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ELECTROMAGNETIC FIELDS OF A SHORT ELECTRIC DIPOLE IN FREE SPACE — REVISITED

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Abstract—Maxwell's equations specify that electromagnetic radiation fields are generated by accelerating charges. However, the electromagnetic radiation fields of an accelerating charge are seldom used to derive the electromagnetic fields of radiating systems. In this paper, the equations pertinent to the electromagnetic fields generated by accelerating charges are utilized to evaluate the electromagnetic fields of a current path of length l for the case when a pulse of current propagates with constant velocity. According to these equations, radiation is generated only at the end points of the channel where charges are being accelerated or decelerated. The electromagnetic fields of a short dipole are extracted from these equations when $r \gg l$, where r is the distance to the point of observation. The speed of propagation of the pulse enters into the electromagnetic fields only in the terms that are second order in l and they can be neglected in the dipole approximation. The results illustrate how the radiation fields emanating from the two ends of the dipole give rise to field terms varying as $1/r$ and $1/r^2$, while the time-variant stationary charges at the ends of the dipole contribute to field terms varying as $1/r^2$ and $1/r^3$.

1. INTRODUCTION

Short electric dipole is a classical text book example used to illustrate the nature of radiation fields in electromagnetism [1]. The standard procedure to derive the electromagnetic fields of a dipole is to first derive the vector and scalar potential associated with the time varying current and subsequently extract the electromagnetic fields from these potentials.

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Maxwell's equations predict that electromagnetic radiation fields are generated only when electric charges are accelerated. However, to the best of our knowledge electromagnetic fields generated by accelerating charges have never been used to derive the electromagnetic fields of a dipole. Here we start with a current channel of length l through which a current pulse propagates with constant speed. The electric and magnetic fields pertinent to this system are evaluated using the field equations corresponding to accelerating charges. Electromagnetic fields corresponding to a short electric dipole are extracted from the resulting equations when $r \gg l$, where r is the distance to the point of observation. When using this procedure, one can observe that electromagnetic radiation emanates from the end points of the dipole where the charges are being accelerated or decelerated. The radiation fields generated at the two ends of the dipole give rise to field terms that vary as $1/r$ and $1/r^2$, while the time varying stationary charges at the ends of the dipole contribute to field terms varying as $1/r^2$ and $1/r^3$, where r is the distance from the source to the point of observation. The final expressions for the dipole fields are, of course, identical to those obtained using the standard technique.

In the derivation of the electromagnetic fields of short electric dipole it is usually assumed that the current flowing along the axis of the dipole is constant. In principle, this assumes instantaneous transfer of information along the dipole axis. In the present derivation, this approximation is relaxed and it is assumed that the current flowing in the dipole propagates with a finite speed along the dipole axis. As shown in this paper, this speed enters into the dipole fields only in the terms that are second order in l, where l is the length of the short electric dipole.

When calculating electromagnetic fields from extended sources, such as lightning return strokes, the channel is divided into small elements and the contribution from each element is determined by assuming that it is equivalent to a short electric dipole [2]. The physical problem is somewhat similar to the one under consideration in this paper because the current enters into the dipole from one end, travels along it, and exits from the other end. In this paper, the conditions that have to be satisfied by the length of the channel element, the current and its speed of propagation for the channel element to be regarded as a short electric dipole are investigated.

2. ELECTROMAGNETIC FIELDS OF A MOVING CHARGE

The theory of electromagnetic fields generated by moving charges is described in any standard text book on electromagnetic theory, and it suffices to quote the results directly [1]. The geometry relevant to the problem under consideration is depicted in Figure 1. A charged particle is moving with speed $\mathbf{u}$ and acceleration $\dot{\mathbf{u}}$. We assume that the direction of $\mathbf{u}$ does not change with time; that is, both $\mathbf{u}$ and $\dot{\mathbf{u}}$ are acting in the same direction. The electric field produced by this charge at point P (with $\boldsymbol{\beta} = \frac{\mathbf{u}}{c}$ and $\mathbf{a_r} = \frac{\mathbf{r}}{r}$) is given by

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi\varepsilon_o r^2}\frac{1}{(1-\boldsymbol{\beta}\cdot\mathbf{a_r})^3}(\mathbf{a_r}-\boldsymbol{\beta})\left(1-\beta^2\right) \\ &\quad +\frac{q}{4\pi\varepsilon_o}\frac{1}{cr(1-\boldsymbol{\beta}\cdot\mathbf{a_r})^3}\left[\mathbf{a_r}\times\left(\mathbf{a_r}\times\dot{\boldsymbol{\beta}}\right)\right] \qquad (1)\end{aligned}$$

$$\begin{aligned}\mathbf{B} &= \frac{q}{4\pi\varepsilon_o c}\frac{1}{r^2(1-\boldsymbol{\beta}\cdot\mathbf{a_r})^3}(\boldsymbol{\beta}\times\mathbf{a_r})\left(1-\beta^2\right) \\ &\quad +\frac{q}{4\pi\varepsilon_o}\frac{1}{c^2(1-\boldsymbol{\beta}\cdot\mathbf{a_r})^3 r}\left\{\mathbf{a_r}\times\left[\mathbf{a_r}\times\left(\mathbf{a_r}\times\dot{\boldsymbol{\beta}}\right)\right]\right\} \qquad (2)\end{aligned}$$

Note that the expressions for $\mathbf{E}$ and $\mathbf{B}$ both consist of two terms. The second term, which depends on the acceleration of the charge, is

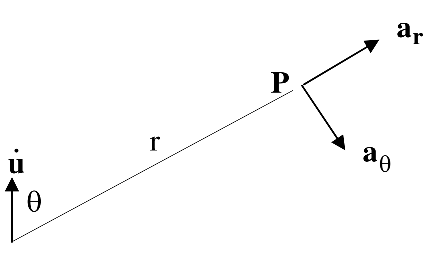

Figure 1. Definition of the parameters that appear in Equations (1) and (2).

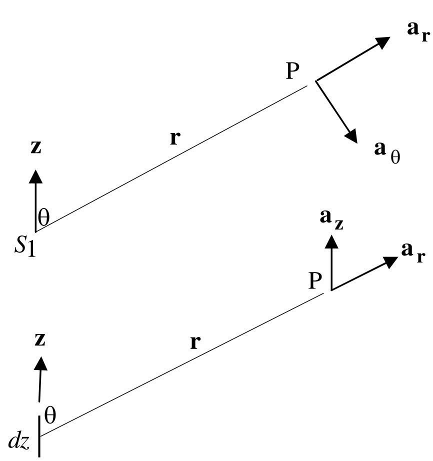

Figure 2. Geometry relevant to the parameters in Equations (3) and (4) (top diagram) and (5) and (6) (bottom diagram).

the radiation field and the first term is called the velocity field. Note that when the term for the velocity field becomes zero, the speed of propagation of the charge is equal to the speed of light.

Consider the geometry shown in the top diagram of Figure 2. A pulse of current originates at point S_1 and travels along the z-axis with constant speed and without any attenuation or dispersion. At the initiation of the current, charges will be accelerated from rest to a speed u. Once they attain this speed, they travel with constant velocity along the z-axis. The acceleration of charge at S_1 generates a radiation field and the uniform propagation of charge along the z-axis generates a velocity field. Recently, using Equations (1) and (2), Cooray and Cooray [3] derived expressions for the radiation field produced by the acceleration of charge at S_1 and for the velocity field produced by the uniform motion. According to their results, the expressions for the electric and magnetic radiation fields generated by the charge acceleration are

$$\mathbf{e}_{rad}(t) = \frac{i(t-r/c)u\sin\theta}{4\pi\varepsilon_o c^2 r}\frac{1}{\left[1-\dfrac{u\cos\theta}{c}\right]}\mathbf{a}_{\boldsymbol{\theta}} \tag{3}$$

$$\mathbf{b}_{rad}(t) = \frac{i(t-r/c)u\sin\theta}{4\pi\varepsilon_o c^3 r}\frac{1}{\left[1-\dfrac{u\cos\theta}{c}\right]}\mathbf{a}_{\boldsymbol{\varphi}} \tag{4}$$

respectively. In these equations, $i(t)$ is the temporal variation of the current emanating from S_1.

Second, consider a spatial element of length dz through which a current pulse $i(t)$ is moving with speed u (see the diagram at the bottom of Figure 2). The velocity fields generated by this element at point P are given by [3]

$$d\mathbf{e}_{vel} = \frac{i(t-r/c)dz}{4\pi\varepsilon_o r^2\left[1-\dfrac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\left[\frac{\mathbf{a_r}}{u}-\frac{\mathbf{a_z}}{c}\right] \tag{5}$$

$$d\mathbf{b}_{vel} = \frac{i(t-r/c)dz}{4\pi\varepsilon_o r^2 c^2\left[1-\dfrac{u}{c}\cos\theta\right]^2}\left[1-\frac{u^2}{c^2}\right]\mathbf{a}_{\boldsymbol{\varphi}} \tag{6}$$

In the next section, these equations together with Coulomb's law are used to derive the electric and magnetic fields of a current channel of length l for the case that a current pulse propagates with constant velocity.

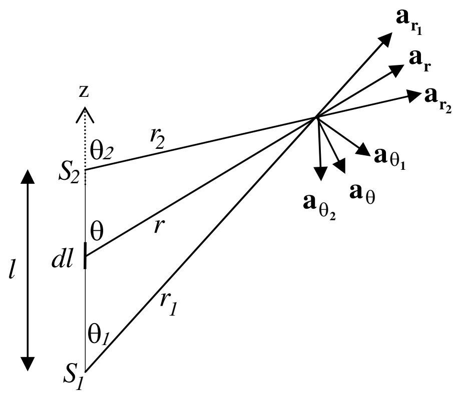

Figure 3. Geometry used in deriving the electromagnetic fields of a current channel.

3. ELECTROMAGNETIC FIELDS GENERATED BY THE CURRENT CHANNEL

The geometry under consideration is shown in Figure 3. A current pulse originates at point S_1 and it travels without attenuation or dispersion towards S_2. At S_2, the current is terminated. The total electric field at point P, generated by this process has five components. They are as follows: (i) the radiation field generated from S_1 as the charge accelerates when the current is initiated, (ii) the radiation field generated from S_2 during the charge deceleration as the current is terminated, (iii) the electrostatic field generated by the negative charge accumulated at S_1 when the positive charge travels towards S_2, (iv) the electrostatic field generated by the accumulation of positive charge at S_2, and (v) the velocity field generated as the current pulse moves along the element. The magnetic field generated by the element consists of three terms, namely, two radiation fields generated by S_1 and S_2, and the velocity field generated as the current propagates along the path. Let us now write down the expressions for these field components.

3.1. The Electric Radiation Field Generated from S_1

Let us assume that the current pulse leaving S_1 can be represented by $I(t)$. The radiation field at point P is given by Equation (3), and with the geometry under consideration here, one can rewrite this as

$$\mathbf{e}_{rad,S_1} = \frac{I(t - r_1/c)u \sin\theta_1}{4\pi\varepsilon_o c^2 r_1} \frac{1}{\left[1 - \frac{u\cos\theta_1}{c}\right]} \mathbf{a}_{\boldsymbol{\theta}_1} \tag{7}$$

3.2. The Electric Radiation Field Generated from S_2

The radiation field generated from S_2 as the charges decelerate is given by

$$\mathbf{e}_{rad,S_2} = -\frac{I(t - l/u - r_2/c)u \sin\theta_2}{4\pi\varepsilon_o c^2 r_2} \frac{1}{\left[1 - \frac{u\cos\theta_2}{c}\right]} \mathbf{a}_{\boldsymbol{\theta}_2} \tag{8}$$

3.3. The Static Field Generated by the Accumulation of Charge at S_1

The charge accumulation at S_1 is equal to the integral of the current, and the field component generated by the charges is given by

$$\mathbf{e}_{stat,S_1} = -\frac{\int_0^{t-r_1/c} I(\xi)d\xi}{4\pi\varepsilon_o r_1^2} \mathbf{a}_{\mathbf{r_1}} \tag{9}$$

3.4. The Static Field Generated by the Accumulation of Positive Charge at S_2

The component of the static field generated by the accumulation of positive charge at S_2 is given by

$$\mathbf{e}_{stat,S_2} = \frac{\int_0^{t-l/u-r_2/c} I(\xi)d\xi}{4\pi\varepsilon_o r_2^2} \mathbf{a}_{\mathbf{r_2}} \tag{10}$$

3.5. The Velocity Field Generated as the Current Pulse Propagates along the Channel Element

The component attributable to the velocity field generated as the current pulse propagates along the channel element can be written directly using Equation (5). The result is

$$\mathbf{e}_{vel} = \int_0^l \frac{I(t - \xi/u - r/c)\left\{1 - \frac{u^2}{c^2}\right\}}{4\pi\varepsilon_o r^2 \left[1 - \frac{u}{c}\cos\theta\right]^2} \left[\frac{\mathbf{a_r}}{u} - \frac{\mathbf{a_z}}{c}\right] d\xi \tag{11}$$

Note, though, that in writing down this equation, we have assumed that the current pulse does not vary as it travels along the element l.

3.6. Magnetic Radiation Field Generated from S_1

The magnetic radiation field generated from S_1 is given by

$$\mathbf{b}_{rad,S_1} = \frac{I(t - r_1/c)u \sin\theta_1}{4\pi\varepsilon_o c^3 r_1} \frac{1}{\left[1 - \dfrac{u\cos\theta_1}{c}\right]} \mathbf{a}_{\boldsymbol{\varphi}} \tag{12}$$

Note that the magnetic field is in the azimuthal direction.

3.7. Magnetic Radiation Field Generated from S_2

The magnetic radiation field generated from S_2 is given by

$$\mathbf{b}_{rad,S_2} = -\frac{I(t - l/u - r_2/c)u \sin\theta_2}{4\pi\varepsilon_o c^3 r_2} \frac{1}{\left[1 - \dfrac{u\cos\theta_2}{c}\right]} \mathbf{a}_{\boldsymbol{\varphi}} \tag{13}$$

3.8. Magnetic Velocity Field Generated as the Current Pulse Propagate along the Channel Element

The velocity field generated as the current pulse propagate along the channel element is given by

$$\mathbf{b}_{vel} = \int_0^l \frac{I(t - \xi/u - r/c)\left\{1 - \dfrac{u^2}{c^2}\right\}\sin\theta}{4\pi\varepsilon_o r^2 c^2 \left[1 - \dfrac{u}{c}\cos\theta\right]^2} \mathbf{a}_{\boldsymbol{\varphi}} d\xi \tag{14}$$

Again note that, in writing down this equation, it was assumed that the current pulse does not vary as it travels along the element l.

The field components given by Equations (7) to (14) provide a complete description of the electric and magnetic fields generated by the current channel. In the next section, we will utilize these fields in conjunction with appropriate approximations to derive the electric and magnetic fields of a dipole.

4. ELECTROMAGNETIC FIELDS GENERATED BY A SHORT ELECTRIC DIPOLE

The geometry for the problem under consideration, including the relevant geometrical parameters, is shown in Figure 4. Since the dipole fields are usually given in the frequency domain, let us assume that the current waveform in the current channel is given by

$$i(t) = I_o e^{j\omega t} \tag{15}$$

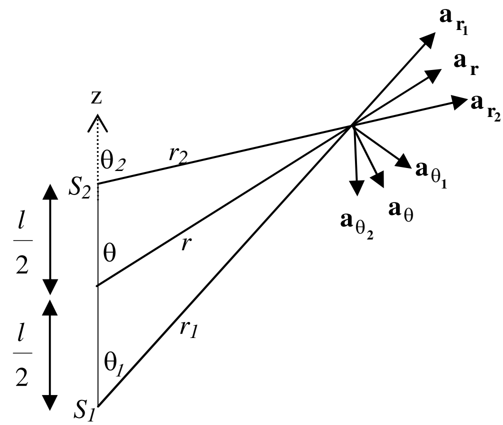

Figure 4. Geometry pertinent to the derivation of the electromagnetic fields of a current element.

where ω is the angular frequency. In order to derive the electromagnetic fields of a dipole from the equations given in the previous section, we have to assume that $r \gg l$. When this condition is satisfied, one can also make the following simplifications:

$$\delta\theta_1 = (\theta - \theta_1) = \frac{l \sin\theta}{2r}; \quad \delta\theta_2 = (\theta_2 - \theta) = \frac{l \sin\theta}{2r} \tag{16}$$

$$\cos\delta\theta_1 \approx 1; \quad \cos\delta\theta_2 \approx 1 \tag{17}$$

$$r_1 = r + \frac{l\cos\theta}{2}; \quad r_2 = r - \frac{l\cos\theta}{2} \tag{18}$$

$$\frac{1}{r_1} = \frac{1}{r}\left\{1 - \frac{l\cos\theta}{2r}\right\}; \quad \frac{1}{r_2} = \frac{1}{r}\left\{1 + \frac{l\cos\theta}{2r}\right\} \tag{19}$$

$$\sin\theta_1 = \sin\theta\left\{1 - \frac{l\cos\theta}{2r}\right\}; \quad \sin\theta_2 = \sin\theta\left\{1 + \frac{l\cos\theta}{2r}\right\} \tag{20}$$

$$\cos\theta_1 = \cos\theta + \frac{l\sin^2\theta}{2r}; \quad \cos\theta_2 = \cos\theta - \frac{l\sin^2\theta}{2r} \tag{21}$$

$$1 - \frac{u\cos\theta_1}{c} = \left\{1 - \frac{u}{c}\cos\theta\right\}\left\{1 - \frac{ul\sin^2\theta}{2rc\left(1 - \frac{u}{c}\cos\theta\right)}\right\} \tag{22}$$

$$1 - \frac{u\cos\theta_2}{c} = \left\{1 - \frac{u}{c}\cos\theta\right\}\left\{1 + \frac{ul\sin^2\theta}{2rc\left(1 - \frac{u}{c}\cos\theta\right)}\right\} \tag{23}$$

$$\frac{1}{1-\dfrac{u\cos\theta_1}{c}}=\frac{1}{1-\dfrac{u\cos\theta}{c}}\left\{1+\frac{ul\sin^2\theta}{2rc\left(1-\dfrac{u}{c}\cos\theta\right)}\right\} \tag{24}$$

$$\frac{1}{1-\dfrac{u\cos\theta_2}{c}}=\frac{1}{1-\dfrac{u\cos\theta}{c}}\left\{1-\frac{ul\sin^2\theta}{2rc\left(1-\dfrac{u}{c}\cos\theta\right)}\right\} \tag{25}$$

Let us now consider each individual field component derived in the previous section and simplify them using the geometrical approximations given in Equations (16) to (25).

4.1. The Radiation Field Generated as the Charges Are Accelerated from S_1

The radiation field generated by the initiation of current can be obtained directly by substituting the expression for the current in Equation (7). This gives

$$E_{rad,S_1,\theta_1}=\frac{I_o e^{j\omega(t-r_1/c)}u\sin\theta_1}{4\pi\varepsilon_o c^2 r_1}\frac{1}{\left[1-\dfrac{u\cos\theta_1}{c}\right]}\mathbf{a}_{\boldsymbol{\theta}_1} \tag{26}$$

The above expression is exact, that is, it does not contain any approximations. In order to extract the electric fields of an infinitesimal current element we will write down the components of this electric field in the direction of $\mathbf{a_r}$ and $\mathbf{a}_\theta$ using the geometrical approximations listed in Equations (16) to (25), which are valid when $r \gg l$. Moreover, we also need to assume that

$$\frac{\omega r}{c} \ll 1.0 \tag{27}$$

which makes it possible for us to write

$$e^{j\omega(t-r_1/c)}=e^{j\omega(t-r/c)}\left\{1-\frac{j\omega l\cos\theta}{2c}\right\} \tag{28}$$

Using these approximations and keeping only the first order terms with respect to l, the component of this field in the direction of $\mathbf{a_r}$ and $\mathbf{a}_\theta$ becomes

$$E_{rad,S_1,\theta}$$
$$=\frac{I_o e^{j\omega(t-r/c)}u\sin\theta}{4\pi\varepsilon_o c^2 r\left(1-\dfrac{u}{c}\cos\theta\right)}\left\{1-\frac{j\omega l\cos\theta}{2c}-\frac{l\cos\theta}{r}+\frac{ul\sin^2\theta}{2rc\left(1-\dfrac{u}{c}\cos\theta\right)}\right\}\mathbf{a}_{\boldsymbol{\theta}} \tag{29}$$

$$E_{rad,S_1,r} = \frac{I_o e^{j\omega(t-r/c)}}{4\pi\varepsilon_o c^2 r^2}\left\{\frac{ul\sin^2\theta}{2\left(1-\frac{u}{c}\cos\theta\right)}\right\}\mathbf{a}_r \tag{30}$$

4.2. The Radiation Field Generated as the Charges Are De-accelerated at S_2

During the de-acceleration of the charges at S_2 the radiation field generated is

$$E_{rad,S_2,\theta_2} = -\frac{I_o e^{j\omega(t-l/u-r_2/c)} u\sin\theta_2}{4\pi\varepsilon_o c^2 r_2}\frac{1}{\left[1-\frac{u\cos\theta_2}{c}\right]}\mathbf{a}_{\theta_2} \tag{31}$$

The above expression is exact, that is, its derivation does not rely upon any assumptions. In order to derive the electric fields of an infinitesimal current element we will write down the components of this electric field in the direction of $\mathbf{a_r}$ and $\mathbf{a}_\theta$ using the geometrical approximations specified in Equations (16) to (25). Moreover, we also assume that

$$\frac{\omega r}{c} \ll 1.0; \quad \frac{\omega l}{u} \ll 1.0 \tag{32}$$

which makes it possible for us to write

$$e^{j\omega(t-l/u-r_2/c)} = e^{j\omega(t-r/c)}\left\{1-\frac{j\omega l}{u}+\frac{j\omega l\cos\theta}{2c}\right\} \tag{33}$$

Using these approximations and keeping only the first order terms with respect to l, the components of this field in the directions of $\mathbf{a_r}$ and $\mathbf{a}_{\boldsymbol{\theta}}$ are given by

$$\begin{aligned} E_{rad,S_2,\theta} &= -\frac{I_o e^{j\omega(t-r/c)} u\sin\theta}{4\pi\varepsilon_o c^2 r\left(1-\frac{u}{c}\cos\theta\right)} \\ &\quad \left\{1-\frac{j\omega l}{u}+\frac{j\omega l\cos\theta}{2c}+\frac{l\cos\theta}{r}-\frac{ul\sin^2\theta}{2rc\left(1-\frac{u}{c}\cos\theta\right)}\right\}\mathbf{a}_{\boldsymbol{\theta}} \end{aligned} \tag{34}$$

$$E_{rad,S_2,r} = \frac{I_o e^{j\omega(t-r/c)}}{4\pi\varepsilon_o c^2 r^2}\left\{\frac{ul\sin^2\theta}{2\left(1-\frac{u}{c}\cos\theta\right)}\right\}\mathbf{a_r} \tag{35}$$

4.3. The Electrostatic Field Generated by the Accumulation of Negative Charge at S_1

As the positive current leaves S_1, negative charge starts to accumulate there and this gives rise to a static field. This static field is given by

$$E_{stat,S_1,r_1}(t) = -\frac{I_o e^{j\omega(t-r_1/c)}}{4\pi\varepsilon_o r_1^2 j\omega}\mathbf{a_{r_1}} \tag{36}$$

After using the approximations given earlier, and noting that

$$\frac{1}{r_1^2} = \frac{1}{r^2}\left\{1 - \frac{l\cos\theta}{r}\right\} \tag{37}$$

the components of this expression in the direction of $\mathbf{a_r}$ and $\mathbf{a}_{\boldsymbol{\theta}}$ can be written as

$$E_{stat,S_1,r} = -\frac{I_o e^{j\omega(t-r/c)}}{4\pi\varepsilon_o r^2 j\omega}\left\{1 - \frac{j\omega l\cos\theta}{2c} - \frac{l\cos\theta}{r}\right\}\mathbf{a_r} \tag{38}$$

$$E_{stat,S_1,\theta}(t) = \frac{I_o e^{j\omega(t-r/c)}}{4\pi\varepsilon_o r^2 j\omega}\left\{\frac{l\sin\theta}{2r}\right\}\mathbf{a}_{\boldsymbol{\theta}} \tag{39}$$

4.4. The Electrostatic Field Generated by the Accumulation of Negative Charge at S_2

The static field generated by the accumulation of positive charge at S_2 is given by

$$E_{stat,S_2,r_2}(t) = \frac{I_o e^{j\omega(t-l/u-r_2/c)}}{4\pi\varepsilon_o r_2^2 j\omega}\mathbf{a_{r_2}} \tag{40}$$

After using the approximations given in Equations (16) to (25), and noting that

$$\frac{1}{r_2^2} = \frac{1}{r^2}\left\{1 + \frac{l\cos\theta}{r}\right\} \tag{41}$$

the components of this expression in the direction of $\mathbf{a_r}$ and $\mathbf{a}_{\boldsymbol{\theta}}$ are given by

$$E_{stat,S_2,r}(t) = \frac{I_o e^{j\omega(t-r/c)}}{4\pi\varepsilon_o r^2 j\omega}\left\{1 - \frac{j\omega l}{u} + \frac{j\omega l\cos\theta}{2c} + \frac{l\cos\theta}{r}\right\}\mathbf{a_r} \tag{42}$$

$$E_{stat,S_2,\theta}(t) = \frac{I_o e^{j\omega(t-r/c)}}{4\pi\varepsilon_o r^2 j\omega}\left\{\frac{l\sin\theta}{2r}\right\}\mathbf{a}_{\boldsymbol{\theta}} \tag{43}$$

4.5. The Velocity Field Generated by the Charges Moving from S_1 to S_2

The velocity field generated as the current pulse propagates along the element can be written directly from (5), and the result is

$$E_{vel,r} = \frac{I_o e^{j\omega(t-r/c)} l}{4\pi\varepsilon_o r^2 u \left[1 - \frac{u}{c}\cos\theta_1\right]^2} \left[1 - \frac{u^2}{c^2}\right] \mathbf{a_r} \tag{44}$$

$$E_{vel,z} = -\frac{I_o e^{j\omega(t-r/c)} l}{4\pi\varepsilon_o r^2 c \left[1 - \frac{u}{c}\cos\theta_1\right]^2} \left[1 - \frac{u^2}{c^2}\right] \mathbf{a_z} \tag{45}$$

After some mathematical manipulations, the components of this field in the directions of $\mathbf{a_r}$ and $\mathbf{a_\theta}$ are given by

$$E_{vel,r,total} = \frac{I_o e^{j\omega(t-r/c)} l}{4\pi\varepsilon_o r^2 u \left[1 - \frac{u}{c}\cos\theta\right]} \left[1 - \frac{u^2}{c^2}\right] \mathbf{a_r} \tag{46}$$

$$E_{vel,\theta} = -\frac{I_o e^{j\omega(t-r/c)} l}{4\pi\varepsilon_o r^2 c \left[1 - \frac{u}{c}\cos\theta\right]^2} \left[1 - \frac{u^2}{c^2}\right] \sin\theta\, \mathbf{a_\theta} \tag{47}$$

4.6. The Total Electric Field of a Short Electric Dipole

The two values for the total electric field of the current element in the directions $\mathbf{a_r}$ and $\mathbf{a_\theta}$ is given by the sum of (29), (34), (38), (42) and (46), and (30), (35), (39), (43) and (47), respectively. The resulting field can be written as

$$E_\theta = \frac{I_o e^{j\omega(t-r/c)} l \sin\theta}{4\pi\varepsilon_o} \left[\frac{j\omega u}{uc^2 r\left(1 - \frac{u}{c}\cos\theta\right)} - \frac{j\omega u \cos\theta}{uc^3 r\left(1 - \frac{u}{c}\cos\theta\right)} \right.$$

$$\left. + \frac{u^2 \sin^2\theta}{c^3 r^2 \left(1 - \frac{u}{c}\cos\theta\right)^2} - \frac{2u\cos\theta}{c^2 r^2 \left(1 - \frac{u}{c}\cos\theta\right)} + \frac{1}{r^3 j\omega} + \frac{\left(1 - u^2/c^2\right)}{cr^2\left(1 - \frac{u}{c}\cos\theta\right)^2} \right] \mathbf{a_\theta} \tag{48}$$

$$E_r = \frac{I_o e^{j\omega(t-r/c)} l}{4\pi\varepsilon_o} \left[\frac{u \sin^2\theta}{c^2 r^2 \left(1 - \frac{u}{c}\cos\theta\right)} + \frac{\cos\theta}{r^2 c} \right.$$

$$\left. + \frac{2\cos\theta}{j\omega r^3} - \frac{1}{ur^2} - \frac{\left(1 - u^2/c^2\right)}{ur^2 \left(1 - \frac{u}{c}\cos\theta\right)} \right] \mathbf{a_r} \tag{49}$$

After rearranging the equations given above one obtains

$$E_\theta = \frac{I_o e^{j\omega(t-r/c)} l \sin\theta}{4\pi\varepsilon_o} \left[\frac{j\omega}{c^2 r} + \frac{1}{cr^2} + \frac{1}{j\omega r^3} - \frac{1}{cr^2} \right.$$

$$\left. + \frac{u^2 \sin^2\theta}{c^3 r^2 \left(1 - \frac{u}{c}\cos\theta\right)^2} - \frac{2u\cos\theta}{c^2 r^2 \left(1 - \frac{u}{c}\cos\theta\right)} + \frac{\left(1 - u^2/c^2\right)}{cr^2 \left(1 - \frac{u}{c}\cos\theta\right)^2} \right] \mathbf{a}_{\boldsymbol{\theta}} \quad (50)$$

$$E_r = \frac{I_o e^{j\omega(t-r/c)} l}{4\pi\varepsilon_o} \left[\frac{2\cos\theta}{cr^2} + \frac{2\cos\theta}{j\omega r^3} - \frac{\cos\theta}{cr^2} - \frac{1}{ur^2} \right.$$

$$\left. + \frac{u\sin^2\theta}{c^2 r^2 \left(1 - \frac{u}{c}\cos\theta\right)} + \frac{\left(1 - u^2/c^2\right)}{ur^2 \left(1 - \frac{u}{c}\cos\theta\right)} \right] \mathbf{a}_{\mathbf{r}} \quad (51)$$

With some algebraic manipulations, one can show that the last four terms of Equations (50) and (51) each add up to zero, leaving the total electric field in the direction of $\mathbf{a}_{\mathbf{r}}$ and $\mathbf{a}_{\boldsymbol{\theta}}$ as

$$E_\theta = \frac{I_o e^{j\omega(t-r/c)} l \sin\theta}{4\pi\varepsilon_o} \left[\frac{j\omega}{c^2 r} + \frac{1}{cr^2} + \frac{1}{j\omega r^3} \right] \mathbf{a}_{\boldsymbol{\theta}} \quad (52)$$

and

$$E_r = \frac{I_o e^{j\omega(t-r/c)} \cos\theta}{4\pi\varepsilon_o c r j\omega} \left[\frac{2}{r} + \frac{2c}{j\omega r^2} \right] \mathbf{a}_{\mathbf{r}} \quad (53)$$

These are identical to the short electric dipole fields derived using the standard technique.

4.7. The Magnetic Radiation Field Generated at S_1 during the Initiation of the Current

The magnetic radiation field generated at S_1 during the initiation of the current

$$B_{rad,S_1,\phi} = \frac{I_o e^{j\omega(t-r_1/c)} u \sin\theta_1}{4\pi\varepsilon_o c^3 r_1} \frac{1}{\left[1 - \frac{u\cos\theta_1}{c}\right]} \mathbf{a}_\phi \quad (54)$$

utilizing the geometrical approximations laid out in Equations (16) to (25) and keeping only the second order terms in l, one obtains

$$B_{rad,S_1,\phi} = \frac{I_o e^{j\omega(t-r/c)} u \sin\theta}{4\pi\varepsilon_o c^3 r} \frac{1}{\left[1 - \frac{u\cos\theta}{c}\right]}$$

$$\left\{ 1 - \frac{j\omega l \cos\theta}{2c} - \frac{l\cos\theta}{r} + \frac{ul\sin^2\theta}{2rc\left(1 - \frac{u}{c}\cos\theta\right)} \right\} \mathbf{a}_{\boldsymbol{\varphi}} \quad (55)$$

4.8. The Magnetic Radiation Field Generated at S_2 during the Cessation of the Current

The magnetic radiation field generated at S_1 during the initiation of the current

$$B_{rad,S_2,\varphi} = -\frac{I_o e^{j\omega(t-l/u-r_2/c)} u \sin\theta_2}{4\pi\varepsilon_o c^3 r_2} \frac{1}{\left[1 - \dfrac{u\cos\theta_2}{c}\right]} \mathbf{a}_\phi \qquad (56)$$

Using the approximations (16) to (25) and keeping only the first order terms in l one obtains

$$B_{rad,S_2,\phi} = -\frac{I_o e^{j\omega(t-r/c)} u \sin\theta}{4\pi\varepsilon_o c^3 r} \frac{1}{\left[1 - \dfrac{u\cos\theta}{c}\right]} \left\{1 - \frac{j\omega l}{u} + \frac{j\omega l\cos\theta}{2c} + \frac{l\cos\theta}{r} - \frac{ul\sin^2\theta}{2rc\left(1 - \dfrac{u}{c}\cos\theta\right)}\right\} \mathbf{a}_{\boldsymbol{\varphi}} \qquad (57)$$

4.9. The Magnetic Velocity Field Generated as the Current Pulse Propagates along the Axis of the Dipole

The magnetic velocity field generated as the current pulse propagates along the dipole is given by

$$B_{vel,\phi} = \frac{e^{j\omega(t-r/c)} l \sin\theta}{4\pi\varepsilon_o r^2 c^2 \left[1 - \dfrac{u}{c}\cos\theta\right]^2} \left[1 - \frac{u^2}{c^2}\right] \mathbf{a}_{\boldsymbol{\varphi}} \qquad (58)$$

4.10. The Total Magnetic Field

The total magnetic field of the dipole is given by the sum of (55), (57) and (58). The result can be written as

$$B_\phi = \frac{I_o e^{j\omega(t-r/c)} l \sin\theta}{4\pi\varepsilon_o} \left[\frac{j\omega}{c^3 r} + \frac{1}{c^2 r^2} - \frac{1}{c^2 r^2} - \frac{2u\cos\theta}{c^3 r^2\left(1 - \dfrac{u}{c}\cos\theta\right)} + \frac{u^2\sin^2\theta}{r^2 c^4\left(1 - \dfrac{u}{c}\cos\theta\right)^2} + \frac{\left(1 - u^2/c^2\right)}{r^2 c^2\left(1 - \dfrac{u}{c}\cos\theta\right)^2}\right] \mathbf{a}_{\boldsymbol{\varphi}} \qquad (59)$$

The last four terms in the above equation add up to zero, making the total magnetic field equal to

$$B_\varphi = \frac{I_o e^{j\omega(t-r/c)} l \sin\theta}{4\pi\varepsilon_o c^2 r}\left[\frac{j\omega}{c}+\frac{1}{r}\right]\mathbf{a}_{\boldsymbol{\varphi}} \tag{60}$$

This expression is identical to the magnetic field for a short electric dipole derived using the standard technique.

5. DISCUSSION

In the derivation of the electromagnetic fields of a short electric dipole given in this paper, one can observe that the speed of propagation of the charges, i.e., u, along the dipole axis enters into the results only in the second order terms of l. The two conditions $r \gg l$ and $\frac{j\omega l}{u} \ll 1$ make it possible to neglect the terms containing u thereby leaving the final result independent of this speed. Also, it is of interest to observe that if the speed of propagation of the current pulse is assumed to be c, the velocity term becomes zero and the total field is comprised of pure radiation and static contributions. It is educational to observe that, in the expression for the total electric field of the dipole, the term in $1/r$ is contributed by the radiation fields, and that in $1/r^3$ is contributed by the electrostatic fields. The $1/r^2$ term is attributable to both the radiation and the static fields.

Now let us consider the radiation fields in the time domain. If the current in the dipole is $I(t)$, then the radiation field generated by the acceleration of charge at S_1 is given by

$$E_{rad,S_1,\theta}(t) = \frac{I(t-r_1/c)u\sin\theta_1}{4\pi\varepsilon_o c^2 r_1}\frac{1}{\left[1-\dfrac{u\cos\theta_1}{c}\right]}\mathbf{a}_{\boldsymbol{\theta_1}} \tag{61}$$

The component of the radiation field in the same direction produced by the deceleration of charges at S_2 is

$$E_{rad,S_2,\theta}(t) = -\frac{I(t-l/u-r_2/c)u\sin\theta_2}{4\pi\varepsilon_o c^2 r_2}\frac{1}{\left[1-\dfrac{u\cos\theta_2}{c}\right]}\mathbf{a}_{\boldsymbol{\theta_2}} \tag{62}$$

Re-writing the terms $\sin\theta_1$, $\cos\theta_1$, $\sin\theta_2$, $\cos\theta_2$, r_1 and r_2 in terms of $\sin\theta$, $\cos\theta$ and r, and separating out the terms varying as $1/r$, one finds:

$$E_{rad}(t) = \frac{u\sin\theta}{4\pi\varepsilon_o r c^2\left[1-\dfrac{u\cos\theta}{c}\right]}\left\{I(t-r/c-l\cos\theta/c)-I(t-r/c-l/u+l\cos\theta/c)\right\}\mathbf{a}_{\boldsymbol{\theta}} \tag{63}$$

Expanding both terms inside the bracket using Taylor series and neglecting the terms containing l^2 and higher powers one obtains

$$E_{rad}(t)=\frac{l\sin\theta}{4\pi\varepsilon_o rc^2}\left\{\frac{dI(t-r/c)}{dt}-\frac{1}{2}\frac{d^2I(t-r/c)}{dt^2}\frac{l}{u}\left(1-\frac{u}{c}\cos\theta\right)\right\} \quad (64)$$

The above equation shows that, in the time domain, the short electric dipole approximation is valid in addition to $r \gg l$ (which was necessary to convert Equations (61) and (62) to (63)) when

$$\left\{\left|\frac{dI(t)}{dt}\right| \gg \left|\frac{1}{2}\frac{d^2I(t)}{dt^2}\frac{l}{u}\left(1-\frac{u}{c}\cos\theta\right)\right|\right\} \quad (65)$$

when these conditions are satisfied, the radiation field becomes

$$E_{rad}(t)=\frac{\sin\theta}{4\pi\varepsilon_o rc^2}\frac{l dI(t-r/c)}{dt} \quad (66)$$

The above is the time domain radiation field of a short electric dipole of length l supporting a current $I(t)$.

6. CONCLUSIONS

Equations pertinent to the electromagnetic fields generated by accelerating charges have been utilized to evaluate the electromagnetic fields of a current path of length l for the case that a current pulse propagates at constant velocity. The electromagnetic fields of a short electric dipole are extracted from these equations when $r \gg l$, where r is the distance to the point of observation. Unlike the classical treatment, which assumes a constant current along the dipole, the present treatment starts with field equations that are valid for a current propagating at a finite speed along the dipole channel. It has been shown that the speed at which the pulse is propagating only enters into the electromagnetic fields in the terms that are second order in l, and they can be neglected in the short electric dipole approximation. The results illustrate how the radiation fields emanating from the two ends of a dipole give rise to field terms varying as $1/r$ and $1/r^2$, while the time varying stationary charges at the ends of the dipole contribute to field terms varying as $1/r^2$ and $1/r^3$. This physical insight is absent in the classical treatment.

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