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GENERATION OF A WIDE-BAND RESPONSE USING EARLY-TIME AND MIDDLE-FREQUENCY DATA THROUGH THE USE OF ORTHOGONAL FUNCTIONS

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Abstract—Generation of a wide-band response using partial information from the time domain (TD) data and frequency domain (FD) data has been accomplished in this paper through the use of three different orthogonal functions, such as the continuous Laguerre functions, the Bessel-Chebyshev functions, and the associate Hermite functions. In this hybrid approach, one can generate the early-time response using the method of marching-on-in-time (MOT) and use the method of moment (MOM) to generate the middle-frequency response, as the low-frequency data may be unstable. Since the early-time and the middle-frequency data are mutually complimentary, they can provide the missing low- and high-frequency response and the late-time response, respectively. Even though obtaining middle-frequency response from an object needs more computation time than the low-frequency response, this approach has better performance for the interpolation and extrapolation of a wide-band response.

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1. INTRODUCTION

Typically, one can perform the electromagnetic analysis using Maxwell's equation either in the time domain (TD) or in the frequency domain (FD). The method of moments (MOM), which uses an integral-equation formulation, can be applied to perform the electromagnetic analysis in the frequency domain. The disadvantage of the MOM is that it needs a large matrix equation to solve electromagnetic problems as the electrical dimension of the structure increases. Therefore, this FD approach is computationally demanding to perform a wide-band analysis, especially to obtain the high-frequency response. A TD approach is preferred to a wide-band analysis because of relatively small computational time based on fewer arithmetic operations. For a time-domain integral equation formulation, the method of marching-on-in-time (MOT) is usually employed. A serious drawback of this algorithm is the occurrence of late-time instabilities in the form of high-frequency oscillations.

For overcoming the disadvantages and strengthening the advantages of each of the approaches, the hybrid TD-FD method has been proposed to interpolate and extrapolate data in both domains simultaneously using only early-time and low-frequency data [1–6]. The MOM approach can efficiently generate low-frequency data, while the MOT algorithm can be used to obtain stable early-time data quite quickly. The basic principle is that the early-time and low-frequency data provide the missing high-frequency response and the missing late-time response, respectively. From this basic principle, one can also extend the hybrid TD-FD method to generate a wide-band response in both domains using early-time and middle-frequency data instead of using early-time and low-frequency data [7]. There are two reasons for this extended hybrid TD-FD method. One reason is that the low-frequency data may be unreliable. Another reason is that one does not need much more computation time to obtain middle-frequency data.

The objective is to generate a wide-band electromagnetic response with high accuracy using the orthogonal functions from the extended hybrid TD-FD. We will use not only the continuous Laguerre functions but also the associate Hermite and Bessel-Chebyshev functions to interpolate and extrapolate the wide-band response using the hybrid TD-FD data such as early-time and middle-frequency data, in an electromagnetic analysis.

This paper is organized as follows. In Section 2, we will explain the definitions and properties of the orthogonal functions, the process of interpolating and extrapolating the wide-band response. Section 3 shows one numerical example (bow-tie antenna) to evaluate the performance of the wide-band response using three orthogonal functions (continuous Laguerre, associate Hermite, and Bessel-Chebyshev functions). Finally, some conclusions are presented in Section 4.

2. FORMULATIONS

2.1. Continuous Laguerre Functions

The continuous orthonormal Laguerre functions and its Laplace transform can be defined by [8]

$$\phi_{cn}(t, l_1) = \frac{1}{\sqrt{l_1}} e^{-\frac{t}{2l_1}} L_n(t/l_1) \tag{1}$$

$$\Phi_{cn}(s, l_1) = \frac{1}{\sqrt{l_1}} \frac{\left(s - \frac{1}{2l_1}\right)^n}{\left(s + \frac{1}{2l_1}\right)^{n+1}} \quad l_1 > 0; \quad n \geq 0 \tag{2}$$

where l_1 is the scaling factor with only positive values because its poles should be all on the negative real axis of the s-plane to have stability. The continuous Laguerre polynomials $L_n(t)$ can be defined by

$$L_n(t) = \frac{e^t}{n!} \frac{d^n \left(t^n e^{-t}\right)}{dt^n}, \quad n \geq 0; \quad t \geq 0 \tag{3}$$

They are causal, i.e., they are nonzero only for $t \geq 0$ where the continuous Laguerre polynomials can be computed in a stable fashion recursively through

$$\begin{aligned} L_0(t) &= 1 \\ L_1(t) &= 1 - t \\ L_n(t) &= \frac{2n - 1 - t}{n} L_{n-1}(t) - \frac{n-1}{n} L_{n-2}(t), \quad n \geq 2; \quad t \geq 0 \end{aligned} \tag{4}$$

A causal electromagnetic response $x(t)$ at a particular location in space for $t \geq 0$ can be expanded by a Laguerre series as

$$x(t) = \sum_{n=0}^{\infty} c_n \phi_{cn}(t, l_1) \tag{5}$$

The Fourier transform of (1) can be evaluated as

$$\Phi_{cn}(f, l_2) = \frac{\left(-\frac{1}{2} + j\frac{f}{l_2}\right)^n}{\sqrt{2\pi l_2} \left(\frac{1}{2} + j\frac{f}{l_2}\right)^{n+1}} \tag{6}$$

where $l_2 = 1/(2\pi l_1)$ and $j = \sqrt{-1}$.

If we consider the discrete Laguerre functions, one can also use the scale factor a as [9]

$$a = e^{-\frac{\Delta t}{2l_1}} \tag{7}$$

2.2. Bessel-Chebyshev Functions

The orthogonal basis function is $\phi_{bn}(t, l_1) = (t/l_1)^{-1} J_n(t, l_1)$, where $J_n(t, l_1)$ is a Bessel function of the first kind of degree n. A signal with compact time support can be expanded as

$$x(t) \approx \sum_{n=0}^{N} c_n (t/l_1)^{-1} J_n(t, l_1) \tag{8}$$

The Fourier transform of the above expression can be given as

$$X(f) = \begin{cases} \sum_{n=0}^{N} c_n \frac{i}{n} (-i)^n l_1 \left[1 - (f/l_2)^2\right]^{1/2} U_{n-1}(f/l_2), & |f| < l_2 \\ 0, & |f| > l_2 \end{cases} \tag{9}$$

where $U_n(f)$ is the Chebyshev polynomial of the second kind defined by [10]

$$U_n(f) = \frac{\sin\left[(n+1)\cos^{-1} f\right]}{(1-f^2)^{1/2}} \tag{10}$$

In this expression, the causality in time is not forced, whereas the signals we are dealing with are causal. The relationship between the first kind of Chebyshev polynomial and the second kind of Chebyshev polynomial are related by the Hilbert transform, i.e.,

$$RV \int_{-1}^{1} \frac{\sqrt{1-y^2}}{y-x} U_{n-1}(y) dy = -\pi T_n(x) \tag{11}$$

where $RV \int_a^b$ is a Cauchy principle value integral and $T_n(f)$ the Chebyshev polynomial of the first kind defined by

$$T_n(f) = \cos\left[n \cos^{-1}(f)\right] \tag{12}$$

Therefore, the causal time signal and its Fourier transform can be rewritten as

$$x(t) \approx \sum_{n=0}^{N} c_n (t/l_1)^{-1} J_n(t, l_1), \qquad t \geq 0 \tag{13}$$

$$X(f) = \begin{cases} \sum_{n=0}^{N} c_n \frac{1}{n} (-i)^n l_1 \left\{ i\left[1 - (f/l_2)^2\right]^{1/2} U_{n-1}(f/l_2) + T_n(f/l_2) \right\}, & |f| < l_2 \\ 0, & |f| > l_2 \end{cases} \tag{14}$$

2.3. Associate Hermite Functions

The orthonormal associate Hermite functions can be defined as [11]

$$h_n\left(t,l_1\right)=\frac{H_n(t/l_1)}{\sqrt{2^n n!}}\frac{e^{\frac{-t^2}{2l_1^2}}}{\sqrt{\sqrt{\pi}l_1}},\qquad n\geq 0 \tag{15}$$

where $H_n(t)$ is Hermite polynomial of order n, with l_1 as a scaling factor, and $n!$ represents factorial of n. The Hermite polynomial can be computed recursively through

$$\begin{aligned}&H_0(t)=1\\&H_1(t)=2t\\&H_n(t)=2tH_{n-1}(t)-2(n-1)H_{n-2}(t)\quad n\geq 2\end{aligned} \tag{16}$$

Using (15)–(16), the recursive formula for the associate Hermite function can be expressed as

$$h_n\left(t\right)=\frac{1}{\sqrt{n}}\left[\sqrt{2}th_{n-1}(t)-\sqrt{n-1}h_{n-2}(t)\right],\quad n\geq 2 \tag{17}$$

A signal $x(t)$ can be expanded into an associate Hermite series as

$$x(t)=\sum_{n=0}^{\infty}c_nh_n(t,l_1)=\sum_{n=0}^{\infty}c_n\phi_{hn}(t,l_1)=\sum_{n=0}^{\infty}\frac{c_n}{\sqrt{l_1}}\phi_{hn}(t/l_1) \tag{18}$$

Since the associate Hermite functions are the eigenfunctions of the Fourier transform operator, their Fourier transform are given by

$$h_n\left(t/l_1\right)\Leftrightarrow\left(-i\right)^n h_n\left(f/l_2\right) \tag{19}$$

Therefore, the Fourier transform of $x(t)$ is given by

$$\begin{aligned}X(f)&=\sum_{n=0}^{\infty}(-i)^nc_nh_n(f,l_2)\\&=\sum_{n=0}^{\infty}(-i)^nc_n\phi_{hn}(f,l_2)=\sum_{n=0}^{\infty}(-i)^n\frac{c_n}{\sqrt{l_2}}\phi_{hn}(f/l_2)\end{aligned} \tag{20}$$

2.4. Process of Interpolation and Extrapolation

Let M_1 and M_2 be the number of TD and FD samples that are given for the functions $x(t)$ and $X(f)$, respectively. The total number of available samples is M_t in the TD and M_f in the FD. It means one can utilize only early-time data (x_1) and low- or middle-frequency data

(X_1). From these relationships, TD and FD samples can be defined as follows:

$$\begin{cases} x_1 = \{x(t_0), x(t_1), \ldots, x(t_{M_1-1})\}^T \\ x_2 = \{x(t_{M_1}), x(t_{M_1+1}), \ldots, x(t_{M_t-1})\}^T \\ X_1 = \{X(f_P), X(f_{P+1}), \ldots, X(f_{P+M_2-1})\}^T \\ X_2 = \begin{bmatrix} \{X(f_{M_2}), X(f_{M_2+1}), \ldots, X\left(f_{M_f-1}\right)\}^T, \text{ at } P = 0 \\ \{X(f_0), \ldots, X(f_{P-1}), X(f_{P+M_2}), \ldots, X\left(f_{M_f-1}\right)\}^T, \\ \quad \text{at } P = 1, 2, \ldots, M_f - M_2 - 1 \end{bmatrix} \end{cases} \tag{21}$$

where x_2 is late time data, X_2 the low- or high-frequency data, and P the starting point of the FD samples to apply the hybrid TD-FD method. The matrix representation for this hybrid TD-FD data would be

$$\begin{bmatrix} \begin{matrix} \phi_0(t_0, a) & \ldots & \phi_{N-1}(t_0, a) \\ \vdots & \ddots & \vdots \\ \phi_0(t_{M-1}, a) & \ldots & \phi_{N-1}(t_{M_1-1}, a) \end{matrix} \\ \text{Re}\begin{pmatrix} \Phi_0(f_0, a) & \ldots & \Phi_{N-1}(f_0, a) \\ \vdots & \ddots & \vdots \\ \Phi_0(f_{M_2-1}, a) & \ldots & \Phi_{N-1}(f_{M_2-1}, a) \end{pmatrix} \\ \text{Im}\begin{pmatrix} \Phi_0(f_0, a) & \ldots & \Phi_{N-1}(f_0, a) \\ \vdots & \ddots & \vdots \\ \Phi_0(f_{M_2-1}, a) & \ldots & \Phi_{N-1}(f_{M_2-1}, a) \end{pmatrix} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} x(t_0) \\ \vdots \\ x(t_{M_1-1}) \\ \text{Re}\begin{pmatrix} X(f_0) \\ \vdots \\ X(f_{M_2-1}) \end{pmatrix} \\ \text{Im}\begin{pmatrix} X(f_0) \\ \vdots \\ X(f_{M_2-1}) \end{pmatrix} \end{bmatrix} \tag{22}$$

where 'Re' and 'Im' in the matrix equation are the real and imaginary parts of the transfer function, respectively. The unknown coefficients c_n can be obtained by solving this matrix equation with the total least-square implementation of the Singular Value Decomposition (SVD) method [12].

3. NUMERICAL SIMULATION: A BOW-TIE ANTENNA

In this example, a bow-tie antenna is used to validate the hybrid TD-FD data using the three different orthogonal functions. The main goal of this example is to interpolate and extrapolate the S-parameters on the feed line of the bow-tie antenna using early TD and middle FD data and compare them with the complete TD and FD solutions. To evaluate the performance of the interpolation and extrapolation, we compute the *estimated error* following the normalized mean square errors (MSEs) in the time and frequency domain as

$$E_{est} = \frac{\|\hat{x} - x\|_2}{\|x\|_2} + \frac{\left\|\hat{X} - X\right\|_2}{\|X\|_2} \tag{23}$$

where $\|\bullet\|_2$ is the L^2-norm of a vector. $\hat{x}$ and $\hat{X}$ are the estimated TD and FD data. Typically, we use a Gaussian input pulse as the excitation for solving the TD problem as

$$g(t) = \frac{4}{\sigma\sqrt{\pi}} U_0 e^{-\left(\frac{4(t-t_0)}{\sigma}\right)^2} \tag{24}$$

where U_0 is the amplitude of the input pulse, σ is the width of the Gaussian pulse, and t_0 is the delay to make $g(t) \approx 0$ for $t < 0$. In FD, the Gaussian pulse is given by

$$G(f) = U_0 e^{-\left(\frac{(2\pi f\sigma)^2}{64} + j2\pi f t_0\right)} \tag{25}$$

In our computation, U_0 is chosen to be 1 V. The Gaussian excitation voltage has the parameters with $\sigma = 2.604$ ns and $t_0 = 4.883$ ns. Thus, the bandwidth (BW) of the original data is approximately 1.53 GHz. We obtained the S-parameter (S_{11}) on the feed line of the bow-tie antenna using the HOBBIES (Higher Order Basis Based Integral Equation Solver) simulation program as shown Fig. 1 [13]. The computed frequency range is from dc to 1.53 GHz ($\Delta f = 6$ MHz, 256 data points). We can obtain the FD data of the bow-tie antenna with the Gaussian pulse excitation by multiplying the response of the bow-tie antenna obtained in the frequency domain by the electromagnetic simulator HOBBIES and the spectrum of the Gaussian plane wave. Theoretically, the TD data generated by the MOT method should be the same as the TD data from IDFT of the FD data. Thus, one can obtain the TD data from $t = 0$ to $t = 32.39$ ns ($\Delta t = 0.1628$ ns, 200 data points) using IDFT of the FD data.

We evaluate the performance of the interpolation and extrapolation using the same TD data (the first 50 points) and different FD

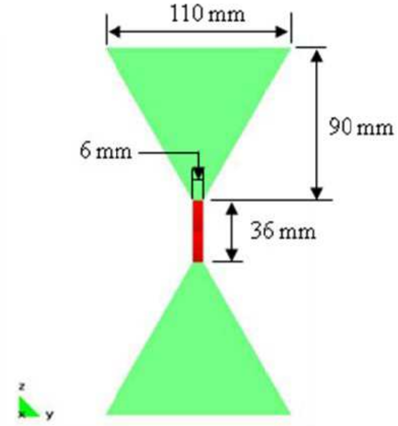

Figure 1. HOBBIES simulation model for the bow-tie antenna.

data (65 points, shifted to the right with 30 incremental points) versus the scaling factor a for the continuous Laguerre functions, Bessel-Chebyshev functions, and associate Hermite functions. The performance of the hybrid TD-FD method is sensitive to the scaling factor a and the expansion order N of orthogonal polynomials [14]. To compare the performance of the hybrid TD-FD method with the seven different dataset, we utilize the scaling factor a under the fixed expansion order of orthogonal polynomials. We choose the degree of the orthonormal functions as $N = 350$ for the continuous Laguerre functions, $N = 300$ for the Bessel-Chebyshev functions, and $N = 200$ for the associate Hermite functions, respectively. When we solve (22) by singular value decomposition (SVD), we set the tolerance level as 10^{-9} for the continuous Laguerre functions, 10^{-14} for the Bessel-Chebyshev functions, and 10^{-5} for the associate Hermite functions, respectively. Fig. 2 shows the results of the MSE for seven different hybrid TD-FD dataset using the three orthogonal functions.

From Fig. 2(a), one can get better performance if one utilizes the hybrid early TD data and the middle FD data (case 2 and 3) instead of the hybrid early TD and the low FD data (case 1) to apply the continuous Laguerre technique ($0.20 \leq a \leq 0.82$). The estimated MSE for the case 1 and case 2 at $a = 0.8$ for the continuous Laguerre technique are 6.93e-2 and 6.075e-4, respectively. Fig. 2(b) also shows that one can have better performance for the case 2 and 3 than for the case 1 to apply the Bessel-Chebyshev technique ($0.34 \leq a \leq 0.40$). The estimated MSE of the case 1 and case 2 at $a = 0.4$ for the Bessel-Chebyshev technique are 1.862e-3 and 7.155e-4, respectively. One also

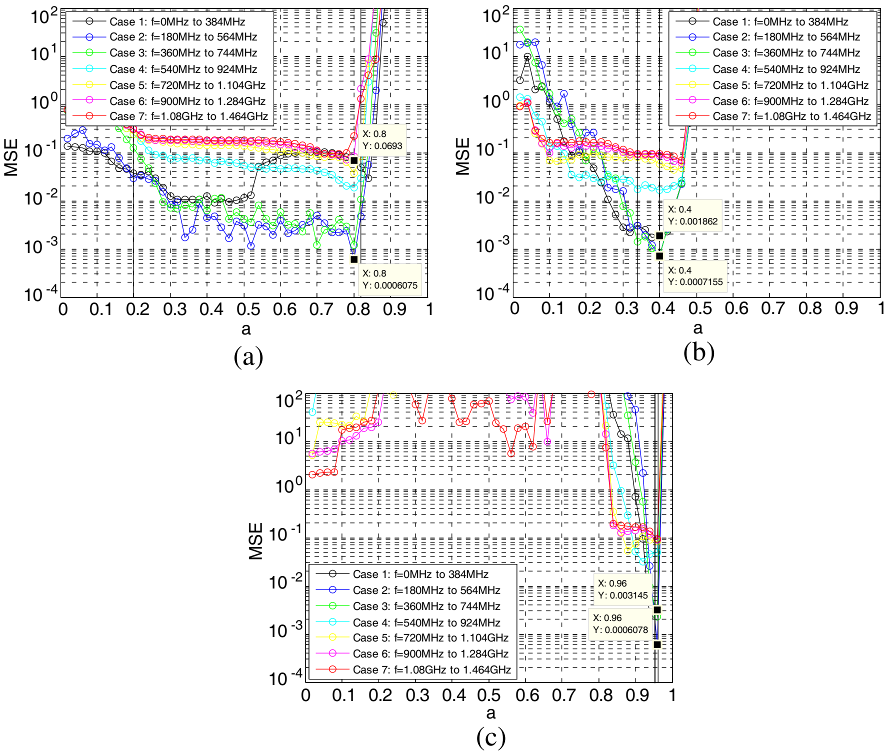

Figure 2. MSE for the same TD data and different FD data: (a) continuous Laguerre functions, (b) Bessel-Chebyshev functions, and (c) associate Hermite functions.

can get lower MSE value for the both case 2 and 3 than the case 1 to apply the associate Hermite technique around $a = 0.96$ as shown in Fig. 2(c). The estimated MSE for the case 1 and case 2 at $a = 0.96$ for the associate Hermite technique are 3.145e-3 and 6.078e-4, respectively. From Figs. 2(a), (b), and (c), one can recognize that the range of the scaling factor a for the continuous Laguerre technique is wider whereas the range of a for the associate Hermite technique is more narrow, to have accurate results. If one use high FD data (case 4–7), instead of using low- or middle-frequency data, the hybrid TD-FD method does not work too well, as the early-time data and the high-frequency data may contain redundant information and thus may not be mutually complementary.

The TD of the S-parameter (S_{11}) on the feed line of the bow-tie antenna is shown by the blue solid line in Fig. 3. The real and

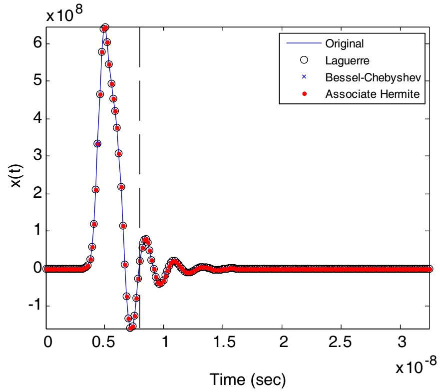

Figure 3. Estimated TD data using the three orthogonal functions.

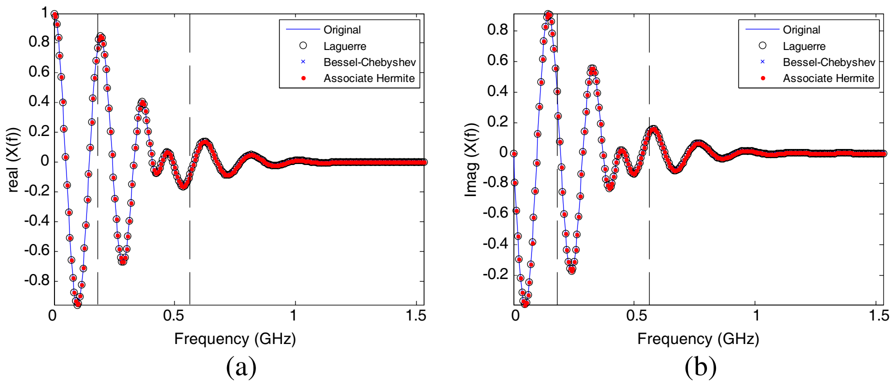

Figure 4. Estimated (a) real and (b) imaginary FD data using the three orthogonal functions.

imaginary parts of the FD of the S_{11} are shown in Fig. 4. From the partial information of the TD and FD data (case 2, 50 TD points: 0 to 7.9754 ns, 65 FD points: 180 to 564 MHz) one can interpolate and extrapolate the original TD and FD data using the continuous Laguerre functions (at $a = 0.8$, black 'o' marker) and Bessel-Chebyshev functions (at $a = 0.4$, blue 'x' marker), and associate Hermite functions (at $a = 0.96$, red '.' marker) as shown in Figs. 3 and 4.

4. CONCLUSION

In this paper, we have presented the continuous Laguerre, Bessel-Chebyshev, and associate Hermite functions to interpolate and extrapolate the wide-band response in both TD and FD simultaneously using seven different hybrid TD-FD dataset. The performance of the hybrid TD-FD method with a fixed expansion order N of the three orthogonal polynomials is sensitive to the scaling factor a. It is possible to find the optimal scaling factor a. At the optimal scaling factor a, the performance of the three orthogonal functions in the case 2 are almost the same. For the range of the scaling factor a satisfied with the cases 2 and 3 getting better performance than the case 1, the continuous Laguerre technique has the widest range whereas the associate Hermite technique has very narrow range. Even though computing the middle-frequency response needs more time and memory than generating the low-frequency response, one can generate the wide-band response with better performance using the continuous Laguerre, Bessel-Chebyshev, and associate Hermite functions in consideration of the range of the scaling factor a from the hybrid early-time data and the middle-frequency data.

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