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TORQUE ON A MOVING ELECTRIC/MAGNETIC DIPOLE

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Abstract—We derive an expression for the torque exerted on an electric/magnetic dipole moving in an electromagnetic field, which contains two new velocity-dependent terms that to our knowledge were not reported before. A physical meaning of various torque components is discussed in terms of Lorentz force law and hidden momentum contribution.

1. INTRODUCTION

It is well known that the torque exerted on a resting small dipole is determined by the expression

$$\mathbf{T}_0 = \mathbf{p}_0 \times \mathbf{E}_0 + \boldsymbol{\mu}_0 \times \mathbf{B}_0, \tag{1}$$

where $\mathbf{p}_0$, $\boldsymbol{\mu}_0$ are the proper electric and magnetic dipole moments, and $\mathbf{E}_0$, $\mathbf{B}_0$ are the electric and magnetic fields, correspondingly, in the rest frame of the dipole. Eq. (1) allows us to write the motion equation of spin $\mathbf{s}$ in a constant (or slowly varied) electromagnetic field for a particle with mass m, Landé factor g and vanishing electric dipole moment $\mathbf{p}_0$ in the form

$$\frac{d\mathbf{s}}{dt} = \frac{ge}{2mc}\mathbf{s} \times \mathbf{B}_0$$

to be valid in the proper particle's frame. A relativistic generalization of this equation yields the BMT equation [1], which also includes the

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Thomas precession [2]. At the same time, the approach used in the derivation of BMT equation (the introduction of spin four-vector with the vanishing time component in the proper frame of particle [1, 3]) occurs convenient to determine the motion of particle's spin in its *proper* frame in terms of the external electromagnetic (EM) field measured in a laboratory. Thus, this approach does not allow us to determine explicitly the torque on a *moving* dipole via the interaction of its electric $\mathbf{p}$ and magnetic $\boldsymbol{\mu}$ dipole moments with an external EM field, when all these quantities are defined in a laboratory frame.

One of the possible ways to derive the torque on a moving dipole is to use Eq. (1) and to carry out the Lorentz transformation for the torque from the rest frame of the dipole to the laboratory frame (where the dipole is moving), expressing simultaneously the EM fields $\mathbf{E}_0$, $\mathbf{B}_0$ of Eq. (1) via fields $\mathbf{E}$, $\mathbf{B}$ measured in a laboratory. However, we remind that the transformation of torque components is guided by the corresponding torque four-tensor [4], and one can check that its application does not allow obtaining an explicit analytical expression for the torque in the laboratory frame.

Surprisingly enough, the problem of determination of torque exerted on an electric/magnetic dipole moving in an EM field was not solved to the moment in a full extent, and considered only fragmentary (see, e.g., Refs. [3, 4, 6–11]). The latter fact determines the goal of the present paper: to derive explicitly an expression for torque experienced by a moving dipole in an external EM field, to determine the physical meaning of various torque components in the approximation of a small dipole and to clarify the limits on application of the obtained equation.

In Section 2, we present the general expression for the torque exerted on a bunch of charged particles in an arbitrary EM field, summing up the contributions due to the Lorentz force and the force component, emerging due to hidden momentum. Then we achieve a compact representation for this torque in the case of small dipole. In Section 3, we verify the correctness of the obtained expression with a number of model physical problems, and in Section 4, we present the conclusion.

2. TORQUE EXERTED ON A MOVING BUNCH OF CHARGES

The total torque on an electric/magnetic dipole can be presented in the form

$$\mathbf{T} = \int_V (\mathbf{r} \times \mathbf{f}_{total})\, dV, \tag{2}$$

where $\mathbf{f}_{total}$ is the total force density, $\mathbf{r}$ the position vector of any point inside the dipole, and V the volume of dipole.

We mention that, as for the total force on a dipole (see, e.g., [11]), the force density $\mathbf{f}_{total}$ should be presented as the sum

$$\mathbf{f}_{total} = \mathbf{f}_L + \mathbf{f}_h, \tag{3}$$

where $\mathbf{f}_L$ is the density of Lorentz force

$$\mathbf{f}_L = \rho_{total}\mathbf{E} + \frac{1}{c}\mathbf{j}_{total} \times \mathbf{B} \tag{4}$$

(see, e.g., Refs. [3, 5]), and $\mathbf{f}_h$ is the density of force due to hidden momentum contribution, whose physical meaning will be discussed below. Here ρ_{total}, $\mathbf{j}_{total}$ are the total charge density and current density, correspondingly.

Hereinafter we consider the case, when the free charges are absent, so that for a material medium we have the known relationships [3, 6]:

$$\rho = -\nabla \cdot \mathbf{P}, \tag{5}$$

$$J = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}, \tag{6}$$

where $\mathbf{M}$, $\mathbf{P}$ are respectively the magnetization and polarization of medium.

Next problem is to find an expression for the force density $\mathbf{f}_h$, based on the known equation for the force on a magnetic dipole, caused by time variation of hidden momentum [12–14]

$$\mathbf{F}_h = -\frac{1}{c}\frac{d}{dt}\left(\boldsymbol{\mu} \times \mathbf{E}\right), \tag{7}$$

where all quantities in Eq. (7) are evaluated in a laboratory frame. As known, the force contribution (7) is required for the system "material medium plus EM field" to maintain the balance of momenta, as for the first time was pointed out in Ref. [12]. Hence we can conjecture the force density due to variation of hidden momentum in the form:

$$\mathbf{f}_h = -\frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{M} \times \mathbf{E}\right). \tag{8}$$

We point out that the use of partial time derivative in Eq. (8) instead of total time derivative in Eq. (7) is related to the fact (usually omitting in the literature) that the measurement of magnetization (polarization) is carried out at a point *fixed* in a laboratory, whereas the measurement of magnetic (electric) dipole moment of a moving bunch of charges is carried out for a volume V *co-moving* to this bunch. Further, using the operator equality $\frac{\partial}{\partial t} = \frac{d}{dt} - (\mathbf{v} \cdot \nabla)$ (where $\mathbf{v}$ stands for velocity), and taking into account that the volume integral $\int\limits_V (\mathbf{v} \cdot \nabla)(\mathbf{M} \times \mathbf{E}) dV$ via the Gauss theorem can be transformed onto a surface integral (where $\mathbf{M}$ disappears), we obtain that

$$\int\limits_V \mathbf{f}_h dV = -\frac{1}{c}\frac{d}{dt}\left(\boldsymbol{\mu}\times\mathbf{E}\right) = \mathbf{F}_h.$$

Thus, combining Eqs. (3)–(6), (8) we derive the total force density in the form:

$$\mathbf{f}_{total} = -\left(\nabla\cdot\mathbf{P}\right)\mathbf{E} + \left(\nabla\times\mathbf{M}\right)\times\mathbf{B} + \frac{1}{c}\frac{\partial\mathbf{P}}{\partial t}\times\mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{M}\times\mathbf{E}\right). \quad (9)$$

Further substituting Eq. (9) into Eq. (2), we derive the expression for total torque exerted on an electric/magnetic dipole in an external electromagnetic field:

$$\mathbf{T}_{total} = \int\limits_V dV\left[\mathbf{r}\times\left(-(\nabla\cdot\mathbf{P})\mathbf{E} + (\nabla\times\mathbf{M})\times\mathbf{B} + \frac{1}{c}\frac{\partial\mathbf{P}}{\partial t}\times\mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{M}\times\mathbf{E})\right)\right]. \quad (10)$$

In order to clarify the physical meaning of the torque (10) and its components, let us simplify Eq. (10) for a small dipole. This approximation implies, first of all, that the spatial variation of electric and magnetic fields at the location of the dipole can be practically ignored. This statement is expressed in the form of inequalities [15]

$$\frac{\partial E_i}{\partial r_j}\Delta r_j \ll E_i, \quad \frac{\partial B_i}{\partial r_j}\Delta r_j \ll B_i \quad (i, j = 1\ldots 3), \quad (11)$$

where Δr_j is the typical size of the dipole along the dimension j. We can add that in the evaluation of the electric/magnetic fields generated by the dipole, it is additionally adopted that distance r between a dipole and a point of observation is much larger than Δr_j, i.e., $r \gg \Delta r_j$ at all j. However, we stress that for our purpose the requirement (11) is the most important, and its implementation may happen for a dipole of macroscopic size, if the electric and magnetic field slow enough vary in space; of course, for elementary particles processed in the classical way, the inequalities (11) are always fulfilled.

Next, we have to notice that in the evaluation of the torque contributions of Eq. (10) for *small* dipole, we can take the fields $\mathbf{E}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$ to be constant *within the volume* of such a dipole, because any terms, which include partial spatial derivatives of the electric and magnetic field become negligible, when the inequalities (11) are adopted†. Thus, in the subsequent integrals over the volume of dipole,

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† We notice that the approximation of constant fields cannot be adopted in calculation of force acting on a small dipole, because some of the force components are vanishing at $\mathbf{E}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})=$ *constant*, and we have to involve the terms, containing their spatial derivatives (see, e.g., [5, 15]).

the fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are taken to be constant. At the same time, when the motion of dipoles is considered, the variation of electric and magnetic fields along *a trajectory* of a moving dipole, in general, cannot be ignored, so that the approximation of static field becomes inapplicable.

Thus, we apply Eq. (10) to a small electric/magnetic dipole, moving at the velocity $\mathbf{v}$ in the external electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields of a laboratory frame. In such a derivation, we assume that the electric $\mathbf{p} = \int\limits_V \mathbf{P}dV$ and magnetic $\boldsymbol{\mu} = \int\limits_V \mathbf{M}dV$ dipole moments of such dipole are the constant values in its rest frame. With these limitations we obtain for the first term in integrand of Eq. (10)

$$-\int\limits_V \mathbf{r} \times (\nabla \cdot \mathbf{P})\, \mathbf{E}dV = \mathbf{p} \times \mathbf{E}, \tag{12}$$

which describes the contribution to torque due to Coulomb interaction. Eq. (12) can be directly proven in components. For example for the z-component we have:

$$\begin{aligned}
&-E_y \int\limits_V x\left(\frac{\partial P_x}{\partial x}+\frac{\partial P_y}{\partial y}+\frac{\partial P_z}{\partial z}\right) dV + E_x \int\limits_V y\left(\frac{\partial P_x}{\partial x}+\frac{\partial P_y}{\partial y}+\frac{\partial P_z}{\partial z}\right) dV \\
&= -E_y \int\limits_V \left(\frac{\partial (xP_x)}{\partial x} - P_x + \frac{\partial (xP_y)}{\partial y} + \frac{\partial (xP_z)}{\partial z}\right) dV \\
&\quad + E_x \int\limits_V \left(\frac{\partial (yP_x)}{\partial x} + \frac{\partial (yP_y)}{\partial y} - P_y + \frac{\partial (yP_z)}{\partial z}\right) dV \\
&= p_x E_y - p_y E_x = (\mathbf{p} \times \mathbf{E})_z
\end{aligned}$$

Here we have taken into account that the integration of the terms $\partial(r_i P_j)/\partial r_l$ $(i,\ j,\ l = 1 \ldots 3)$ over the volume of the dipole gives the value of P_j on its surface, which is equal to zero.

Next, we evaluate the torque component due to interaction of magnetization currents of a dipole with a magnetic field (second term in the integrand of Eq. (10)). Using the vector identities $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ and $\mathbf{a} \times (\nabla \times \mathbf{b}) = \nabla(\mathbf{a} \cdot \mathbf{b}) - \mathbf{b} \times (\nabla \times \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} - (\mathbf{b} \cdot \nabla)\mathbf{a}$, this term can be presented in the form:

$$\begin{aligned}
&\int\limits_V \mathbf{r} \times ((\nabla \times \mathbf{M}) \times \mathbf{B})\, dV = -\int\limits_V \mathbf{r} \times (\mathbf{B} \times (\nabla \times \mathbf{M}))\, dV \\
&= -\int\limits_V \mathbf{r} \times (\nabla (\mathbf{B} \cdot \mathbf{M}) - (\mathbf{B} \cdot \nabla)\, \mathbf{M} - (\mathbf{M} \cdot \nabla)\, \mathbf{B} - \mathbf{M} \times (\nabla \times \mathbf{B}))\, dV
\end{aligned}$$

$$= \int_V \left[\mathbf{r} \times (\mathbf{B} \cdot \nabla) \mathbf{M} \right] dV. \tag{13}$$

Here, we have taken into account that the integral $\int_V [\mathbf{r} \times \nabla(\mathbf{B} \cdot \mathbf{M})] dV$ can be transformed to the surface integral, where the magnetization $\mathbf{M}$ is vanishing; also we notice that in the adopted approximation ($\mathbf{B} \approx$ constant), the terms $(\mathbf{M} \cdot \nabla)\mathbf{B}$ and $\mathbf{M} \times (\nabla \times \mathbf{B})$ are vanishing, too. Integrating by parts the remaining integral in Eq. (13), we obtain:

$$\int_V \left[\mathbf{r} \times (\mathbf{B} \cdot \nabla) \mathbf{M} \right] dV = \boldsymbol{\mu} \times \mathbf{B}, \tag{14}$$

We again prove this equality in components. For example, for the z-component we have:

$$\begin{aligned}
M_{2z} = & \int_V \left[x \left(B_x \frac{\partial M_y}{\partial x} + B_y \frac{\partial M_y}{\partial y} + B_z \frac{\partial M_y}{\partial z} \right) \right. \\
& \left. - y \left(B_x \frac{\partial M_x}{\partial x} + B_y \frac{\partial M_x}{\partial y} + B_z \frac{\partial M_x}{\partial z} \right) \right] dV \\
= & \int_V \left[\left(B_x \frac{\partial (x M_y)}{\partial x} - B_x M_y + B_y \frac{\partial (x M_y)}{\partial y} + B_z \frac{\partial (x M_y)}{\partial z} \right) \right. \\
& \left. - \left(B_x \frac{\partial (y M_x)}{\partial x} + B_y \frac{\partial (y M_x)}{\partial y} - B_y M_x + B_z \frac{\partial (y M_x)}{\partial z} \right) \right] dV \\
= & \mu_x B_y - \mu_y B_x = (\boldsymbol{\mu} \times \mathbf{B})_z
\end{aligned}$$

Further, we evaluate the term responsible for the interaction of polarization currents of a dipole with a magnetic field (third term in the integrand of Eq. (10)). Here we notice that due to the adopted constancy of proper electric dipole moment, we get $\partial \mathbf{P}/\partial t = 0$ in the rest frame of a dipole. However, for a moving dipole the stationary distribution of its charges yields $d\mathbf{P}/dt = 0$, and hence $\partial \mathbf{P}/\partial t = -(\mathbf{v} \cdot \nabla)\mathbf{P}$. With the latter equality we derive:

$$\begin{aligned}
& -\frac{1}{c} \int_V \mathbf{r} \times ((\mathbf{v} \cdot \nabla) \mathbf{P} \times \mathbf{B}) \, dV \\
& = -\frac{1}{c} \int_V \mathbf{r} \times ((\mathbf{v} \cdot \nabla)(\mathbf{P} \times \mathbf{B})) \, dV = \frac{1}{c} \mathbf{v} \times (\mathbf{p} \times \mathbf{B}).
\end{aligned} \tag{15}$$

(Here we used the equality $(\mathbf{v} \cdot \nabla)\mathbf{P} \times \mathbf{B} = (\mathbf{v} \cdot \nabla)(\mathbf{P} \times \mathbf{B})$, which reflects the adopted constancy of $\mathbf{B}$ within the volume of small dipole).

For example, let us demonstrate the validity of Eq. (15) for the y-component:

$$\int_V \left[z\left(\left(\mathbf{v}\cdot\nabla\right)\left(\mathbf{P}\times\mathbf{B}\right)_x\right) - x\left(\left(\mathbf{v}\cdot\nabla\right)\left(\mathbf{P}\times\mathbf{B}\right)_z\right)\right] dV$$

$$= \int_V \left[z\left(v_x\frac{\partial\left(\mathbf{P}\times\mathbf{B}\right)_x}{\partial x} + v_y\frac{\partial\left(\mathbf{P}\times\mathbf{B}\right)_x}{\partial y} + v_z\frac{\partial\left(\mathbf{P}\times\mathbf{B}\right)_x}{\partial z}\right)\right.$$

$$\left. -x\left(v_x\frac{\partial\left(\mathbf{P}\times\mathbf{B}\right)_z}{\partial x} + v_y\frac{\partial\left(\mathbf{P}\times\mathbf{B}\right)_z}{\partial y} + v_z\frac{\partial\left(\mathbf{P}\times\mathbf{B}\right)_z}{\partial z}\right)\right] dV$$

$$= \int_V \left(v_x\frac{\partial\left[z\left(\mathbf{P}\times\mathbf{B}\right)_x\right]}{\partial x} + v_y\frac{\partial\left[z\left(\mathbf{P}\times\mathbf{B}\right)_x\right]}{\partial y} + v_z\frac{\partial\left[z\left(\mathbf{P}\times\mathbf{B}\right)_x\right]}{\partial z} - v_z\left(\mathbf{P}\times\mathbf{B}\right)_x\right) dV$$

$$- \int_V \left(v_x\frac{\partial\left[x\left(\mathbf{P}\times\mathbf{B}\right)_z\right]}{\partial x} - v_x\left(\mathbf{P}\times\mathbf{B}\right)_z + v_y\frac{\partial\left(x\left(\mathbf{P}\times\mathbf{B}\right)_z\right)}{\partial y} + v_z\frac{\partial\left(x\left(\mathbf{P}\times\mathbf{B}\right)_z\right)}{\partial z}\right) dV$$

$$= \int_V \left[\left(v_x\left(\mathbf{P}\times\mathbf{B}\right)_z - v_z\left(\mathbf{P}\times\mathbf{B}\right)_x\right)\right] dV = -\left[\mathbf{v}\times\left(\mathbf{p}\times\mathbf{B}\right)\right]_y .$$

Here we again take into account that the volume integrals, where the functions of spatial coordinates are subjected to differentiation, can be transformed to the surface integrals, where polarization $\mathbf{P}$ is vanishing.

Finally, addressing to the last term of integrand of Eq. (10) (the hidden momentum contribution), and taking into account that for stationary magnetization $\frac{\partial}{\partial t}(\mathbf{M}\times\mathbf{E}) = -(\mathbf{v}\cdot\nabla)(\mathbf{M}\times\mathbf{E})$, we obtain by analogy with Eq. (15):

$$\mathbf{T}_h = -\frac{1}{c}\int_V \left(\mathbf{r}\times\frac{\partial}{\partial t}\left(\mathbf{M}\times\mathbf{E}\right)\right) dV$$

$$= \frac{1}{c}\int_V \left(\mathbf{r}\times\left(\mathbf{v}\cdot\nabla\right)\left(\mathbf{M}\times\mathbf{E}\right)\right) dV = -\frac{1}{c}\mathbf{v}\times\left(\boldsymbol{\mu}\times\mathbf{E}\right). \qquad (16)$$

Finally, substituting Eqs. (12), (14)–(16) into Eq. (10), we derive the expression for total torque exerted on a small dipole in an EM field:

$$\mathbf{T}_{total} = \mathbf{p}\times\mathbf{E} + \boldsymbol{\mu}\times\mathbf{B} + \frac{1}{c}\mathbf{v}\times\left(\mathbf{p}\times\mathbf{B}\right) - \frac{1}{c}\mathbf{v}\times\left(\boldsymbol{\mu}\times\mathbf{E}\right), \qquad (17)$$

where all quantities are evaluated in a laboratory frame.

The first and second terms in rhs of Eq. (17) look similar to corresponding terms of Eq. (1), though now they include the electric $\mathbf{p}$ and magnetic $\boldsymbol{\mu}$ dipole moments of a moving dipole. The third term is responsible for the interaction of polarization currents of a dipole with the magnetic field, while the fourth term stands for the hidden momentum contribution to the torque on a dipole. To our recollection, the two last terms in Eq. (17) (presented separately by the rhs of Eqs. (15) and (16), correspondingly) seems were not reported before.

Further, we notice that in experiments, the directly measured values of any bunch of charges are the proper electric $\mathbf{p}_0$ and magnetic $\boldsymbol{\mu}_0$ dipole moments. However, this fact does not create any difficulties in comparison of Eq. (17) with experimental results, since the dipole moments $\mathbf{p}$, $\boldsymbol{\mu}$ of the moving dipole are directly related with $\mathbf{p}_0$, $\boldsymbol{\mu}_0$ measured in a laboratory for the same dipole at rest via the known relativistic transformations [6, 16]

$$\mathbf{p} = \mathbf{p}_0 - \frac{(\gamma-1)}{\gamma v^2}(\mathbf{p}_0 \cdot \mathbf{v})\,\mathbf{v} + \frac{\mathbf{v}\times\boldsymbol{\mu}_0}{c}, \tag{18a}$$

$$\boldsymbol{\mu} = \boldsymbol{\mu}_0 - \frac{(\gamma-1)}{\gamma v^2}(\boldsymbol{\mu}_0 \cdot \mathbf{v})\,\mathbf{v} + \frac{\mathbf{p}_0\times\mathbf{v}}{c}, \tag{18b}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor. The last term in rhs of Eq. (18a) describes the known effect of relativistic polarization of a moving magnetic dipole, while the last term in rhs of Eq. (18b) is responsible for the development of the magnetic dipole moment by the moving electric dipole. One can add that the second terms in the rhs of both of these equations take into account the scale contraction effect along the velocity $\mathbf{v}$ of the moving dipole. Thus all of the mentioned relativistic effects are accounted for, by the first and second terms of rhs of Eq. (17).

Next, it is worth to mention the results of Ref. [10], where Namias considered the torque on an electric dipole ($\mathbf{p}_0 \neq 0$, $\boldsymbol{\mu}_0 = 0$) and torque on a magnetic dipole ($\mathbf{p}_0 = 0$, $\boldsymbol{\mu}_0 \neq 0$) separately. Applying the electric-charge model and magnetic-charge model, he subsequently derived the expressions for torque as follows (in Gaussian units):

$$\mathbf{T}_p = \mathbf{p}\times\mathbf{E} + \frac{1}{c}\mathbf{p}\times(\mathbf{v}\times\mathbf{B}) \quad (p_0 \neq 0,\ \boldsymbol{\mu}_0 = 0), \tag{19a}$$

and

$$\mathbf{T}_\mu = \boldsymbol{\mu}\times\mathbf{B} - \frac{1}{c}\boldsymbol{\mu}\times(\mathbf{v}\times\mathbf{E}) \quad (p_0 = 0,\ \boldsymbol{\mu}_0 \neq 0). \tag{19b}$$

Thus, Namias did not formulate the general problem, which would be geared to obtain the torque on a moving compact dipole with $p_0 \neq 0$, $\boldsymbol{\mu}_0 \neq 0$, where Eq. (17) has been derived. At the same time, to make a comparison of Eq. (17) with the results of Ref. [10], we can apply Eq. (17) to the particular cases $\mathbf{p}_0 \neq 0$, $\boldsymbol{\mu}_0 = 0$ and $\mathbf{p}_0 = 0$, $\boldsymbol{\mu}_0 \neq 0$.

In the former case Eq. (17) reads:

$$\begin{aligned}\mathbf{T}_{total} &= \mathbf{p}\times\mathbf{E}+\frac{1}{c}\left(\mathbf{p}_0\times\mathbf{v}\right)\times\mathbf{B}+\frac{1}{c}\mathbf{v}\times\left(\mathbf{p}\times\mathbf{B}\right)\\ &\quad -\frac{1}{c^2}\mathbf{v}\times\left(\left(\mathbf{p}_0\times\mathbf{v}\right)\times\mathbf{E}\right)\quad\left(\mathbf{p}_0\neq 0,\ \boldsymbol{\mu}_0=0\right),\end{aligned}\tag{20a}$$

where we have used Eq. (18b). Using the Jacobi identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})=0$ and the equality $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$, we can present the equation by Namias (19a) in the form more convenient for further analysis:

$$\mathbf{T}_p=\mathbf{p}\times\mathbf{E}+\frac{1}{c}\left(\mathbf{p}\times\mathbf{v}\right)\times\mathbf{B}+\frac{1}{c}\mathbf{v}\times\left(\mathbf{p}\times\mathbf{B}\right).\tag{21a}$$

We see that, in general, Eq. (21a) disagrees with our Eq. (20a) in the terms of order $(v/c)^2$ and higher. We add that the physical meaning of the torque component $-\frac{1}{c^2}\mathbf{v}\times((\mathbf{p}_0\times\mathbf{v})\times\mathbf{E})$, which is absent in Eq. (21a), is clarified in Section 3, where our Eq. (17) is applied to various physical situations.

Next, consider the particular case $\mathbf{p}_0=0$, $\boldsymbol{\mu}_0\neq 0$, where Eq. (17) takes the form

$$\begin{aligned}\mathbf{T}_{total} &= \boldsymbol{\mu}\times\mathbf{B}+\frac{1}{c}\left(\mathbf{v}\times\boldsymbol{\mu}_0\right)\times\mathbf{E}-\frac{1}{c}\mathbf{v}\times\left(\boldsymbol{\mu}\times\mathbf{E}\right)\\ &\quad +\frac{1}{c^2}\mathbf{v}\times\left(\left(\mathbf{v}\times\boldsymbol{\mu}_0\right)\times\mathbf{B}\right)\quad\left(\mathbf{p}_0=0,\ \boldsymbol{\mu}_0\neq 0\right).\end{aligned}\tag{20b}$$

Here we also have used Eq. (18a). Applying again the Jacobi identity, we transform the Eq. (19b) by Namias to the form convenient for comparison with Eq. (20b):

$$\mathbf{T}_\mu=\boldsymbol{\mu}\times\mathbf{B}+\frac{1}{c}\left(\mathbf{v}\times\boldsymbol{\mu}\right)\times\mathbf{E}-\frac{1}{c}\mathbf{v}\times\left(\boldsymbol{\mu}\times\mathbf{E}\right).\tag{21b}$$

Now we see that the Namias Eq. (22) coincides with our Eq. (20b) only to first order in (v/c) and, in general, disagrees with Eq. (20b) in the terms of order $(v/c)^2$ and higher.

Thus we conclude that the approach by Namias to the derivation of torque on a moving dipole in the framework of the electric-charge and magnetic-charge models, correspondingly, is not fully correct, because the relativistic transformations (18) for the electric and magnetic dipole moments are not properly accounted for the electric-charge and magnetic-charge models, when these models are analysed separately. This can explain the deviation of Namias Eqs. (21a), (21b) from the respective Eqs. (20a), (20b) in the order $(v/c)^2$ and higher.

In the next section, we apply the newly derived Eq. (17) to calculation of torque on small electric and magnetic dipoles, paying special attention to the contribution of the third and fourth terms of this equation.

3. TORQUE ON MOVING MAGNETIC AND ELECTRIC DIPOLES: ILLUSTRATIVE EXAMPLES

In this section, we verity Eq. (17) for some selected problems.

First, we consider the motion of magnetic dipole in a static constant electric field $\mathbf{E}$ for the configuration, when the velocity of dipole $\mathbf{v}$ and vector $\mathbf{E}$ are both parallel to the axis x, while the proper magnetic dipole moment $\boldsymbol{\mu}_0$ is parallel to the axis y (Fig. 1(a)).

For this configuration the relativistic polarization

$$\mathbf{p} = \frac{\mathbf{v} \times \boldsymbol{\mu}_0}{c} \tag{22}$$

of magnetic dipole emerges, which gives the torque component

$$\mathbf{T}_{rel} = \mathbf{p} \times \mathbf{E} = \frac{1}{c} \left(\mathbf{v} \times \boldsymbol{\mu}_0 \right) \times \mathbf{E}, \tag{23}$$

and for the case of Fig. 1(a), the torque (23) is directed against the axis y.

One more torque component stems from the fourth term of Eq. (17) (hidden momentum contribution), which for parallel vectors

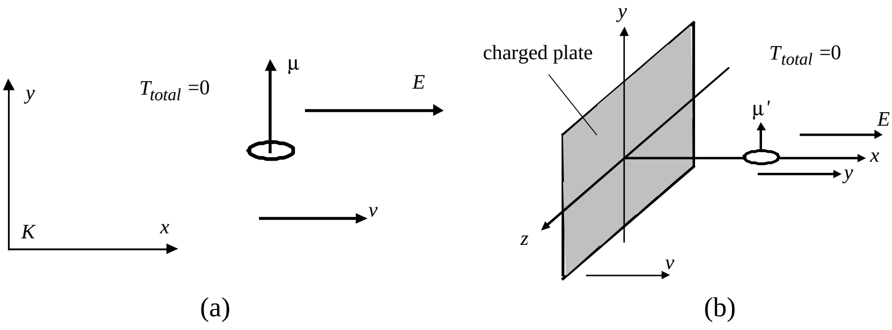

Figure 1. (a) Magnetic dipole $\boldsymbol{\mu}$ lying in the positive y-direction moves at the constant velocity $\mathbf{v}$ along the lines of constant electric field $\mathbf{E}$ (the axis x). Such an electric field can be produced by a charged plate, moving synchronously with the dipole at the same velocity v along the axis x. (b) Since in the rest frame of the plate and magnetic dipole the torque on the dipole is equal to zero, it must be also equal to zero in the frame of observation K, and Eq. (17) confirms this result.

$\mathbf{v}$ and $\mathbf{E}$ reads:

$$\mathbf{T}_h = -\frac{1}{c}\mathbf{v}\times(\boldsymbol{\mu}\times\mathbf{E}) = \frac{1}{c}\mathbf{E}\times(\mathbf{v}\times\boldsymbol{\mu}) + \frac{1}{c}\boldsymbol{\mu}\times(\mathbf{E}\times\mathbf{v}) = -\frac{1}{c}\left(\mathbf{v}\times\boldsymbol{\mu}\right)\times\mathbf{E}, \quad (24)$$

where we have used the Jacobi identity. Taking also into account that for orthogonal vectors $\boldsymbol{\mu}_0$, $\mathbf{v}$, and $p_0 = 0$, $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ (see Eq. (18b)), we derive that the torque components (23) and (24) mutually cancel each other, so that the net torque on the magnetic dipole is equal to zero.

We point out that this result for the problem of Fig. 1(a), where the vectors $\mathbf{E}$, $\mathbf{v}$ are collinear to each other, is a single possibility from the relativistic viewpoint. Indeed, we can suppose that the constant electric field $\mathbf{E}$ along the axis x is created by a large rectangular homogeneously charged plate, lying in the plane yz (see Fig. 1(b)) and the boundaries of the plate are very far from the magnetic dipole. What is more, we can admit the case, where the plate moves along the axis x at the same constant velocity $\mathbf{v}$, like the magnetic dipole. For such configuration no change of the field occurs with the motion of place, because $\mathbf{B}' \sim \mathbf{v}\times\mathbf{E} = 0$, and $\mathbf{E}' = \mathbf{E}$. Further we observe that in the rest frame of the plate and dipole, no torque acts on the dipole due to a resting plate. Therefore, in any other inertial frame, where both the plate and dipole move along the axis x at any constant velocity, the torque must be equal to zero, too.

One can add that the problem of Fig. 1(a) is closely related to the problem considered in [18] on the interaction of point-like charge q and magnetic dipole $\boldsymbol{\mu}$, which are both rest in some inertial frame K', and the direction of $\boldsymbol{\mu}$ is orthogonal to the line joining charge and dipole. In this frame the mutual force between charge and magnetic dipole is equal to zero, and no torque is exerted on the dipole by the resting charge. Then Mansuripur considers the situation, where the frame K' is moving in the laboratory frame K at the constant velocity v along the line, joining charge and dipole (the axis x in Fig. 1). In these conditions he shows that the application of Lorentz force law (4) yields a non-vanishing torque exerted on the moving magnetic dipole by the moving charge (presented by our Eq. (23), which obviously represents a non-adequate result, since in the proper frame of charge and dipole K', the torque is equal to zero. On the other hand, when the Einstein-Laub formula [19, 20] is applied, both the force and the torque are equal to zero in the frames K' and K. Based on this result, Mansuripur concluded that for material media, the Lorentz force law must be abandoned in favor of the Einstein-Laub law.

However, he did not take into account the hidden momentum contribution (24), which recovers the consistency with relativistic requirements of the Lorentz force approach.

In the next illustrative example, we continue to analyse physical implications of Eq. (17), again focusing attention on its last term, presented separately by Eq. (16), and ask a question as follows: can we prescribe the hidden momentum to a dipole, where its magnetic dipole moment has a purely relativistic origin, i.e.,

$$\boldsymbol{\mu} = \frac{1}{c}\left(\mathbf{p}_0 \times \mathbf{v}\right), \tag{25}$$

and $\boldsymbol{\mu}_0 = 0$?

In order to answer this question, we substitute Eq. (25) into Eq. (16) and derive:

$$\mathbf{T}_h = -\frac{1}{c^2}\mathbf{v}\times\left(\left(\mathbf{p}_0\times\mathbf{v}\right)\times\mathbf{E}\right) = \frac{1}{c^2}\mathbf{v}\times\left(\mathbf{E}\times\left(\mathbf{p}_0\times\mathbf{v}\right)\right) = \frac{1}{c^2}\mathbf{v}\times\mathbf{p}_0\left(\mathbf{E}\cdot\mathbf{v}\right). \tag{26}$$

This torque component, presented also in the above Eq. (20a), shows that it is not zero, when the vectors $\mathbf{p}_0$, $\mathbf{v}$ are not collinear to each other (otherwise the magnetic dipole moment (25) disappears), and $\mathbf{E}$ has a non-vanishing projection into $\mathbf{v}$. Next we have to check, whether the torque component (26) is adequate from the physical viewpoint. In this respect the most interesting case is realized, when the vectors $\mathbf{p}_0$, $\mathbf{E}$ are collinear to each other and, for simplicity, $\mathbf{B} = 0$ in the frame of observation.

This situation can be modelled by the following problem (see Fig. 2(a)). There is a parallel plate charged capacitor with the plates lying in the plane xz, and its inner electric field $\mathbf{E}$ is directed along the axis y. An electric dipole with the proper moment $\mathbf{p}_0$ to be parallel to $\mathbf{E}$ is moving inside the capacitor with the constant velocity $\mathbf{v}$, constituting the angle α with the axis y. Due to the constancy of electric field, the

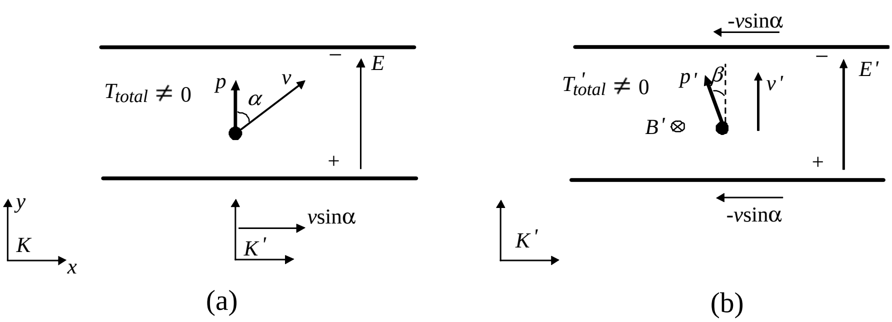

Figure 2. (a) Electric dipole $\mathbf{p}$ moves in the laboratory frame K inside a parallel plate charged capacitor at the constant velocity $\mathbf{v}$, lying in the plane xy and constituting the angle α with the axis y. (b) The same problem, as seen by an observer in the inertial frame K'.

force on electric dipole $\mathbf{p}$ is equal to zero, while the torque is presented as the sum of two components:

$$\mathbf{T} = \mathbf{p} \times \mathbf{E} - \frac{1}{c}\mathbf{v} \times (\boldsymbol{\mu} \times \mathbf{E}) = \mathbf{p} \times \mathbf{E} - \frac{1}{c^2}\mathbf{v} \times \mathbf{p}_0 (\mathbf{E} \cdot \mathbf{v}), \tag{27}$$

where Eq. (26) has been used. Here we also take into account that the electric dipole moment $\mathbf{p}$ of the moving dipole is no longer parallel to the vector $\mathbf{E}$ (as it was the case for the vector $\mathbf{p}_0$), and has a non-vanishing component onto the axis x due to transformation (18a), making the product $\mathbf{p} \times \mathbf{E}$ to be different from zero. From the physical viewpoint, the change of spatial orientation of $\mathbf{p}$ in comparison with $\mathbf{p}_0$ is explained by the scale contraction effect for the dipole along its velocity $\mathbf{v}$. Indeed, designating the length of the dipole as Δl, we obtain that its projection onto the vector $\mathbf{v}$ is contracted by $\sqrt{1 - v^2/c^2}$ times ($\Delta l \cos\alpha\sqrt{1 - v^2/c^2}$), whereas its projection onto the orthogonal direction remains unchanged ($\Delta l \sin\alpha$). As a result, the entire dipole experiences a spatial turn at some angle β, which is straightforwardly calculated to the accuracy c^{-2}:

$$\beta \approx \frac{v^2}{2c^2} \sin\alpha \cos\alpha. \tag{28}$$

Taking also into account that the second term in rhs of Eq. (27) can be presented in the form

$$\mathbf{T}_h = -\frac{1}{c^2}\mathbf{v} \times \mathbf{p}_0 (\mathbf{E} \cdot \mathbf{v}) = \hat{\mathbf{z}}\frac{v^2}{c^2} pE \sin\alpha \cos\alpha,$$

we obtain the entire torque as

$$\mathbf{T} = -\hat{\mathbf{z}} pE \sin\beta + \hat{\mathbf{z}}\frac{v^2}{c^2} pE \sin\alpha \cos\alpha, \tag{29}$$

where $\hat{\mathbf{z}}$ is the unit vector along the axis z.

The existence of both torque components in Eq. (29) can be understood via the relativistic requirements, if we introduce into consideration an inertial observer K', moving along the axis x with the constant velocity $v \sin\alpha$ in a laboratory frame. In the frame K' the electric dipole $\mathbf{p}'$ moves only along the axis y with the velocity $\mathbf{v}' = v'\hat{\mathbf{y}}$ (which can be found from the Einetein law of velocity composition); the capacitor plates move at the velocity $-v \sin\alpha$ along the axis x and generate the electric $\mathbf{E}' = E'\hat{\mathbf{y}}$ and magnetic $\mathbf{B}' = -B'\hat{\mathbf{z}}$ fields, which can be found via the Lorentz transformation for electromagnetic fields [3]. The directions of vectors $\mathbf{v}'$, $\mathbf{p}'$, $\mathbf{E}'$, $\mathbf{B}'$, as seen in the frame K', are shown in Fig. 2(b). Here we have taken into account the Thomas-Wigner rotation of coordinate axes of the proper frame of electric dipole with respect to the axes of the frame K' [3], which yields a spatial turn of the vector $\mathbf{p}'$ with respect $\mathbf{p}_0$, and the related angle of rotation coincides with the angle (25) to the accuracy of calculations c^{-2}. As a result, an observer in K' frame fixes the torque component $\mathbf{p}' \times \mathbf{E}'$, which to the accuracy c^{-2} coincides with the first term in rhs of Eq. (29).

In addition, in the frame K' the torque on the dipole contains one more component, described by the third term in rhs of Eq. (17),

$$\mathbf{T}'_B = \frac{1}{c}\mathbf{v}' \times \left(\mathbf{p}' \times \mathbf{B}'\right) = \frac{v'p'B'}{c}\hat{\mathbf{z}}. \tag{30}$$

The correctness of Eq. (30) can be directly verified via the Lorentz force law, if we imagine the electric dipole as two mechanically bound charges $-q$ and $+q$ separated by the distance Δl. Then, according to the Lorentz force law, the magnetic force on the charge $+q$ $(q(\mathbf{v}' \times \mathbf{B}')/c = -q(v'B')/c(\hat{\mathbf{y}} \times \hat{\mathbf{z}}))$ is directed in the negative x-direction, whereas the magnetic force on the opposite charge $(-q(\mathbf{v}' \times \mathbf{B}')/c = q(v'B')/c(\hat{\mathbf{y}} \times \hat{\mathbf{z}}))$ lies in the positive x-direction. Hence the resultant torque is equal to

$$T_B = \Delta l' q v' B'/c\hat{\mathbf{z}} = p'v'B'/c\hat{\mathbf{z}},$$

which coincides with the result (30). Concurrently, we have found that the third term of Eq. (17), not reported before, does agree with the Lorentz force law.

Further taking into account that in the sufficient accuracy of calculations c^{-2}, $v' \approx v\cos\alpha$, $B' \approx vE\sin\alpha/c$, $p' \approx p_0$, we see that the torque component (30) fixed in the frame K' and resulting from the third term of Eq. (17), is equal to the second torque component in rhs of Eq. (29), resulting from the fourth term of Eq. (17), as detremined in the frame K. We add that for the chosen model of electric dipole (two mechanically bound charges $-q$ and $+q$), the origin of this torque in the laboratory frame K has a non-electromagnetic origin, and is defined by the mechanical stresses in the moving dipole with their further transformation to the frame of observation. Here we omit the corresponding derivation, which is similar to the analysis of mechanical stresses in moving dipoles, applied in Ref. [11].

4. CONCLUSION

In this paper, we considered the torque exerted on an electric/magnetic dipole moving in an external EM field. In our approach, we applied the Lorentz force law (4) and additionally involved the contribution to total torque, emerging due to hidden momentum of a dipole. In the framework of this approach, we obtained the general Eq. (10) and reduced it to the form (17) for the case of a compact electric/magnetic dipole and slow spatial variation of the electric $\mathbf{E}(\mathbf{r})$ and magnetic $\mathbf{B}(\mathbf{r})$ fields, when both fields can be taken constant within the volume of the dipole due to the inequalities (11).

Comparing Eq. (17) with other known expressions for the torque on an electric/magnetic dipole [3, 4, 6–11], we notice that the first and second terms in rhs of this equation keep the form of the known components of torque exerted on a resting dipole (compare with Eq. (1)). However, we stress that, unlike Eq. (1), the electric $\mathbf{p}$ and magnetic $\boldsymbol{\mu}$ dipole moments represent the result of relativistic transformation of corresponding proper moments to the frame of observation. The third and fourth terms in rhs of Eq. (17), as we are aware, were not reported to the moment.

The illustrative examples, considered in Section 3, confirm the validity of Eq. (17). In these examples, we focused our attention to the analysis of problems, where the third and fourth terms of this equation explicitly manifest themselves. In particular, in the illustrative example presented in Fig. 1, we have shown that the torque contribution (16) due to hidden momentum is strongly required to provide the consistency of Eq. (17) with repativistic requirements and, in particular, invalidates the conclusion made in Ref. [18] on the non-applicability of Lorentz force approach to material media. Further, analyzing the illustrative example of Fig. 2, we confirmed the validity of the torque component $\frac{1}{c}\mathbf{v} \times (\mathbf{p} \times \mathbf{B})$ (the third term of Eq. (17)), and also found that the hidden momentum contribution should be prescribed to magnetic dipoles of a purely relativistic origin (25).

Discussing the limits of applicability of Eq. (17), we first highlight the inequalities (11), which allow us to define the typical size of a dipole in the given EM field, when the approximation of small dipole is relevant. In these conditions we omit the contributions to torque, caused by the non-vanishing spatial partial derivatives of EM field. Besides, we remind that in the derivation of Eq. (17) we supposed the stationary distribution of charges/currents in the moving dipole. In general, our approach can be straightforwardly extended to the case of non-stationary charges/current distributions inside the dipole. However, this problem falls outside of the scope of the present paper.

Finally, we would like to stress that in this paper we focused our attention to the derivation of expression for the torque, but not on the motional equation for magnetic/electric dipole moments in an external EM field. For the case of a point-like dipole, the motion equation of the proper magnetic dipole moment $\boldsymbol{\mu}_0 = \boldsymbol{\mu}_0(t)$ is described by the BMT equation, which can be also generalized to the case of non-vanishing proper electric dipole moment $\mathbf{p}_0$ (see, e.g., Ref. [16]). In practical applications of elementary particle physics, the motion equation for particle's spin in its *rest frame* (or for related proper magnetic dipole moment $\boldsymbol{\mu}_0$) is the most convenient way. At the same time, we are confident that in the framework of classical electrodynamics the obtained Eqs. (10) and (17) derived in a *laboratory frame* are useful and allow us to better understand the origin of torque exerted on moving compact bunches of charges.

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