+\delta\cos\chi) \quad \text{where,} \quad \delta \equiv \frac{b}{c}. \tag{22}$$

The poloidal energy W_χ can then be found analytically as shown below.

For the toroidal current density $J_\varphi(\chi)$ there is no constraint of this kind. We will here mainly consider a constant (χ-independent) J_φ and the energy minimising $J_\varphi(\chi)$ of Eqs. (54) and (56) below of the ideally conducting torus.

One possible set of helices on a torus can be defined, on parameter form, by,

$$\mathbf{r}(t) = \rho(\chi(t))\mathbf{u}_\rho(\varphi(t)) + b\,\sin\chi(t)\,\mathbf{u}_z, \tag{23}$$

where, $\rho(\chi)$ is given in Eq. (22). The unit vectors are defined in Eq. (5). The velocity along the helix, if t is time, is

$$\dot{\mathbf{r}}(t) = \mathbf{v}(t) = \rho(\chi(t))\dot{\varphi}(t)\,\mathbf{u}_\varphi(\varphi(t)) + b\,\dot{\chi}(t)\mathbf{u}_\chi(\chi(t),\varphi(t)), \tag{24}$$

where $\mathbf{u}_\chi$ is given in Eq. (6). To get explicit helices the angular functions $\varphi(t)$, $\chi(t)$, must be specified. The simplest choice corresponds to constant angular velocities $\dot{\varphi}$, $\dot{\chi}$, which gives,

$$\varphi(t) = \dot{\varphi}t = \frac{2\pi m}{T}t, \quad \text{and} \quad \chi(t) = \dot{\chi}t = \frac{2\pi l}{T}t, \tag{25}$$

where T is the period. To get closed differentiable curves m and l should be positive integers. A plot of such a helical curve on a torus is shown in Fig. 2.

A current density on the torus is obtained by assuming that a surface charge density σ is moving on the torus surface with velocity field (24), $\mathbf{v} = \rho\,\dot{\varphi}\mathbf{u}_\varphi + b\dot{\chi}\mathbf{u}_\chi$,

$$\mathbf{J} = \sigma(\rho\dot{\varphi}\mathbf{u}_\varphi + b\dot{\chi}\mathbf{u}_\chi). \tag{26}$$

Assuming $\dot{\chi}$ constant and using (22) we find that we must have $\sigma = \sigma_0 c/\rho$, where σ_0 is constant. Eq. (26) then gives the current density

$$\mathbf{J} = \sigma_0[c\,\dot{\varphi}\mathbf{u}_\varphi + (c/\rho)b\dot{\chi}\mathbf{u}_\chi]. \tag{27}$$

Also assuming constant $\dot{\varphi}$ this corresponds to a surface current density with $\mathbf{J}_\varphi = \sigma_0 c\dot{\varphi} = \text{constant}$ and $\mathbf{J}_\chi = j_\chi c/\rho$ where $j_\chi = \sigma_0 b\dot{\chi} =$

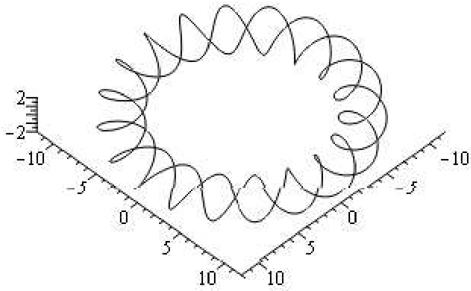

Figure 2. Plot of a torus helix on a torus with major and minor radius $c = 10$, $b = 2$, respectively. The helix has $l = 19$, $m = 2$, and pitch angle $\alpha_0 = 62.2$ degrees as defined in Eq. (47).

constant. This motivates the choice $\mathbf{J}_\varphi$ = constant mentioned above. Note however, that while this choice is mathematically convenient it has no deeper physical motivation.

The density corresponding to geodesic motion of the charge carriers on the torus, or the density from a winding in which the speed of the charge carriers in the wires are constant, would be physically motivated. Unfortunately, these problems do not lead to tractable formulas. Geodesics on the torus have been calculated by Irons [35] but they have complicated expressions. Helical windings have been considered by Bhadra [4] and by Sy [15].

5. ENERGY OF HELICAL CURRENT DENSITY AS A FUNCTION OF TORUS ASPECT RATIO

Let us thus now consider the explicit current density,

$$\mathbf{J}(\chi, \varphi) = \sigma_0 \left[c\,\dot{\varphi}\,\mathbf{u}_\varphi(\varphi) + \frac{c}{\rho(\chi)} b\,\dot{\chi}\,\mathbf{u}_\chi(\chi, \varphi) \right]. \tag{28}$$

Here $\rho = (c + b\cos\chi)$ and the angular velocities are assumed constant. We further assume that the average charge density $\sigma_0 = Q/S$ where Q is the total charge carried round by the velocity field (24) and where $S = 4\pi^2 bc$ is the surface area of the torus (B2). If we introduce the aspect ratio,

$$\delta \equiv b/c, \tag{29}$$

the results of Section 3 then give us

$$W = W_\varphi + W_\chi = \frac{\mu_0}{8\pi} \frac{1}{(2\pi)^4} \frac{Q^2}{c} \left[c^2 \dot{\varphi}^2 f_\varphi(\delta) + b^2 \dot{\chi}^2 f_\chi(\delta) \right], \tag{30}$$

where,

$$f_\varphi(\delta)=\int\frac{\cos(\varphi_1-\varphi_2)(1+\delta\cos\chi_1)(1+\delta\cos\chi_2)\,\mathrm{d}\chi_1\mathrm{d}\chi_2\mathrm{d}\varphi_1\mathrm{d}\varphi_2}{\Delta(\varphi_1-\varphi_2,\chi_1,\chi_2;\,\delta)} \quad (31)$$

and

$$f_\chi(\delta)=\int\frac{[\sin\chi_1\sin\chi_2\cos(\varphi_1-\varphi_2)+\cos\chi_1\cos\chi_2]\mathrm{d}\chi_1\mathrm{d}\chi_2\mathrm{d}\varphi_1\mathrm{d}\varphi_2}{\Delta(\varphi_1-\varphi_2,\chi_1,\chi_2;\delta)}. \quad (32)$$

The integral signs here imply a quadruple integral over the domains of the four coordinates. We have also used the shorthand notation ($\varphi=\varphi_1-\varphi_2$)

$$\begin{aligned}&\frac{|\mathbf{r}(\varphi_1,\chi_1)-\mathbf{r}(\varphi_2,\chi_2)|}{c}\\ &\equiv\Delta(\varphi,\chi_1,\chi_2;\,\delta)\\ &=\sqrt{2}\times\sqrt{(1-\cos\varphi)[1+\delta(\cos\chi_1+\cos\chi_2)]+\delta^2(1-\cos\chi_1\cos\chi_2\cos\varphi-\sin\chi_1\sin\chi_2)}\end{aligned} \quad (33)$$

In Appendix C we discuss how this type of integral can be handled, as regards the symmetries and the Coulomb singularity.

Defining the currents,

$$I_\varphi=\frac{Q\dot{\varphi}}{2\pi},\quad I_\chi=\frac{Q\dot{\chi}}{2\pi}, \quad (34)$$

we can write the energy (30) as

$$W=\frac{1}{2}L_\varphi I_\varphi^2+\frac{1}{2}L_\chi I_\chi^2 \quad (35)$$

with the inductances,

$$L_\varphi=\frac{\mu_0}{16\pi^3}c\,f_\varphi(\delta), \quad (36)$$

and,

$$L_\chi=\frac{\mu_0}{16\pi^3}c\,\delta^2 f_\chi(\delta). \quad (37)$$

Here $\delta=b/c<1$.

6. AN EXACT RESULT FOR L_χ

For the case of a purely poloidal surface current density

$$\mathbf{J}_\chi(\chi,\varphi)=\sigma_0\left[\frac{bc}{\rho}\dot{\chi}\,\mathbf{u}_\chi(\chi,\varphi)\right]=j_\chi\frac{c}{\rho}\mathbf{u}_\chi, \quad (38)$$

see (28), one can find an exact expression for the magnetic field from Ampère's law,

$$\oint\mathbf{H}\cdot\mathrm{d}\mathbf{r}=I, \quad (39)$$

where I is the current going through the closed path of integration. Using the cylindrical symmetry and choosing circular paths one easily finds that the magnetic field is (Hayt and Buck [36])

$$\mathbf{H} = j_\chi \frac{c}{\rho}\, \mathbf{u}_\varphi, \tag{40}$$

where now $\rho = c + \beta \cos\chi$, inside the torus $(0 \leq \beta \leq b)$ and $\mathbf{H} = 0$ outside $(b < \beta)$. We then also have $\mathbf{B} = \mu_0 \mathbf{H}$ so the energy of this current distribution is,

$$W_\chi = \int \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \mathrm{d}V, \tag{41}$$

where the integration is over the volume of the torus. Using the coordinates, φ, χ, β defined by Eq. (2), with volume element (B3), integration gives,

$$W_\chi = \frac{\mu_0}{2} j_\varphi^2 c^2 \int_{-\pi}^{\pi} \mathrm{d}\varphi \int_0^b \mathrm{d}\beta \int_{-\pi}^{\pi} \mathrm{d}\chi \, \frac{\beta}{c + \beta \cos\chi}, \tag{42}$$

for the magnetic energy. Doing the trivial φ-integration, the χ integration gives for the integral in the above formula

$$\mathrm{Int}_\chi = 4\pi^2 \int_0^b \frac{\beta \, \mathrm{d}\beta}{c^2 - \beta^2} = 4\pi^2 \left(c - \sqrt{c^2 - b^2} \right) = 4\pi^2 (c - a). \tag{43}$$

This integral can also be done by means of standard results and the final outcome is, using $I_\chi = \frac{Q\dot{\chi}}{2\pi}$,

$$W_\chi = \frac{\mu_0}{2} j_\varphi^2 c^2 \mathrm{Int}_\chi = \frac{1}{2} \left[\mu_0 \left(c - \sqrt{c^2 - b^2} \right) \right] I_\chi^2 = \frac{\mu_0}{2} (c - a) I_\chi^2 \tag{44}$$

The quantity inside the brackets is thus the inductance L_χ. This result can be found in a number of texts, e.g., Grover [37], Snow [38], and Knoepfel [39]. Snow gives the expression $b^2/(c + \sqrt{c^2 - b^2})$ which is algebraically equivalent to $c - \sqrt{c^2 - b^2} = c - a$, where a is the length parameter. None of these authors present an explicit derivation, but one can be found in Kovetz [40]. The more recent book by Paul [41] only gives the approximate result $L_\chi = (\mu_0/2)\, b^2/c$ for a thin torus.

This also means that the function f_χ defined in (32) and (37) is given by $(\delta = b/c)$

$$f_\chi(\delta) = 16\pi^3 \frac{1}{\delta^2} \left(1 - \sqrt{1 - \delta^2} \right) \tag{45}$$

The limiting value for a thin ring is then $f_\chi(0) = 8\pi^3 \approx 248.050$, while the value for $b = c$ is $16\pi^3$.

7. OPTIMAL HELIX PITCH ANGLE

The angle α that the current vector (28) makes with the azimuthal (toroidal) direction ($\mathbf{u}_\varphi$) is given by

$$\tan\alpha(\chi) = \frac{J_\chi}{J_\varphi} = \frac{b\dot{\chi}}{\rho\dot{\varphi}} = \frac{b}{\rho}\frac{l}{m}. \tag{46}$$

The angle α_0 at $\chi = \pm\pi/2$ may be defined as the pitch angle of the toroidal helix and is given by,

$$\tan\alpha_0 = \frac{b}{c}\frac{l}{m} = \delta\frac{\dot{\chi}}{\dot{\varphi}}, \tag{47}$$

see (25). Fig. 3 shows how the energy varies with δ for different number of turns l when $bc = 1$ and $m = 1$. The case $l = 0$ corresponds to a purely toroidal current (a current loop), which has the minimum energy of all current configurations. Now, if a poloidal current component is introduced ($l > 0$), an energy minimum will be reached at a certain aspect ratio $b/c < 1$. The more turns l the current makes around the torus, the smaller the aspect ratio will be (and the thinner the torus). Simultaneously, the cross-section area πb^2 of the torus diminishes and its solenoidal inductance is $\sim b^2/c$. Hence, the energy minimising pitch angle grows, approaching 90°, as a function of l (when $m = 1$), as shown in Fig. 4.

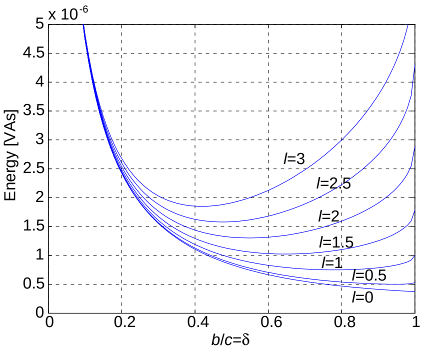

Figure 3. Plot of the energy for different combinations of currents for a varying number turns l as a function of c/b when $m = 1$ and $bc = 1$.

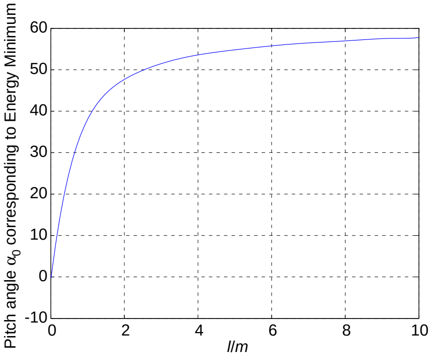

Figure 4. Pitch angle α_0 for a torus as a function of the ratio l/m. The angle approaches 90° monotonically, but very slowly.

8. SOLUTIONS IN TERMS OF SERIES OF TOROIDAL FUNCTIONS

The field problem is sometimes conveniently analysed in terms of the orthogonal system of toroidal coordinates η, ψ, φ, which are dimensionless and real valued in the intervals

$$0 \leq \eta < \infty, \quad -\pi < \psi \leq \pi, \quad \text{and} \quad 0 \leq \varphi < 2\pi. \tag{48}$$

They are related to the Cartesian coordinates through the transformation

$$x=\frac{a \sinh \eta \cos \varphi}{\cosh \eta - \cos \psi}, \quad y=\frac{a \sinh \eta \sin \varphi}{\cosh \eta - \cos \psi} \quad \text{and} \quad z=\frac{a \sin \psi}{\cosh \eta - \cos \psi}, \tag{49}$$

where $a = \sqrt{c^2 - b^2} = b \sinh \eta$ is the length parameter of the particular system and $c = a \coth \eta$. The angles ψ and φ designate any point on a torus, characterised by a constant coordinate $\eta = \eta_0$. The aspect ratio corresponding to η_0 is $b/c = 1/\cosh \eta_0$. Field theory in terms of toroidal coordinates can be used as described by Moon and Spencer [42]. Explicit solutions in terms of toroidal coordinates can be found in Carter et al. [5], Bhadra [4], and Belevitch and Boersma [29].

The φ-component of the vector Eq. (A3) is given by,

$$\nabla^2 A_\varphi - \frac{A_\varphi}{\rho^2} = -\mu_0 J_\varphi \tag{50}$$

where ∇^2 is the scalar Laplacian. Expressing this Laplacian in terms of the toroidal coordinates one can solve the above equations in terms of series of toroidal functions for certain boundary conditions. Below we present the solutions most relevant for our study.

8.1. Inductance for Constant Surface Current

In [5] one can find a formula for the inductance of a torus with a constant (χ-independent) azimuthal surface current density, i.e., the L_φ of (36). Using Eq. (52) of their work [5] one finds for the function f_φ of Eqs. (31) and (36) the expression,

$$f_\varphi(1/\xi) = -512\pi \frac{\sqrt{(\xi^2-1)^3}}{\xi} {\sum_{n=0}^{\infty}}' 2 \frac{\left[\mathrm{Q}^1_{n-\frac{1}{2}}(\xi)\right]^3 \mathrm{P}^1_{n-\frac{1}{2}}(\xi)}{4n^2-1}. \tag{51}$$

Here $\xi = 1/\delta = c/b$ and the notation is from Belevitch and Boersma [29]. The functions $\mathrm{P}^1_{n-\frac{1}{2}}(x)$ and $\mathrm{Q}^1_{n-\frac{1}{2}}(x)$ are associated Legendre functions (defined for $1 < x$) and $\sum'$ means that the term with $n = 0$ is to be multiplied by $\frac{1}{2}$. The series converges rapidly

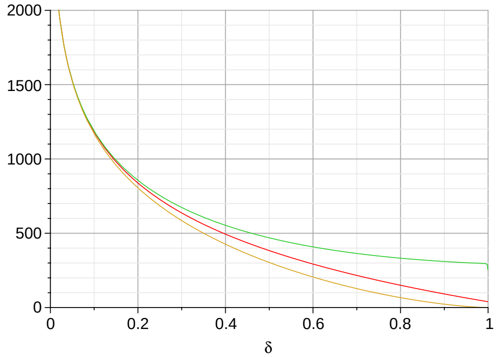

Figure 5. Plot of functions f_φ giving the inductance for surface current densities on a torus. The upper curve is for a constant current density on the torus. The lowest curve is for an energy minimising (superconducting) current density. The curve in between is f_a giving the standard approximation for the inductance of a thin ring. On the horizontal axis, $\delta = b/c$, the minor radius divided by the major. The series (51) (upper curve) and (58) (lower curve) converge rapidly except near $\delta = 1$.

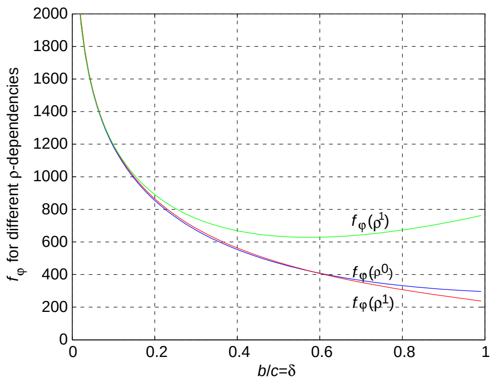

Figure 6. Plot of the functions f_φ giving the inductance for various surface current densities on a torus. Note that the curve $f_\varphi(\rho^0)$ is the same as the upper curve in Fig. 5.

except very near $\xi = 1$. We have compared this formula with the integral expression of Eq. (31) and found that they agree. In the plot Fig. 5 below we compare this function with the corresponding function f_a of Eq. (64) for a thin ring.

8.2. Current on a Ideally Conducting Torus

The problem of a current on a superconducting or perfectly conducting torus has been treated by many authors, e.g., Fock [32]. The results below are from Belevitch and Boersma [29]. Here we consider a such a torus without external magnetic field. In this case the magnetic flux Φ through the torus is conserved, and the current flows on the surface of the torus in such a way that the internal magnetic field is zero and the magnetic energy is minimised. This means that the vector potential inside and on the surface of the torus is

$$\mathbf{A} = \frac{\Phi}{2\pi\rho}\mathbf{u}_\varphi. \tag{52}$$

Outside the torus one then finds ($\mathbf{A} = A_\varphi \mathbf{u}_\varphi$),

$$A_\varphi(\eta,\psi) = \frac{4\Phi}{\pi^2 c}\left[\frac{\cosh\eta - \cos\psi}{2(1-\xi^{-2})}\right]^{\frac{1}{2}} {\sum_{n=0}^{\infty}}' \frac{\mathrm{Q}^1_{n-\frac{1}{2}}(\xi)\mathrm{P}^1_{n-\frac{1}{2}}(\cosh\eta)\cos(n\psi)}{(4n^2-1)\mathrm{P}^1_{n-\frac{1}{2}}(\xi)}. \tag{53}$$

The current density is given by,

$$j_\varphi(\psi) = \frac{-\Phi}{\sqrt{2}\pi^2\mu_0 b^2}\left[\frac{\xi - \cos\psi}{\xi^2-1}\right]^{\frac{3}{2}} {\sum_{n=0}^{\infty}}' \frac{\cos(n\psi)}{\mathrm{P}^1_{n-\frac{1}{2}}(\xi)}. \tag{54}$$

The total current by

$$I_\varphi = \frac{\Phi}{\mu_0 b}\frac{1}{\xi}\left(\frac{1}{\pi^2}\frac{\xi}{\sqrt{\xi^2-1}} {\sum_{n=0}^{\infty}}' \frac{-4\,\mathrm{Q}^1_{n-\frac{1}{2}}(\xi)}{(4n^2-1)\mathrm{P}^1_{n-\frac{1}{2}}(\xi)}\right). \tag{55}$$

The normalised current density j_φ/I_φ as a function of χ, using,

$$\psi(\chi) = \arccos\left(\frac{\xi\cos\chi + 1}{\cos\chi + \xi}\right), \tag{56}$$

is plotted for $b = 1$ and various values of $\xi = c/b$ in Fig. 7. It is clear that the current becomes increasingly concentrated on the inner radius of the torus as the major radius c approaches the minor radius b. This was pointed out by Tayler [21] in 1960, who found that for aspect ratio $\xi = 3$ the current density on the inside is 8 times greater than on the outside. Formulas (54)–(56) give $j_\varphi(\chi = \pi)/j_\varphi(\chi = 0) = 8.059984$ for $\xi = 3$, $b = 1$, in good agreement with Tayler.

Finally the inductance is given by

$$L_\varphi(\xi) = \mu_0 c\left(\frac{1}{\pi^2}\frac{\xi}{\sqrt{\xi^2-1}} {\sum_{n=0}^{\infty}}' \frac{-4\,\mathrm{Q}^1_{n-\frac{1}{2}}(\xi)}{(4n^2-1)\mathrm{P}^1_{n-\frac{1}{2}}(\xi)}\right)^{-1}. \tag{57}$$

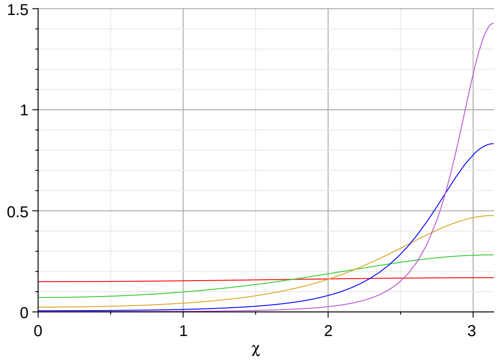

Figure 7. The normalised current density $J_\varphi(\chi) = j_\varphi(\psi(\chi))$ on the superconducting torus as a function of χ which is zero on the outside of the torus and π on the inside. The current peaks on the inside. The values of ξ shown are $c/b = \xi = 1.1, 1.3, 2.0, 5.0, 100$, the slowly varying curves corresponding to the larger ξ.

This means that the function defined in (36) is given by $(\xi = 1/\delta)$

$$f_\varphi(\delta) = \frac{16\pi^3}{\mu_0 c} L_\varphi(\xi). \tag{58}$$

A plot of this function can be found in Fig. 5.

9. INDUCTANCE OF THIN RING

The inductance for an azimuthal current in a thin ring of radius c with circular cross section of radius b is given by a number of authors. Different texts, however, give different expressions and different ranges of validity of their formulae. Here we try to summarise and harmonise the various expressions. We also indicate their accuracy by comparing them with numerical or exact results.

Several texts give the inductance in question as,

$$L_a = \mu_0 c \left(\ln \frac{8c}{b} - \frac{7}{4} \right). \tag{59}$$

The derivation can be found in Becker [43] or in Landau and Lifshitz [44]. The number 7/4 is valid for a homogeneous current distribution in the circular cross section. If the current is a constant surface current in the ring shaped cross section of the torus surface Becker's derivation can be modified and one finds instead,

$$L_a = \mu_0 c \left(\ln \frac{8c}{b} - 2 \right), \tag{60}$$

a result which some sources present as the inductance of a thin ring. Essén ([45], Appendix) derives the inductance of a thin ring as the limit of many charged particles moving in a circle and finds a discrepancy between results derived from Neumann's Formula (A7) and a corresponding one derived using the Darwin Lagrangian approach.

More accurately the handbook by Cohen [46] gives for a thin ($c > 5b$) ring,

$$L_a = \mu_0 c \left(\ln \frac{8c}{b} - 2 + \mu_{\mathrm{r}} g(\lambda) \right), \tag{61}$$

where μ_{r} is the relative permeability and $g(\lambda)$ is a function which is $0.25 = 1/4$ for large skin depth and decreases to zero with vanishing skin depth. Large skin depth and $\mu_{\mathrm{r}} = 1$ corresponds to (59) while the limit of small penetration gives (60). Frank and Tobocman [47] give,

$$L_a = \mu_0 c \left(\ln \frac{8c}{b} - 2 + \frac{\mu_{\mathrm{r}}}{4} \right), \tag{62}$$

in agreement with (59) when $\mu_{\mathrm{r}} = 1$ and with (60) when $\mu_{\mathrm{r}} = 0$ (no interior magnetic field). One of very few authors that go beyond the thin ring approximation is Snow [38], who derives the ring inductance,

$$\begin{aligned} L_a \;=\; \mu_0 c \Bigg\{ & \left[1 + \frac{2\gamma + 1}{8} \left(\frac{b}{c} \right)^2 \right] \ln \frac{8c}{b} - \frac{7}{4} \\ & + \frac{(\gamma - 1)\left(\gamma - \frac{2}{3}\right)}{16} \left(\frac{b}{c} \right)^2 + \mathcal{O}\left[\left(\frac{b}{c} \right)^3 \ln \frac{c}{b} \right] \Bigg\}, \end{aligned} \tag{63}$$

assuming an azimuthal current density inside the ring with $J_\varphi \sim \rho^\gamma$ for arbitrary γ. The notation indicates that the result is accurate to order $\delta^3 \ln(1/\delta)$. For the case $\gamma = -1$ this formula has also been derived by Haas [17].

Taking (60) as the most relevant result in the present study implies that the function corresponding to f_φ of Eqs. (31) and (36) is

$$f_a(\delta) = 16\pi^3 \left[\ln(8/\delta) - 2 \right]. \tag{64}$$

For a graph indicating its accuracy, see Fig. 5 above. The functions f corresponding to (63) of Snow are,

$$f_\gamma(\delta) = 16\pi^3 \left\{ \left[1 + \frac{2\gamma + 1}{8} \delta^2 \right] \ln \frac{8}{\delta} - \frac{7}{4} + \frac{(\gamma - 1)\left(\gamma - \frac{2}{3}\right)}{16} \delta^2 \right\}. \tag{65}$$

These are plotted in Fig. 8.

Very good analytical approximations to the functions (51) and (58) can be constructed by starting from (65). One must first

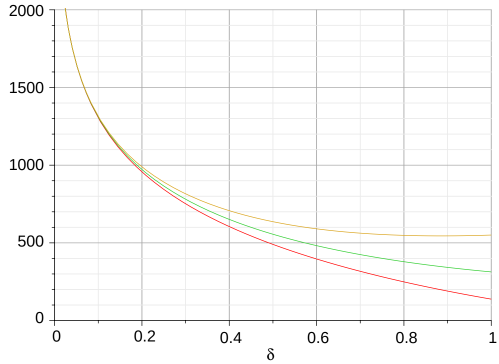

Figure 8. Plot of the functions f_γ of Eq. (65) giving the inductance for current densities $\sim \rho^\gamma$ in a thin ring. The top curve corresponds to $\gamma = 1$, the middle one to $\gamma = 0$, and the bottom curve to $\gamma = -1$. Note that the γ-ordering here is the same as in Fig. 6 for surface currents.

replace $-7/4$ by -2 and then fit the parameter γ to make the curve optimal in some way. Such an analytical expression together with the analytical results of Section 6 facilitates the study of the energy of various helical current distributions on the torus.

10. CONCLUSIONS

In this article we have attempted to organize coherently results from a large and confusing literature on the inductance for surface currents on a torus. Helical winding of a torus is found to be analytically non-trivial, even in the limit of dense winding. Some results can only be obtained by numerical integration, but even then it is important to use symmetry and qualitative features of the problem to one's advantage. The Coulomb singularity is one of these features. There are careful treatments of the problem using field theory for toroidal coordinates scattered in the literature, but these do not refer to each other and seem to have been done completely independently. We therefore think that the review of these given here should be of value. Even for the case of thin rings there is a large literature and results sometimes do not seem to agree. We try to summarise the reasons for these apparent discrepancies above.

APPENDIX A. GENERAL RESULTS FOR MAGNETIC ENERGY

Here we present some general formulas and definitions relating to magnetic energy. More detailed results on magnetic energy and current density can be found in Fiolhais et al. [48].

The electrodynamic field equations are

$$\nabla \times \mathbf{H} = \mathbf{J}, \quad \text{(A1)}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \text{(A2)}$$

where $\mathbf{B} = \mu_0 \mathbf{H}$ is the magnetic flux density (or induction), $\mathbf{H}$ the magnetic field, μ_0 the permeability of vacuum and $\mathbf{J}$, the surface current density made up by the drifting particles. In view of (A2), $\mathbf{B}$ is expressed by means of the divergenceless (Coulomb gauge: $\nabla \cdot \mathbf{A} = 0$) vector potential $\mathbf{A}$ through $\mathbf{B} = \nabla \times \mathbf{A}$. Since

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \quad \text{(A3)}$$

(A1) yields,

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \quad \text{(A4)}$$

the solution of which is the well-known expression

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\partial V} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}S', \quad \text{(A5)}$$

where $\mathrm{d}S'(\mathbf{r}')$ is the surface element and ∂V, the entire surface of the torus. The divergence of (A5) is zero. The magnetic energy W associated with a current $\mathbf{J}$ is the volume integral of the energy density of the field $\frac{1}{2}\mathbf{B} \cdot \mathbf{H}$. On account of Gauss' law we can further write

$$W = \frac{1}{2} \int_V \nabla \times \mathbf{A} \cdot \mathbf{H} \, \mathrm{d}V = \frac{1}{2} \int_V \nabla \times \mathbf{H} \cdot \mathbf{A} \, \mathrm{d}V. \quad \text{(A6)}$$

Use of (A1) and (A5) then finally gives

$$W = \frac{1}{2} \int_{\partial V} \mathbf{J} \cdot \mathbf{A} \, \mathrm{d}S = \frac{\mu_0}{8\pi} \int_{\partial V} \int_{\partial V} \frac{\mathbf{J}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}S' \, \mathrm{d}S. \quad \text{(A7)}$$

The last expression gives Neumann's formula [49] for the inductance L through

$$W = \frac{1}{2} L I^2, \quad \text{(A8)}$$

where I is the net current.

APPENDIX B. TORUS TERMINOLOGY, AREA AND VOLUME

Referring to the parameterisation in Eq. (2) and Fig. 1, only a torus for which $c > b$ is really a ring, and such a torus is called a *ring torus*. The degenerate case when $b = c$ is called a *horn torus* [50]. Here we are not interested in the case $c < b$ (spindle torus) when the surface intersects itself. The surface area element of the torus in terms of the coordinates χ, φ, defined in Eq. (2) with $\beta = b$, is

$$\mathrm{d}S = b\rho(\chi)\mathrm{d}\varphi\,\mathrm{d}\chi = b(c + b\cos\chi)\mathrm{d}\varphi\,\mathrm{d}\chi. \tag{B1}$$

For the torus surface area S one finds,

$$S = \int_{\partial V} \mathrm{d}S = 4\pi^2 bc. \tag{B2}$$

To get the volume we must integrate over all points inside the torus $(0 \leq \beta \leq b)$ and we again use the parametrization of (2). The volume element is then

$$\mathrm{d}V = \beta(c + \beta\cos\chi)\mathrm{d}\varphi\mathrm{d}\chi\mathrm{d}\beta \tag{B3}$$

Using this we obtain,

$$V = \int_V \mathrm{d}V = 2\pi^2 b^2 c, \tag{B4}$$

for the volume of the torus.

APPENDIX C. HANDLING THE COULOMB SINGULARITY IN THE INDUCTANCE INTEGRALS

Straightforward numerical integration is often the fastest way to quantitative results for integrals. When there is a Coulomb singularity, however, the straightforward method usually have trouble with convergence. Here we indicate how the Coulomb singularity can be handled.

We first note that the double integration over the torus surface that is implied in the Eqs. (31) and (32) can be simplified by noting that,

$$\int_0^{2\pi}\mathrm{d}\varphi_1\int_0^{2\pi}\mathrm{d}\varphi_2\int_0^{2\pi}\mathrm{d}\chi_1\int_0^{2\pi}\mathrm{d}\chi_2\; g(\varphi_1-\varphi_2,\chi_1,\chi_2)$$
$$= 2\pi\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\chi\int_0^{2\pi}\mathrm{d}\xi\; g(\varphi,\chi+\xi,\xi), \tag{C1}$$

where the coordinate transformations ($\varphi = \varphi_1 - \varphi_2$, $\psi = \varphi_2$, $\chi = \chi_1 - \chi_2$, $\xi = \chi_2$) of Fig. C1 have been employed. The integration over

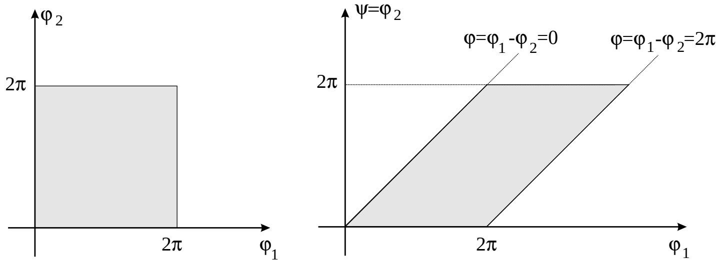

Figure C1. Since the functions involved in our induction integrals are periodic, with period 2π, in the variables φ_1, φ_2, χ_1, χ_2 one can change the integration over these to integration over $\varphi = \varphi_1 - \varphi_2$, $\psi = \varphi_2$ with $0 \leq \varphi$, $\psi < 2\pi$, as indicated in the figure above. Analogously for χ_1, χ_2 we put $\chi = \chi_1 - \chi_2$, $\xi = \chi_2$.

$\varphi_2 = \psi$ is trivial since it does not appear in the integrand. Here, and below, we put $\chi_2 = \xi$.

We now note that the distance expression, Δ of Eq. (33), in the denominator or the integrals can be written,

$$\begin{aligned}\Delta^2 &= 2\left([1-\cos\varphi]\{1+\delta[\cos(\chi+\xi)+\cos\xi]\}\right.\\ &\quad \left.+\delta^2[1-\cos(\chi+\xi)\cos\xi\cos\varphi-\sin(\chi+\xi)\sin\xi]\right)\\ &\approx (1+\delta\cos\xi)^2\varphi^2+\delta^2\chi^2+\ldots \end{aligned} \tag{C2}$$

when squared. The approximation of the last line is valid for small φ and χ.

Our integrals of type (C1) can now be written by subtracting and adding a function with identical behavior at the singularity. One finds;

$$2\pi\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\chi\int_0^{2\pi}\mathrm{d}\xi\; g(\varphi,\chi+\xi,\xi)$$

$$\equiv 2\pi\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\chi\int_0^{2\pi}\mathrm{d}\xi\;\frac{h(\varphi,\chi+\xi,\xi,\delta)}{\Delta(\varphi,\chi+\xi,\xi;\,\delta)} \tag{C3}$$

$$= 2\pi\int_{-\pi}^{\pi}\mathrm{d}\varphi\int_{-\pi}^{\pi}\mathrm{d}\chi\int_0^{2\pi}\mathrm{d}\xi\left[\frac{h(\varphi,\chi+\xi,\xi,\delta)}{\Delta(\varphi,\chi+\xi,\xi;\delta)}-\frac{h(0,\xi,\xi,\delta)}{\sqrt{(1+\delta\cos\xi)^2\varphi^2+\delta^2\chi^2}}\right] \tag{C4}$$

$$+2\pi\int_0^{2\pi}\mathrm{d}\xi\,F(\xi,\delta) \tag{C5}$$

where,

$$F(\xi,\delta)=h(0,\xi,\xi,\delta)\int_{-\pi}^{\pi}\mathrm{d}\varphi\int_{-\pi}^{\pi}\mathrm{d}\chi\frac{1}{\sqrt{(1+\delta\cos\xi)^2\varphi^2+\delta^2\chi^2}}$$

$$
= h(0,\xi,\xi,\delta) 2\pi \left[\frac{1}{(1+\delta\cos\xi)} \ln\left(\frac{\sqrt{\delta^2+(1+\delta\cos\xi)^2}+(1+\delta\cos\xi)}{\sqrt{\delta^2+(1+\delta\cos\xi)^2}-(1+\delta\cos\xi)} \right) + \frac{1}{\delta} \ln\left(\frac{\sqrt{\delta^2+(1+\delta\cos\xi)^2}+\delta}{\sqrt{\delta^2+(1+\delta\cos\xi)^2}-\delta} \right) \right]. \quad \text{(C6)}
$$

The integral over the Coulomb singularity in (C5) can thus be made analytically, as shown in Eq. (C6), while the integral (C4) is non-singular.

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