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SCALAR POTENTIAL DEPOLARIZING DYAD ARTIFACT FOR A UNIAXIAL MEDIUM

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Abstract—A scalar potential formulation for a uniaxial anisotropic medium is succinctly derived through the exclusive use of Helmholtz's theorem and subsequent identification of operator orthogonality. The resulting formulation is shown to be identical to prior published research, with the notable exception that certain scalar potential fields not considered in previous work are rigorously demonstrated to be unimportant in the field recovery process, thus ensuring uniqueness. In addition, it is revealed that both a physically expected and unexpected depolarizing dyad contribution appears in the development. Using a Green's function spectral domain analysis and subsequent careful application of Leibnitz's rule it is shown that, for an unbounded homogeneous uniaxial medium, the unexpected depolarizing dyad term is canceled, leading to a mathematically and physically consistent and correct theory.

1. INTRODUCTION

Vector potentials are often employed to aid the solution of electromagnetic problems involving simple (i.e., linear, homogeneous, and isotropic) media [1–4]. In the past couple decades, new scalar and vector potential formulations have been heavily investigated and utilized in the electromagnetic analysis of complex media [5–16] such as anisotropic and bianisotropic materials [17–20]. This recent interest has been greatly renewed by significant developments in material fabrication capability and the phenomena exclusively associated with complex media [21–23]. One of the many pioneers to investigate scalar potential techniques for complex media was Weiglhofer [6–13, 15, 19]. Weiglhofer's analysis in [11], involving the scalar potential

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formulation for uniaxial anisotropic media, is especially interesting from an application viewpoint due to the relative ease of manufacturing this type of material [24].

The goal of this paper is to first briefly review the pioneering uniaxial anisotropic scalar potential work in [11]. This review is important because it allows identification of a scalar field contribution that was not considered in the method of derivation but influences the uniqueness of the field recovery process, and it reveals an expected and unexpected depolarizing dyad contribution. The next objective here is to offer an alternative scalar potential derivation for uniaxial anisotropic media exclusively based upon Helmholtz's theorem and subsequent identification of operator orthogonality. The end result is identical to [11]; however, it is definitively shown in this alternative derivation that the scalar field contribution not previously considered does not influence field calculation, and thus ensures uniqueness. The final goal is to demonstrate that, using a Green's function spectral domain analysis, the unexpected depolarizing dyad term is removable for a homogeneous uniaxial anisotropic medium, resulting in a mathematically and physically consistent theory.

2. UNIAXIAL ANISOTROPIC SCALAR POTENTIAL FORMULATION

The first objective of this section is to briefly review the uniaxial anisotropic scalar potential derivation [11] in order to identify a scalar field contribution not previously considered and to identify both an expected and unexpected depolarizing dyad contribution. The second objective is to offer an alternative derivation that rigorously demonstrates the scalar field contribution (not previously considered) does not impact field uniqueness. Removal of the unexpected depolarizing dyad for a homogeneous uniaxial anisotropic medium is demonstrated in Section 3.

2.1. Prior Uniaxial Anisotropic Scalar Potential Formulation

Following the analysis in [11], Maxwell's equations for a linear, inhomogeneous (in z), uniaxial electric and magnetic anisotropic medium are (with exp $(j\omega t)$ assumed and suppressed)

$$\begin{aligned}\nabla\times\vec{E}(\vec{\rho},z) &= -\vec{J}_h(\vec{\rho},z) - j\omega\overleftrightarrow{\mu}(z)\cdot\vec{H}(\vec{\rho},z)\\ \nabla\times\vec{H}(\vec{\rho},z) &= \vec{J}_e(\vec{\rho},z) + j\omega\overleftrightarrow{\varepsilon}(z)\cdot\vec{E}(\vec{\rho},z)\end{aligned} \tag{1}$$

where the z axis is the principal axis, $\overleftrightarrow{\varepsilon} = \hat{x}\varepsilon_t\hat{x} + \hat{y}\varepsilon_t\hat{y} + \hat{z}\varepsilon_z\hat{z}$ the dyadic permittivity, $\overleftrightarrow{\mu} = \hat{x}\mu_t\hat{x} + \hat{y}\mu_t\hat{y} + \hat{z}\mu_z\hat{z}$ the dyadic permeability, and $\vec{\rho} = \hat{x}x + \hat{y}y$. Decomposing (1) into transverse and longitudinal components and subsequent equating leads to the following relations for Ampere's and Faraday's laws, respectively

$$\nabla_t \times \hat{z}E_z + \hat{z} \times \frac{\partial \vec{E}_t}{\partial z} = -\vec{J}_{ht} - j\omega\mu_t\vec{H}_t \tag{2}$$

$$\nabla_t \times \vec{E}_t = -\hat{z}J_{hz} - \hat{z}j\omega\mu_z H_z \tag{3}$$

$$\nabla_t \times \hat{z}H_z + \hat{z} \times \frac{\partial \vec{H}_t}{\partial z} = \vec{J}_{et} + j\omega\varepsilon_t\vec{E}_t \tag{4}$$

$$\nabla_t \times \vec{H}_t = \hat{z}J_{ez} + \hat{z}j\omega\varepsilon_z E_z \tag{5}$$

where $\nabla_t = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$ is the transverse del operator. The two-dimensional form of Helmholtz's theorem [11], which states that a vector field is completely specified by the superposition of curl-free and divergence-free contributions, is now used to represent the transverse fields and currents, namely

$$\vec{E}_t = \nabla_t\Phi + \nabla_t \times \hat{z}\theta = \nabla_t\Phi - \hat{z} \times \nabla_t\theta \tag{6}$$

$$\vec{H}_t = \nabla_t\pi + \nabla_t \times \hat{z}\psi = \nabla_t\pi - \hat{z} \times \nabla_t\psi \tag{7}$$

$$\vec{J}_{et} = \nabla_t u_e + \nabla_t \times \hat{z}v_e = \nabla_t u_e - \hat{z} \times \nabla_t v_e \tag{8}$$

$$\vec{J}_{ht} = \nabla_t u_h + \nabla_t \times \hat{z}v_h = \nabla_t u_h - \hat{z} \times \nabla_t v_h \tag{9}$$

where Φ, θ, π, ψ are field-based scalar potentials and u_e, v_e, u_h, v_h are current-based scalar potentials. Insertion of (7) into (5) and (6) into (3) and use of standard vector calculus identities leads to the following relations

$$E_z = -\frac{1}{j\omega\varepsilon_z}\left(\nabla_t^2\psi + J_{ez}\right) \tag{10}$$

$$H_z = \frac{1}{j\omega\mu_z}\left(\nabla_t^2\theta - J_{hz}\right) \tag{11}$$

where $\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian operator. Thus, the fields are recovered from the potentials via (6), (7), (10) and (11).

Prompted by Helmholtz's theorem and analysis in [11], the first step in obtaining the governing wave equations for the potentials is to take the transverse divergence and transverse curl of (2) and (4) and substituting in (6)–(9), leading to the preliminary result

$$\nabla_t^2\left(\frac{\partial\theta}{\partial z} + u_h + j\omega\mu_t\pi\right) = 0 \tag{12}$$

$$\nabla_t^2\left(-E_z + \frac{\partial\Phi}{\partial z} - v_h - j\omega\mu_t\psi\right) = 0 \tag{13}$$

$$\nabla_t^2\left(\frac{\partial\psi}{\partial z} - u_e - j\omega\varepsilon_t\Phi\right) = 0 \tag{14}$$

$$\nabla_t^2\left(-H_z + \frac{\partial\pi}{\partial z} + v_e + j\omega\varepsilon_t\theta\right) = 0 \tag{15}$$

It is subsequently inferred in [11] that, based on (12)–(15), the following relations prevail;

$$\frac{\partial\theta}{\partial z}+u_h+j\omega\mu_t\pi = 0 \quad\rightarrow\quad \pi=-\frac{1}{j\omega\mu_t}\left(\frac{\partial\theta}{\partial z}+u_h\right) \tag{16}$$

$$-E_z+\frac{\partial\Phi}{\partial z}-v_h-j\omega\mu_t\psi = 0 \quad\rightarrow\quad E_z=\frac{\partial\Phi}{\partial z}-v_h-j\omega\mu_t\psi \tag{17}$$

$$\frac{\partial\psi}{\partial z}-u_e-j\omega\varepsilon_t\Phi = 0 \quad\rightarrow\quad \Phi=\frac{1}{j\omega\varepsilon_t}\left(\frac{\partial\psi}{\partial z}-u_e\right) \tag{18}$$

$$-H_z+\frac{\partial\pi}{\partial z}+v_e+j\omega\varepsilon_t\theta = 0 \quad\rightarrow\quad H_z=\frac{\partial\pi}{\partial z}+v_e+j\omega\varepsilon_t\theta. \tag{19}$$

Insertion of (10) and (18) into (17), and (11) and (16) into (19) leads to the desired scalar potential wave equations

$$-\frac{\varepsilon_t}{\varepsilon_z}\nabla_t^2\psi-\varepsilon_t\frac{\partial}{\partial z}\left(\frac{1}{\varepsilon_t}\frac{\partial\psi}{\partial z}\right)-k_t^2\psi=-\varepsilon_t\frac{\partial}{\partial z}\left(\frac{1}{\varepsilon_t}u_e\right)+\frac{\varepsilon_t}{\varepsilon_z}J_{ez}-j\omega\varepsilon_t v_h \tag{20}$$

$$-\frac{\mu_t}{\mu_z}\nabla_t^2\theta-\mu_t\frac{\partial}{\partial z}\left(\frac{1}{\mu_t}\frac{\partial\theta}{\partial z}\right)-k_t^2\theta=\mu_t\frac{\partial}{\partial z}\left(\frac{1}{\mu_t}u_h\right)-\frac{\mu_t}{\mu_z}J_{hz}-j\omega\mu_t v_e \tag{21}$$

where ε_t, ε_z, μ_t, μ_z are, in general, functions of z and $k_t^2=\omega^2\varepsilon_t\mu_t$. Summarizing, the scalar potentials ψ and θ are first solved using (20) and (21). Next, π and Φ are computed using (16) and (18). Finally, the field recovery process is completed via (6), (7), (10) and (11).

The above brief review is important in order to bring out three salient points. First, an examination of (10) and (11) readily shows that

$$\vec{E}_z=\hat{z}E_z=-\hat{z}\frac{1}{j\omega\varepsilon_z}\nabla_t^2\psi-\hat{z}\frac{1}{j\omega\varepsilon_z}\hat{z}\cdot\vec{J}_e=-\hat{z}\frac{1}{j\omega\varepsilon_z}\nabla_t^2\psi+\overleftrightarrow{L}^e_{zz}\cdot\vec{J}_e \tag{22}$$

$$\vec{H}_z=\hat{z}H_z=\hat{z}\frac{1}{j\omega\mu_z}\nabla_t^2\theta-\hat{z}\frac{1}{j\omega\mu_z}\hat{z}\cdot\vec{J}_h=\hat{z}\frac{1}{j\omega\mu_z}\nabla_t^2\theta+\overleftrightarrow{L}^h_{zz}\cdot\vec{J}_h \tag{23}$$

where the relations $J_{ez}=\hat{z}\cdot\vec{J}_e$ and $J_{hz}=\hat{z}\cdot\vec{J}_h$ have been used. The terms $\overleftrightarrow{L}^e_{zz}=-\hat{z}\frac{1}{j\omega\varepsilon_z}\hat{z}$ and $\overleftrightarrow{L}^h_{zz}=-\hat{z}\frac{1}{j\omega\mu_z}\hat{z}$ are the well-known electric and magnetic field depolarizing dyads resulting from the longitudinal current densities and are mathematically and physically consistent with prior well-documented findings [25–35]. In these findings, it is discussed that the source region of volume V is split into two subregions $V-V_\delta$ and V_δ, where V_δ is a small cavity excavated around the source point (in the limit as $\delta\rightarrow 0$). This method is used in order to carefully handle the source point singularity. However, in doing this, the excavated region disrupts the current flow causing surface charges to accumulate, which subsequently causes gap (i.e., cavity) fields to exist that are not there in the original continuous volume region V. It is subsequently discussed in these findings that the contribution from the V_δ region, which depends on its shape, produces depolarization fields that cancel the gap fields. In the work considered here in which the z axis is the principal axis, the source excluding region is a

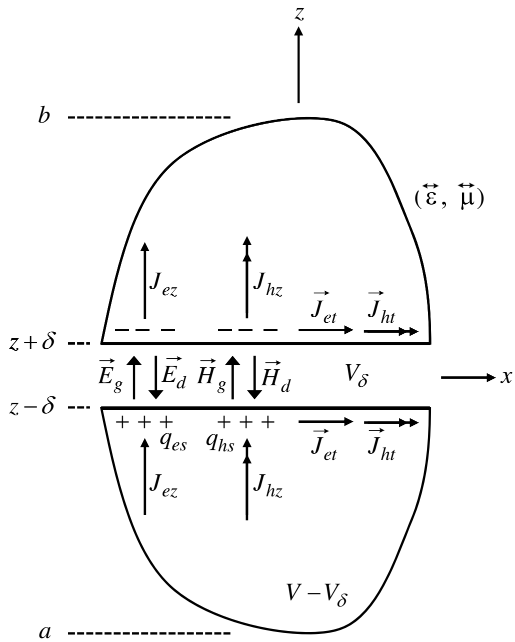

Figure 1. Source region gap and depolarizing fields.

slice gap as shown in Figure 1. Also shown are the expected electric and magnetic surface charges due to the longitudinal (i.e., z-directed) current densities J_{ez}, J_{hz} and the corresponding gap fields $\vec{E}_g$, $\vec{H}_g$ and depolarizing fields $\vec{E}_d = \overleftrightarrow{L}^e_{zz} \cdot \vec{J}_e$ and $\vec{H}_d = \overleftrightarrow{L}^h_{zz} \cdot \vec{J}_h$, which are in agreement with the prior findings.

In a similar manner, an examination of (6) and (7) in conjunction with (16) and (18) reveals that

$$\vec{E}_t = \nabla_t \Phi + \nabla_t \times \hat{z}\theta = \frac{1}{j\omega\varepsilon_t} \nabla_t \frac{\partial\psi}{\partial z} - \frac{1}{j\omega\varepsilon_t} \nabla_t u_e + \nabla_t \times \hat{z}\theta \qquad (24)$$

$$\vec{H}_t = \nabla_t \pi + \nabla_t \times \hat{z}\psi = -\frac{1}{j\omega\mu_t} \nabla_t \frac{\partial\theta}{\partial z} - \frac{1}{j\omega\mu_t} \nabla_t u_h + \nabla_t \times \hat{z}\psi \quad (25)$$

where $\nabla_t u_e$ and $\nabla_t u_h$ are identified as, with the aid of (8) and (9), the curl-free contributions of the transverse electric and magnetic current densities, respectively. Analogous with the previous results in (22) and (23), it appears that based on (24) and (25), there exists transverse electric and magnetic depolarizing dyad fields implicated by the transverse current densities, namely $-\frac{1}{j\omega\varepsilon_t} \nabla_t u_e$ and $-\frac{1}{j\omega\mu_t} \nabla_t u_h$. Based on prior findings in the research literature, these transverse depolarizing dyads are completely unexpected both mathematically and physically. Figure 1, for example, clearly shows that transverse current densities $\vec{J}_{et}$, $\vec{J}_{ht}$ which travel parallel to the gap region V_δ do not cause surface charge buildup at the boundaries $z-\delta$ and $z+\delta$. Thus gap and depolarizing fields are not anticipated. Note, it is this physical picture that prompted the author to explore the mathematical derivation that demonstrates the cancelation of the apparent transverse depolarizing dyad fields in (24)–(25). This observation, which represents the second salient point, will be further discussed in Section 3.

The third crucial point to discuss here in the reviewed analysis of [11] involves Equations (12)–(15) and the relations subsequently inferred in (16)–(19). As an example, consider Equation (12) and the subsequent inferred result in (16). Due to the transverse Laplacian operator in (12), it is more mathematically precise to infer in (16) that

$$\begin{aligned} & \tfrac{\partial\theta}{\partial z} + u_h + j\omega\mu_t\pi = C_1(z) + C_2(z)f(x,y) \\ \Rightarrow \qquad & \pi = -\tfrac{1}{j\omega\mu_t}\left(\tfrac{\partial\theta}{\partial z} + u_h\right) + \tfrac{1}{j\omega\mu_t}\left[C_1(z) + C_2(z)f(x,y)\right] \end{aligned} \tag{26}$$

where $C_1(z)$, $C_2(z)$ are functions of z and $f(x,y)$ is a function of x, y which must satisfy the condition $\nabla_t^2 f(x,y) = 0$. In order to see the ramifications of this observation on the field recovery process, consider an example case where $C_1(z) = 3z$, $C_2(z) = 2$ and $f(x,y) = x+y$. The transverse Laplacian of $C_1(z) + C_2(z)f(x,y)$ is clearly zero and is thus consistent with (12), as expected. However, for the field recovery in (7) which involves $\nabla_t\pi$, one obtains the result

$$\begin{aligned} \nabla_t\pi &= -\tfrac{1}{j\omega\mu_t}\nabla_t\left(\tfrac{\partial\theta}{\partial z} + u_h\right) + \tfrac{1}{j\omega\mu_t}\nabla_t\left[3z + 2(x+y)\right] \\ &= -\tfrac{1}{j\omega\mu_t}\nabla_t\left(\tfrac{\partial\theta}{\partial z} + u_h\right) + \tfrac{1}{j\omega\mu_t}\left[2(\hat{x}+\hat{y})\right]. \end{aligned} \tag{27}$$

Since $\nabla_t C_1(z) = 0$, it does not impact the field recovery process. However, notice the additional contribution $\frac{1}{j\omega\mu_t}[2(\hat{x}+\hat{y})]$ in (27) due to the factor $\nabla_t[C_2(z)f(x,y)]$. Since C_2 and f are, in general, arbitrary terms, they have important implications, namely, the field recovery is not unique — it can take on any arbitrary value! In the next section, an alternative scalar potential derivation will be presented that rigorously demonstrates the unimportance of any such scalars in the field recovery process, and thus ensures uniqueness of the field.

2.2. Alternative Uniaxial Anisotropic Scalar Potential Formulation

In this alternative uniaxial anisotropic scalar potential derivation, calculation of E_z and H_z follows in the exact same manner as in the prior derivation. Namely, (7) is inserted into (5), and (6) is inserted into (3) to obtain (10) and (11). Next, in contrast to the prior scalar potential derivation, the transverse divergence and transverse curl are not performed on (2) and (4). Instead, the second representation of the transverse fields and sources in (6)–(9) are directly substituted into (2) and (4), resulting in

$$\begin{aligned} & -\hat{z}\times\nabla_t E_z + \hat{z}\times\nabla_t \tfrac{\partial\Phi}{\partial z} + \nabla_t \tfrac{\partial\theta}{\partial z} \\ = \; & -\nabla_t u_h + \hat{z}\times\nabla_t v_h - \nabla_t j\omega\mu_t\pi + \hat{z}\times\nabla_t j\omega\mu_t\psi \end{aligned} \tag{28}$$

$$\begin{aligned} & -\hat{z}\times\nabla_t H_z + \hat{z}\times\nabla_t \tfrac{\partial\pi}{\partial z} + \nabla_t \tfrac{\partial\psi}{\partial z} \\ = \; & \nabla_t u_e - \hat{z}\times\nabla_t v_e + \nabla_t j\omega\varepsilon_t\Phi - \hat{z}\times\nabla_t j\omega\varepsilon_t\theta \end{aligned} \tag{29}$$

where standard vector calculus identities have been utilized. It is important here to recognize that the operators ∇_t and $\hat{z}\times\nabla_t$ are orthogonal (i.e., $\nabla_t\cdot\hat{z}\times\nabla_t = 0$) and are thus linearly independent. Note, a Fourier transform on the transverse spatial variables, in which ∇_t gets mapped into $j\vec{\lambda}_\rho = j(\hat{x}\lambda_x + \hat{y}\lambda_y)$, may also be employed as an alternative proof of orthogonality of the operators since $\nabla_t\cdot\hat{z}\times\nabla_t$ transforms to $j\vec{\lambda}_\rho\cdot\hat{z}\times j\vec{\lambda}_\rho = \vec{\lambda}_\rho\times\vec{\lambda}_\rho\cdot\hat{z} = 0$. Therefore, the ∇_t terms and the $\hat{z}\times\nabla_t$ terms can be respectively equated in (28) and in (29), leading to the rigorous result

$$\begin{aligned} \nabla_t\left(\tfrac{\partial\theta}{\partial z} + u_h + j\omega\mu_t\pi\right) &= 0 \;\Rightarrow \\ \tfrac{\partial\theta}{\partial z} + u_h + j\omega\mu_t\pi &= C_1(z) \end{aligned} \tag{30}$$

$$\begin{aligned} \hat{z}\times\nabla_t\left(-E_z + \tfrac{\partial\Phi}{\partial z} - v_h - j\omega\mu_t\psi\right) &= 0 \;\Rightarrow \\ -E_z + \tfrac{\partial\Phi}{\partial z} - v_h - j\omega\mu_t\psi &= C_2(z) \end{aligned} \tag{31}$$

$$\begin{aligned} \nabla_t\left(\frac{\partial\psi}{\partial z} - u_e - j\omega\varepsilon_t\Phi\right) &= 0 \;\Rightarrow \\ \frac{\partial\psi}{\partial z} - u_e - j\omega\varepsilon_t\Phi &= C_3(z) \end{aligned} \tag{32}$$

$$\begin{aligned} \hat{z}\times\nabla_t\left(-H_z + \tfrac{\partial\pi}{\partial z} + v_e + j\omega\varepsilon_t\theta\right) &= 0 \;\Rightarrow \\ -H_z + \tfrac{\partial\pi}{\partial z} + v_e + j\omega\varepsilon_t\theta &= C_4(z) \end{aligned} \tag{33}$$

where C_n $(n = 1, 2, 3, 4)$ are scalar fields that can, in general, be functions of z. Note that the derived relations in (30)–(33) only involve the transverse del operator ∇_t, whereas derived relations (12)–(15) in the previous section involved the transverse Laplacian operator ∇_t^2. Since the field recovery process for the transverse fields involves ∇_t and the longitudinal fields involves ∇_t^2, it is clear upon differentiating that the C_n in (30)–(33) do not influence the fields (i.e., $\nabla_t C_n(z) = 0$). Thus, without loss of generality or uniqueness, the C_n may be set equal to zero, definitively leading to the desired potential relations

$$\pi = -\tfrac{1}{j\omega\mu_t}\left(\tfrac{\partial\theta}{\partial z} + u_h\right) \tag{34}$$

$$E_z = \tfrac{\partial\Phi}{\partial z} - v_h - j\omega\mu_t\psi \tag{35}$$

$$\Phi = \frac{1}{j\omega\varepsilon_t}\left(\frac{\partial\psi}{\partial z} - u_e\right) \tag{36}$$

$$H_z = \frac{\partial\pi}{\partial z} + v_e + j\omega\varepsilon_t\theta \tag{37}$$

which are identical to relations (16)–(19) inferred in the previous section. The final wave equations (20)–(21) follow in the exact same manner as discussed in the previous section. Thus, it has been rigorously shown for the first time in this manner (to the author's knowledge) that the scalars C_n appearing in the potential formulation do not contribute to the fields and consequently ensures a unique field representation.

3. TRANSVERSE FIELD DEPOLARIZING DYAD ARTIFACTS

Based on the uniaxial anisotropic scalar potential formulation previously discussed, it appears that there exists transverse electric and magnetic field depolarizing dyads, namely $-\frac{1}{j\omega\varepsilon_t}\nabla_t u_e$ and $-\frac{1}{j\omega\mu_t}\nabla_t u_h$. These apparent terms originate from (6) and (7) and from the potential fields Φ and π,

$$\Phi = \frac{1}{j\omega\varepsilon_t}\left(\frac{\partial\psi}{\partial z} - u_e\right) \tag{38}$$

$$\pi = -\frac{1}{j\omega\mu_t}\left(\frac{\partial\theta}{\partial z} + u_h\right) \tag{39}$$

where u_e and u_h are related to the transverse electric and magnetic current densities $\vec{J}_{et}$ and $\vec{J}_{ht}$ via (8) and (9), respectively. It was previously mentioned that this result was both mathematically and physically unexpected in lieu of the research literature and Figure 1. The goal of this section is to demonstrate that, via a Green's function spectral domain approach and careful application of Leibnitz's rule, the apparent depolarizing dyads are removable for an unbounded homogeneous uniaxial anisotropic medium, leading to a mathematically and physically consistent theory. Thus, the equivalent goal here is to show that the terms u_e and u_h in (38) and (39) get canceled, which subsequently eliminates the apparent electric and magnetic field depolarizing dyads.

The first step in showing the apparent depolarizing dyads are removable is to seek solutions to the wave equations. The scalar potential wave Equations (20) and (21) for a homogeneous uniaxial anisotropic medium simplify to

$$-\frac{\varepsilon_t}{\varepsilon_z}\nabla_t^2\psi - \frac{\partial^2\psi}{\partial z^2} - k_t^2\psi = -\frac{\partial u_e}{\partial z} + \frac{\varepsilon_t}{\varepsilon_z}J_{ez} - j\omega\varepsilon_t v_h \tag{40}$$

$$-\frac{\mu_t}{\mu_z}\nabla_t^2\theta - \frac{\partial^2\theta}{\partial z^2} - k_t^2\theta = \frac{\partial u_h}{\partial z} - \frac{\mu_t}{\mu_z}J_{hz} - j\omega\mu_t v_e \tag{41}$$

where the functional dependence of the potentials and current densities on x, y, z has been dropped for notational convenience. Since the medium is unbounded, Fourier transformation is prompted on x, y, z. The principal axis is the z axis, thus it is convenient to first transform on the transverse variables x, y followed by transformation on the longitudinal variable z using the respective generic transform pairs

$$\begin{aligned} \tilde{f}\left(\vec{\lambda}_\rho, z\right) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\vec{\rho}, z\right) e^{-j\vec{\lambda}_\rho\cdot\vec{\rho}} dxdy \\ f\left(\vec{\rho}, z\right) &= \tfrac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}\left(\vec{\lambda}_\rho, z\right) e^{j\vec{\lambda}_\rho\cdot\vec{\rho}} d\lambda_x d\lambda_y \end{aligned} \tag{42}$$

$$\begin{aligned} \tilde{\tilde{f}}\left(\vec{\lambda}_\rho, \lambda_z\right) &= \int_{-\infty}^{\infty} \tilde{f}\left(\vec{\lambda}_\rho, z\right) e^{-j\lambda_z z} dz \\ \tilde{f}\left(\vec{\lambda}_\rho, z\right) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\tilde{f}}\left(\vec{\lambda}_\rho, \lambda_z\right) e^{j\lambda_z z} d\lambda_z \end{aligned} \tag{43}$$

where $\vec{\lambda}_\rho = \hat{x}\lambda_x + \hat{y}\lambda_y$. Upon consecutively performing these Fourier transformations, the wave equations in the $\vec{\lambda}_\rho, \lambda_z$ domain become

$$\left(\lambda_z^2 - \lambda_{z\psi}^2\right)\tilde{\tilde{\psi}} = -j\lambda_z\tilde{\tilde{u}}_e + \tfrac{\varepsilon_t}{\varepsilon_z}\tilde{\tilde{J}}_{ez} - j\omega\varepsilon_t\tilde{\tilde{v}}_h \tag{44}$$

$$\left(\lambda_z^2 - \lambda_{z\theta}^2\right)\tilde{\tilde{\theta}} = j\lambda_z\tilde{\tilde{u}}_h - \tfrac{\mu_t}{\mu_z}\tilde{\tilde{J}}_{hz} - j\omega\mu_t\tilde{\tilde{v}}_e \tag{45}$$

where $\lambda_{z\psi}^2 = k_t^2 - \frac{\varepsilon_t}{\varepsilon_z}\lambda_\rho^2$, $\lambda_{z\theta}^2 = k_t^2 - \frac{\mu_t}{\mu_z}\lambda_\rho^2$, $\lambda_\rho^2 = \lambda_x^2 + \lambda_y^2$, and $k_t^2 = \omega^2\varepsilon_t\mu_t$. Solving for $\tilde{\tilde{\psi}}(\vec{\lambda}_\rho, \lambda_z)$ and $\tilde{\tilde{\theta}}(\vec{\lambda}_\rho, \lambda_z)$ produces

$$\tilde{\tilde{\psi}} = \frac{-j\lambda_z\tilde{\tilde{u}}_e + \frac{\varepsilon_t}{\varepsilon_z}\tilde{\tilde{J}}_{ez} - j\omega\varepsilon_t\tilde{\tilde{v}}_h}{(\lambda_z + \lambda_{z\psi})(\lambda_z - \lambda_{z\psi})} \tag{46}$$

$$\tilde{\tilde{\theta}} = \frac{j\lambda_z\tilde{\tilde{u}}_h - \frac{\mu_t}{\mu_z}\tilde{\tilde{J}}_{hz} - j\omega\mu_t\tilde{\tilde{v}}_e}{(\lambda_z + \lambda_{z\theta})(\lambda_z - \lambda_{z\theta})} \tag{47}$$

where the poles of (46)–(47) physically represent the propagation factors $\lambda_z = \mp\lambda_{z\psi}, \mp\lambda_{z\theta}$ of up-going and down-going waves along the z direction, respectively. Note, the poles $+\lambda_{z\psi}$, $+\lambda_{z\theta}$ are located in the lower-half complex λ_z plane and the poles $-\lambda_{z\psi}$, $-\lambda_{z\theta}$ are located in the upper-half complex λ_z plane. Also, an examination of E_z, H_z

in (10), (11) reveals that $\tilde{\tilde{\psi}}$ is associated with TM^z fields and $\tilde{\tilde{\theta}}$ is associated with TE^z fields.

Expressions for $\tilde{\tilde{u}}_e, \tilde{\tilde{v}}_e$ and $\tilde{\tilde{u}}_h, \tilde{\tilde{v}}_h$ in terms of $\vec{\tilde{\tilde{J}}}_{et}$, $\vec{\tilde{\tilde{J}}}_{ht}$ and ultimately $\vec{\tilde{\tilde{J}}}_e$, $\vec{\tilde{\tilde{J}}}_h$ (leading to the identification of spectral-domain potential-based Green's functions) can be found via Fourier transforming the divergence and curl of both (8) and (9), respectively, leading to

$$j\vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_{et} = -\lambda_\rho^2 \tilde{\tilde{u}}_e \ \Rightarrow \ \tilde{\tilde{u}}_e = -\frac{j\vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_{et}}{\lambda_\rho^2} = -\frac{j\vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_e}{\lambda_\rho^2} \tag{48}$$

$$j\vec{\lambda}_\rho \times \vec{\tilde{\tilde{J}}}_{et} = \hat{z}\lambda_\rho^2 \tilde{\tilde{v}}_e \ \Rightarrow \ \tilde{\tilde{v}}_e = \frac{j\hat{z}\cdot\vec{\lambda}_\rho \times \vec{\tilde{\tilde{J}}}_{et}}{\lambda_\rho^2} = \frac{j\hat{z}\times\vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_{et}}{\lambda_\rho^2} = \frac{j\hat{z}\times\vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_e}{\lambda_\rho^2} \tag{49}$$

$$j\vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_{ht} = -\lambda_\rho^2 \tilde{\tilde{u}}_h \ \Rightarrow \ \tilde{\tilde{u}}_h = -\frac{j\vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_{ht}}{\lambda_\rho^2} = -\frac{j\vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_h}{\lambda_\rho^2} \tag{50}$$

$$j\vec{\lambda}_\rho \times \vec{\tilde{\tilde{J}}}_{ht} = \hat{z}\lambda_\rho^2 \tilde{\tilde{v}}_h \ \Rightarrow \ \tilde{\tilde{v}}_h = \frac{j\hat{z}\cdot\vec{\lambda}_\rho \times \vec{\tilde{\tilde{J}}}_{ht}}{\lambda_\rho^2} = \frac{j\hat{z}\times\vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_{ht}}{\lambda_\rho^2} = \frac{j\hat{z}\times\vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_h}{\lambda_\rho^2} \tag{51}$$

where the vector calculus identity $\vec{a}\cdot\vec{b}\times\vec{c} = \vec{a}\times\vec{b}\cdot\vec{c}$ has been used in (49) and (51). Note, $\vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_{et,ht} = \vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_{e,h}$ and $\hat{z} \times \vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_{et,ht} = \hat{z} \times \vec{\lambda}_\rho \cdot \vec{\tilde{\tilde{J}}}_{e,h}$ since $\vec{\lambda}_\rho$ and $\hat{z} \times \vec{\lambda}_\rho$ are purely transverse vectors. Thus, upon substituting (48)–(51) into (46)–(47) and using $\tilde{\tilde{J}}_{ez,hz} = \hat{z} \cdot \vec{\tilde{\tilde{J}}}_{e,h}$ allows $\tilde{\tilde{\psi}}$ and $\tilde{\tilde{\theta}}$ to be written as

$$\tilde{\tilde{\psi}}(\vec{\lambda}_\rho, \lambda_z) = \vec{\tilde{\tilde{G}}}_{\psi e}(\vec{\lambda}_\rho, \lambda_z) \cdot \vec{\tilde{\tilde{J}}}_e(\vec{\lambda}_\rho, \lambda_z) + \vec{\tilde{\tilde{G}}}_{\psi h}(\vec{\lambda}_\rho, \lambda_z) \cdot \vec{\tilde{\tilde{J}}}_h(\vec{\lambda}_\rho, \lambda_z) \tag{52}$$

$$\tilde{\tilde{\theta}}(\vec{\lambda}_\rho, \lambda_z) = \vec{\tilde{\tilde{G}}}_{\theta e}(\vec{\lambda}_\rho, \lambda_z) \cdot \vec{\tilde{\tilde{J}}}_e(\vec{\lambda}_\rho, \lambda_z) + \vec{\tilde{\tilde{G}}}_{\theta h}(\vec{\lambda}_\rho, \lambda_z) \cdot \vec{\tilde{\tilde{J}}}_h(\vec{\lambda}_\rho, \lambda_z) \tag{53}$$

$$\vec{\tilde{\tilde{G}}}_{\psi e}(\vec{\lambda}_\rho, \lambda_z) = \frac{-\frac{\lambda_z}{\lambda_\rho^2}\vec{\lambda}_\rho + \frac{\varepsilon_t}{\varepsilon_z}\hat{z}}{(\lambda_z + \lambda_{z\psi})(\lambda_z - \lambda_{z\psi})} \tag{54}$$

$$\vec{\tilde{\tilde{G}}}_{\psi h}(\vec{\lambda}_\rho, \lambda_z) = \frac{\frac{\omega\varepsilon_t}{\lambda_\rho^2}\hat{z} \times \vec{\lambda}_\rho}{(\lambda_z + \lambda_{z\psi})(\lambda_z - \lambda_{z\psi})} \tag{55}$$

$$\vec{\tilde{\tilde{G}}}_{\theta e}(\vec{\lambda}_\rho, \lambda_z) = \frac{\frac{\omega\mu_t}{\lambda_\rho^2}\hat{z} \times \vec{\lambda}_\rho}{(\lambda_z + \lambda_{z\theta})(\lambda_z - \lambda_{z\theta})} \tag{56}$$

$$\vec{\tilde{\tilde{G}}}_{\theta h}(\vec{\lambda}_\rho, \lambda_z) = \frac{\frac{\lambda_z}{\lambda_\rho^2}\vec{\lambda}_\rho - \frac{\mu_t}{\mu_z}\hat{z}}{(\lambda_z + \lambda_{z\theta})(\lambda_z - \lambda_{z\theta})} \tag{57}$$

where $\vec{\tilde{\tilde{G}}}_{\psi e}$, $\vec{\tilde{\tilde{G}}}_{\psi h}$ and $\vec{\tilde{\tilde{G}}}_{\theta e}$, $\vec{\tilde{\tilde{G}}}_{\theta h}$ are the spectral domain vector Green's functions for $\tilde{\tilde{\psi}}$ and $\tilde{\tilde{\theta}}$ due to electric and magnetic current densities $\tilde{\tilde{\vec{J}}}_e$ and $\tilde{\tilde{\vec{J}}}_h$. Note, a more traditional dyadic Green's function interpretation for the potentials can be readily identified by observing in (6) and (7) that $\tilde{\tilde{\vec{\psi}}} = \hat{z}\tilde{\tilde{\psi}}$ and $\tilde{\tilde{\vec{\theta}}} = \hat{z}\tilde{\tilde{\theta}}$, leading to the dyadic Green's functions $\tilde{\tilde{\overleftrightarrow{G}}}_{\psi e} = \hat{z}\tilde{\tilde{\vec{G}}}_{\psi e}$, $\tilde{\tilde{\overleftrightarrow{G}}}_{\psi h} = \hat{z}\tilde{\tilde{\vec{G}}}_{\psi h}$, $\tilde{\tilde{\overleftrightarrow{G}}}_{\theta e} = \hat{z}\tilde{\tilde{\vec{G}}}_{\theta e}$ and $\tilde{\tilde{\overleftrightarrow{G}}}_{\theta h} = \hat{z}\tilde{\tilde{\vec{G}}}_{\theta h}$. Inverse transforming (52) and (53) using (43) and Cauchy's integral theorem [32, 33], one obtains for $\tilde{\psi}(\vec{\lambda}_\rho, z)$ and $\tilde{\theta}(\vec{\lambda}_\rho, z)$ (upon considering the cases of $z > z'$ and $z < z'$) the result

$$\tilde{\psi}\left(\vec{\lambda}_\rho, z\right) = \int_a^b \tilde{\vec{G}}_{\psi e}\left(\vec{\lambda}_\rho; z - z'\right) \cdot \tilde{\vec{J}}_e\left(\vec{\lambda}_\rho, z'\right) dz' + \int_a^b \tilde{\vec{G}}_{\psi h}\left(\vec{\lambda}_\rho; z - z'\right) \cdot \tilde{\vec{J}}_h\left(\vec{\lambda}_\rho, z'\right) dz' \tag{58}$$

$$\tilde{\theta}\left(\vec{\lambda}_\rho, z\right) = \int_a^b \tilde{\vec{G}}_{\theta e}\left(\vec{\lambda}_\rho; z - z'\right) \cdot \tilde{\vec{J}}_e\left(\vec{\lambda}_\rho, z'\right) dz' + \int_a^b \tilde{\vec{G}}_{\theta h}\left(\vec{\lambda}_\rho; z - z'\right) \cdot \tilde{\vec{J}}_h\left(\vec{\lambda}_\rho, z'\right) dz' \tag{59}$$

where the source currents in the z variable are assumed to exist in the localized region $a \leqslant z \leqslant b$ and

$$\tilde{\vec{G}}_{\psi e}\left(\vec{\lambda}_p; z - z'\right) = -\frac{\frac{j\lambda_{z\psi}}{\lambda_\rho^2}\operatorname{sgn}(z - z')\,\vec{\lambda}_\rho + \frac{j\varepsilon_t}{\varepsilon_z}\hat{z}}{2\lambda_{z\psi}}\, e^{-j\lambda_{z\psi}|z-z'|} \tag{60}$$

$$\tilde{\vec{G}}_{\psi h}\left(\vec{\lambda}_p; z - z'\right) = -\frac{\frac{j\omega\varepsilon_t}{\lambda_\rho^2}\hat{z} \times \vec{\lambda}_\rho}{2\lambda_{z\psi}}\, e^{-j\lambda_{z\psi}|z-z'|} \tag{61}$$

$$\tilde{\vec{G}}_{\theta e}\left(\vec{\lambda}_p; z - z'\right) = -\frac{\frac{j\omega\mu_t}{\lambda_\rho^2}\hat{z} \times \vec{\lambda}_\rho}{2\lambda_{z\theta}}\, e^{-j\lambda_{z\theta}|z-z'|} \tag{62}$$

$$\tilde{\vec{G}}_{\theta h}\left(\vec{\lambda}_p; z - z'\right) = \frac{\frac{j\lambda_{z\theta}}{\lambda_\rho^2}\operatorname{sgn}(z - z')\,\vec{\lambda}_\rho + \frac{j\mu_t}{\mu_z}\hat{z}}{2\lambda_{z\theta}}\, e^{-j\lambda_{z\theta}|z-z'|} \tag{63}$$

with $\lambda_{z\psi} = \sqrt{k_t^2 - \frac{\varepsilon_t}{\varepsilon_z}\lambda_\rho^2}$, $\lambda_{z\theta} = \sqrt{k_t^2 - \frac{\mu_t}{\mu_z}\lambda_\rho^2}$ and $\operatorname{sgn}(z - z') = +1, -1$ for $z > z'$, $z < z'$.

Now that the solutions to the wave equations for potentials $\tilde{\psi}$ and $\tilde{\theta}$ have been found, it is shown next that the current terms u_e and u_h in (38) and (39) are canceled via careful application of Leibnitz's rule [36, 37]:

$$\frac{\partial}{\partial z}\int\limits_{c(z)}^{d(z)} f(z,z')dz' = \int\limits_{c(z)}^{d(z)} \frac{\partial f(z,z')}{\partial z}dz' + \frac{\partial d(z)}{\partial z}f[z,d(z)] - \frac{\partial c(z)}{\partial z}f[z,c(z)] \quad (64)$$

where c, d and f must be continuous and have continuous derivatives in the domain of existence. This analysis is more easily accomplished in the $\vec{\lambda}_\rho$ domain since $\tilde{\psi}$ and $\tilde{\theta}$ are readily available in (58)–(63), thus Fourier transforming (38) and (39) using (42) produces

$$\tilde{\Phi}(\vec{\lambda}_\rho, z) = \frac{1}{j\omega\varepsilon_t}\left[\frac{\partial\tilde{\psi}(\vec{\lambda}_\rho,z)}{\partial z} - \tilde{u}_e(\vec{\lambda}_\rho, z)\right] \quad (65)$$

$$\tilde{\pi}(\vec{\lambda}_\rho, z) = -\frac{1}{j\omega\mu_t}\left[\frac{\partial\tilde{\theta}(\vec{\lambda}_\rho,z)}{\partial z} + \tilde{u}_h(\vec{\lambda}_\rho, z)\right] \quad (66)$$

where $\tilde{u}_e, \tilde{u}_h$ are easily obtained from (48) and (50) via the inverse Fourier transform (43), resulting in

$$\tilde{u}_e(\vec{\lambda}_\rho, z) = -\frac{j\vec{\lambda}_\rho}{\lambda_\rho^2}\cdot\vec{\tilde{J}}_e(\vec{\lambda}_\rho, z) \quad (67)$$

$$\tilde{u}_h(\vec{\lambda}_\rho, z) = -\frac{j\vec{\lambda}_\rho}{\lambda_\rho^2}\cdot\vec{\tilde{J}}_h(\vec{\lambda}_\rho, z) \quad (68)$$

In (65), the derivative with respect to z of the potential $\tilde{\psi}(\vec{\lambda}_\rho,\, z)$ must be performed, namely

$$\begin{aligned}\frac{\partial\tilde{\psi}(\vec{\lambda}_\rho,z)}{\partial z} &= \frac{\partial}{\partial z}\int\limits_a^b \vec{\tilde{G}}_{\psi e}\left(\vec{\lambda}_\rho; z-z'\right)\cdot\vec{\tilde{J}}_e\left(\vec{\lambda}_\rho, z'\right)dz' \\ &\quad + \frac{\partial}{\partial z}\int\limits_a^b \vec{\tilde{G}}_{\psi h}\left(\vec{\lambda}_\rho; z-z'\right)\cdot\vec{\tilde{J}}_h\left(\vec{\lambda}_\rho, z'\right)dz'\end{aligned} \quad (69)$$

where Equation (58) has been implicated. Here, it is assumed that the current densities are continuous and have continuous derivatives. However, careful examination of $\vec{\tilde{G}}_{\psi e}$ reveals it is discontinuous and its derivative is discontinuous at $z' = z$, and $\vec{\tilde{G}}_{\psi h}$ has a discontinuous derivative at the point $z' = z$ (if the observation point z is inside the source region). Thus, the integrals in (69) must be separated into two continuously differentiable subintervals $[a, z)$ and $(z, b]$ to ensure (60)–(61) adhere to the requirements of Leibnitz's rule, leading to

$$\frac{\partial\tilde{\psi}(z)}{\partial z} = \frac{\partial}{\partial z}\int_{a}^{z-\delta}\vec{\tilde{G}}_{\psi e}\left(z-z'\right)\cdot\vec{\tilde{J}}_{e}(z')dz' + \frac{\partial}{\partial z}\int_{z+\delta}^{b}\vec{\tilde{G}}_{\psi e}\left(z-z'\right)\cdot\vec{\tilde{J}}_{e}(z')dz'$$
$$+\frac{\partial}{\partial z}\int_{a}^{z-\delta}\vec{\tilde{G}}_{\psi h}\left(z-z'\right)\cdot\vec{\tilde{J}}_{h}(z')dz' + \frac{\partial}{\partial z}\int_{z+\delta}^{b}\vec{\tilde{G}}_{\psi h}\left(z-z'\right)\cdot\vec{\tilde{J}}_{h}(z')dz' \quad (70)$$

where the $\vec{\lambda}_\rho$ functional dependence and $\lim \delta \to 0$ symbol have been dropped for notational convenience. It is critical to observe in (70) that the limits of integration involving the terms $z-\delta$ and $z+\delta$ are now functions of the variable z, and thus great care must be taken when applying Leibnitz's rule.

The derivatives in (70) are now taken in a straight-forward manner via (64), producing the results

$$\frac{\partial}{\partial z}\int_{a}^{z-\delta}\vec{\tilde{G}}_{\psi e}\left(z-z'\right)\cdot\vec{\tilde{J}}_{e}\left(z'\right)dz'$$
$$=\int_{a}^{z-\delta}\frac{\partial}{\partial z}\vec{\tilde{G}}_{\psi e}\left(z-z'\right)\cdot\vec{\tilde{J}}_{e}\left(z'\right)dz' + \vec{\tilde{G}}_{\psi e}\left(z-z'\right)\cdot\vec{\tilde{J}}_{e}\left(z'\right)\Big|_{z'=z-\delta} \quad (71)$$

$$\frac{\partial}{\partial z}\int_{z+\delta}^{b}\vec{\tilde{G}}_{\psi e}\left(z-z'\right)\cdot\vec{\tilde{J}}_{e}\left(z'\right)dz'$$
$$=\int_{z+\delta}^{b}\frac{\partial}{\partial z}\vec{\tilde{G}}_{\psi e}\left(z-z'\right)\cdot\vec{\tilde{J}}_{e}\left(z'\right)dz' - \vec{\tilde{G}}_{\psi e}\left(z-z'\right)\cdot\vec{\tilde{J}}_{e}\left(z'\right)\Big|_{z'=z+\delta} \quad (72)$$

$$\frac{\partial}{\partial z}\int_{a}^{z-\delta}\vec{\tilde{G}}_{\psi h}\left(z-z'\right)\cdot\vec{\tilde{J}}_{h}\left(z'\right)dz'$$
$$=\int_{a}^{z-\delta}\frac{\partial}{\partial z}\vec{\tilde{G}}_{\psi h}\left(z-z'\right)\cdot\vec{\tilde{J}}_{h}\left(z'\right)dz' + \vec{\tilde{G}}_{\psi h}\left(z-z'\right)\cdot\vec{\tilde{J}}_{h}\left(z'\right)\Big|_{z'=z-\delta} \quad (73)$$

$$\frac{\partial}{\partial z}\int_{z+\delta}^{b}\vec{\tilde{G}}_{\psi h}\left(z-z'\right)\cdot\vec{\tilde{J}}_{h}\left(z'\right)dz'$$

$$= \int_{z+\delta}^{b} \frac{\partial}{\partial z} \vec{\tilde{G}}_{\psi h}\left(z - z'\right) \cdot \vec{\tilde{J}}_h\left(z'\right) dz' - \vec{\tilde{G}}_{\psi h}\left(z - z'\right) \cdot \vec{\tilde{J}}_h\left(z'\right)\Big|_{z'=z+\delta} \quad (74)$$

and, upon combining (71) with (72) and (73) with (74) in the $\lim \delta \to 0$, leads to

$$\begin{aligned} \frac{\partial \tilde{\psi}(z)}{\partial z} = & \int_{a}^{b} \frac{\partial}{\partial z} \vec{\tilde{G}}_{\psi e}\left(z - z'\right) \cdot \vec{\tilde{J}}_e\left(z'\right) dz' \\ & + \lim_{\delta \to 0} \left[\vec{\tilde{G}}_{\psi e}\left(z - z'\right)\Big|_{z'=z-\delta} - \vec{\tilde{G}}_{\psi e}\left(z - z'\right)\Big|_{z'=z+\delta} \right] \cdot \vec{\tilde{J}}_e(z) \\ & + \int_{a}^{b} \frac{\partial}{\partial z} \vec{\tilde{G}}_{\psi h}\left(z - z'\right) \cdot \vec{\tilde{J}}_h\left(z'\right) dz' \\ & + \lim_{\delta \to 0} \left[\vec{\tilde{G}}_{\psi h}\left(z - z'\right)\Big|_{z'=z-\delta} - \vec{\tilde{G}}_{\psi h}\left(z - z'\right)\Big|_{z'=z+\delta} \right] \cdot \vec{\tilde{J}}_h(z) \quad (75) \end{aligned}$$

where the continuity of the current densities has been used. Substitution of (60)–(61) into (75) and careful handling of the sgn $(z - z')$ function reveals that

$$\begin{aligned} & \lim_{\delta \to 0} \left[\vec{\tilde{G}}_{\psi e}\left(z - z'\right)\Big|_{z'=z-\delta} - \vec{\tilde{G}}_{\psi e}\left(z - z'\right)\Big|_{z'=z+\delta} \right] \cdot \vec{\tilde{J}}_e(z) \\ = & -\frac{j}{\lambda_\rho^2} \vec{\lambda}_\rho \cdot \vec{\tilde{J}}_e(z) = \tilde{u}_e(\vec{\lambda}_\rho, z) \quad (76) \end{aligned}$$

$$\lim_{\delta \to 0} \left[\vec{\tilde{G}}_{\psi h}\left(z - z'\right)\Big|_{z'=z-\delta} - \vec{\tilde{G}}_{\psi h}\left(z - z'\right)\Big|_{z'=z+\delta} \right] \cdot \vec{\tilde{J}}_h(z) = 0 \quad (77)$$

where the functional dependence on $\vec{\lambda}_\rho$ in $\tilde{u}_e$ has been reintroduced for the sake of comparison with (65). The result in (76) is remarkable because it clearly demonstrates that a careful handling of Leibnitz's rule when differentiating $\tilde{\psi}$ with respect to z produces an additional term $\tilde{u}_e(\vec{\lambda}_\rho, z)$ that exactly cancels the $-\tilde{u}_e(\vec{\lambda}_\rho, z)$ term in (65). Thus, as a consequence, the unexpected electric field depolarizing dyad artifact is indeed canceled, leading to a mathematically and physically consistent/correct theory! For completeness, the potential $\tilde{\Phi}$ is calculated using (65) and the results in (76)–(77), leading to

$$\tilde{\Phi}(\vec{\lambda}_\rho, z) = \frac{1}{j\omega\varepsilon_t} \int_{a}^{b} \frac{\partial}{\partial z} \vec{\tilde{G}}_{\psi e}(\vec{\lambda}_\rho; z - z') \cdot \vec{\tilde{J}}_e(\vec{\lambda}_\rho, z') dz'$$

$$+\frac{1}{j\omega\varepsilon_t}\int_a^b \frac{\partial}{\partial z}\vec{\tilde{\vec{G}}}_{\psi h}(\vec{\lambda}_\rho; z-z')\cdot\vec{\tilde{J}}_h(\vec{\lambda}_\rho, z')dz' \tag{78}$$

Inverse Fourier transforming (58) with respect to $\tilde{\psi}$, (59) with respect to $\tilde{\theta}$, and (78) with respect to $\tilde{\Phi}$ via (42) and subsequent insertion into (6) and (10) allows correct calculation of the spatial-domain electric field and corresponding field-based Green's functions due to electric and magnetic current densities. In a similar procedure involving (66), it can be shown that the magnetic field depolarizing dyad is also removable and that the correct spatial magnetic field and corresponding Green's functions subsequently follows, the details of which are not shown for the sake of brevity.

4. CONCLUSION

A scalar potential formulation for a uniaxial anisotropic medium was derived through the exclusive use of Helmholtz's theorem and subsequent identification of operator orthogonality. The resulting formulation was shown to be identical to prior published research, with the notable exception that certain scalar fields not considered in previous work were identified to not impact the field recovery process. This derivation is new to the author's knowledge and constitutes an important contribution since it ensures field uniqueness. In addition, it was discussed that both a physically expected and unexpected depolarizing dyad appeared in the development. Based on a Green's function spectral domain analysis, it was shown that, for a homogeneous uniaxial medium, the unexpected depolarizing dyad term is removable (i.e., gets canceled) via careful application of Leibnitz's rule. Demonstrating the removal of the depolarizing dyad artifacts constitutes a crucial contribution since it leads to a mathematically and physically consistent theory and leads to the correct calculation of fields and corresponding Green's functions both externally and internally to the source region. Future work will investigate depolarizing dyad artifacts in gyrotropic media and will explore a possible proof of whether the depolarizing dyad artifacts are generally removable for inhomogeneous media.

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