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FIELD APPROACH IN THE TRANSFORMATION OPTICS CONCEPT

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Abstract—An alternative, field-based formulation of transformation optics is proposed. Field transformations are expressed in the language of boundary conditions for the electromagnetic fields facilitated through the introduction of generalized potential functions. It is shown that the field-based approach is equivalent to the conventional coordinate-transformation approach but is preferable when looking for specific field distribution. A set of example devices such as invisibility cloaks, concentrators, rotators, and transformation optics lenses capable of creating light beams with predetermined field distribution (e.g., Gaussian and sinusoidal) is studied to validate the effectiveness of the field-based formulation. As for the boundary conditions for the cloaked region the absence of the normal component of the Poynting vector is justified. In the frames of the field-based approach the physical reasons behind infinite components (singularities) of the material parameters of transformation optics devices are straightforwardly revealed.

1. INTRODUCTION

Transformation optics (TO) is based on a beautiful idea: the invariance of Maxwell's equations with respect to the coordinate

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transformation [1–7]. If one knows electric and magnetic fields as solutions of Maxwell's equations in some original medium, then any coordinate transformation will alter the spatial dependencies for both material parameters and electromagnetic fields — yet the transformed fields will still obey Maxwell's equations in the transformed medium. The transformed fields are constructed in such manner that allows to exclude the incident fields from the boundary conditions. This prominent property provides a number of exciting applications like invisibility cloaking [8–24], optical concentrators [19, 25], rotators [19, 26], illusion optics devices [27–32], and cylindrical to plane wave converters [33]. Transformation of the fields and material parameters can be applied in computational photonics, too [2, 34–38].

Specifically, the invisibility cloaking has received considerable attention recently. A "cloak" is a device, which guides radiation around a region in space in such way that this region irrespectively to what it contains would appear nonexistent to an outside observer [13, 14]. Such cloaks were proposed for different kinds of waves, i.e., optical [10, 21], plasmonic [39–43], and acoustic [44–47], as well as for different geometries: cylindrical, spherical, carpet-like (so-called ground cloaks), and even arbitrary. It is not just a theoretical concept, because its performance has been validated in several experiments with cylindrical [5, 48, 49] and carpet/ground cloaks [47, 50–56].

A cylindrical cloak (often distinguished because of its symmetry) has singularities in the material parameters, i.e., it requires infinite values of azimuthal components of the dielectric permittivity and magnetic permeability tensors. To avoid infinite values, a simplified set of parameters has been proposed [57–59]. However, in this case the cloak ceases to be truly invisible [60]. Another type of a non-singular cloak is proposed in the "ground cloak" geometry [61]. It is easier in realization [50, 51] and has the potential to hide relatively large objects in the visible wavelength range [52, 53].

All transformation optics devices are inherently magnetic materials; however, a cloak can be made non-magnetic for some special cases [14, 62, 63]. In Ref. [64] it is proved that invisibility is equivalent to the absence of scattered fields. This enabled far-field investigations of a cloak by simply calculating its scattering cross-section [65, 66]. It should also be noted that there is another approach to achieve invisibility, using radiation cancelation with the dipole radiation of the cloaked object [67, 68]. It can be applied to cloak, e.g., a sensor [69].

Conventionally, TO principles are introduced geometrically, i.e., by specifying the desired coordinate transformation. This approach is extremely illustrative, easy to understand, and useful. However, another possible approach is to start from a predetermined form of the fields in a transformation medium and apply proper boundary conditions on its interfaces. Indeed, the knowledge of the field solutions from the outset can be a great advantage in TO. When several transformation media are stacked, the problem becomes analogous to that of multilayered media: in both cases, known general solutions in several layers need to be matched by means of boundary conditions.

In this paper, we present a systematic formulation of the alternative approach to TO, which is grounded on the use of boundary conditions and, therefore, can be called field-based transformation optics. There have been earlier attempts on such field-based approaches [70–72]. In Ref. [70], the general form of transformation fields is applied to classify the media and discuss the properties of reciprocity, chirality, and bi-anisotropy. Another variant of the field-based approach is developed in Ref. [71], where the TO devices are considered in details. Starting from the general formulation, Yaghjian and Maci introduce the boundary conditions for a cloak as vanishing of the normal components of the field inductions at the inner interface. The Maxwell equations are solved for every particular problem, what requires much efforts.

In our paper, we apply the invariance of Maxwell's equations to construct general solutions for the fields in different media and then stitch them at interfaces. In contrary to the Yaghjian and Maci approach, we use the flux boundary conditions: the normal component of the Poynting vector is equal to zero at the inner interface. We do not employ coordinate transformations (only for demonstration purposes), but introduce general potential functions. By revisiting the common TO problems (e.g., invisibility cloaking and optical concentrator), we show that the field-based approach is mostly equivalent to the conventional geometric formulation. However, some cases are identified where the field-based approach provides additional insight into the physics of the transformation media, such as the problem of material parameter singularities of invisibility cloaks. We show that these singularities result from the field discontinuities at the boundaries, and offer a means to circumvent this problem and design a singularity-free cloak. Using the same principles we propose the design of a non-magnetic cloak that can have invisibility without requiring any magnetic materials. We also show how the field-based TO can be helpful in designing devices producing predetermined electromagnetic field distribution, e.g., lenses (plates) capable of converting plane waves into Gaussian beams. This method can be further used to calculate metamaterial's parameters generating non-diffracting (e.g., Bessel [73]) or accelerating (e.g., Airy [74]) light beams.

The rest of the paper is organized as follows. In Section 2, the principles of field-based transformation optics approach are put forth, and the basic set of boundary conditions for the fields is revealed. In Section 3, examples of the transformation optics devices (cloak, concentrator, rotator, lens) are analyzed using the field boundary conditions, i.e., electrodynamically rather than geometrically. Section 4 follows with the analysis of material parameter singularities (infinite values) that can arise during transformation. Electromagnetic fields near the singularity are considered, and finite material parameters for a cloak are derived. Section 5 deals with a specific case of a non-magnetic cloak. Finally, Section 6 summarizes the paper.

2. FIELD TRANSFORMATION OPTICS

2.1. Concept

Conventional TO deals with the transformation of the three-dimensional space (usual electrodynamic applications) or four-dimensional space (space-time cloaking [75–77]). It is extremely convenient, when we intuitively understand how the space should be transformed to get an expectable result, for example, to concentrate electromagnetic energy. The genuine triumph of this approach is an invisibility cloak. It is natural to squeeze the region of the virtual (electromagnetic) space $\bar{r}'$ to get an invisible cavity in the physical space $\bar{r}$ (Appendix A). However, it is not obvious, which space transformations should be applied to receive a prescribed field distribution in the region.

We propose to look at TO from another perspective. Instead of geometrical definitions of devices with squeezing and stretching of space we put forward the requirement of the special field in a spatial region, which can be achieved through the boundary conditions the fields obey at the interfaces. Strictly speaking, both approaches (conventional and field-based) at the end are equivalent, but we expect our approach to be more convenient for working with the field transformation.

TO equations for material parameters and fields follow from the invariance of the Maxwell equations with respect to coordinate transformations (see Appendix A). The inverse Jacobian matrix $J^{-1} = \nabla \otimes \bar{r}'(\bar{r})$ (see Appendix B for details) can be presented in terms of potential functions $\psi_i(\bar{r}) = (\bar{n}_i \bar{r}')$, $i = 1, 2, 3$:

$$J^{-1} = \nabla\psi_1 \otimes \bar{n}_1 + \nabla\psi_2 \otimes \bar{n}_2 + \nabla\psi_3 \otimes \bar{n}_3, \tag{1}$$

where $\bar{a} \otimes \bar{b}$ is a dyad, and $\otimes$ stands for the dyadic (tensor or outer) product of vectors $\bar{a}$ and $\bar{b}$; $\bar{n}_1$, $\bar{n}_2$, and $\bar{n}_3$ are a triple of unit orthogonal vectors. The set of arbitrary orthogonal vectors $\bar{n}_i$ is important for presenting electric field $\bar{E}'$ in an arbitrary way, so that we are not limited by the choice of a special coordinate system. At the same time, in many situations it can be beneficial to specify $\bar{n}_i$ as Cartesian basis vectors. Functions ψ_i can be complex functions dependent on parameters, e.g., frequency ω, thus differing in the general case from the usual coordinates in the electromagnetic space.

Starting from conventional equations of TO (2)–(5), we will interpret them in another way and thus introduce the concept of field transformation optics. Electric $\bar{E}(\bar{r})$ and magnetic $\bar{H}(\bar{r})$ fields in the physical space are transformed from fields $\bar{E}'(\bar{r}')$ and $\bar{H}'(\bar{r}')$ in the electromagnetic space by means of the inverse Jacobian matrix:

$$\begin{aligned} \bar{E}(\bar{r}) &= J^{-1}\bar{E}'\left(\bar{r}'\right) = \sum_{i=1}^{3} \nabla\psi_i \left(\bar{n}_i \bar{E}'\right), \\ \bar{H}(\bar{r}) &= J^{-1}\bar{H}'\left(\bar{r}'\right) = \sum_{i=1}^{3} \nabla\psi_i \left(\bar{n}_i \bar{H}'\right). \end{aligned} \tag{2}$$

Looking for the Jacobian matrix in the form $J = \bar{n}_1 \otimes \bar{a}'_1 + \bar{n}_2 \otimes \bar{a}'_2 + \bar{n}_3 \otimes \bar{a}'_3$ and using the identity $JJ^{-1} = 1$, we further derive

$$J = \frac{\sum_{i=1}^{3} \bar{n}_i \otimes \bar{a}_i}{\nabla\psi_1(\nabla\psi_2 \times \nabla\psi_3)}, \tag{3}$$

where solenoidal vectors $\bar{a}_1 = \nabla\psi_2 \times \nabla\psi_3 = \nabla \times (\psi_2 \nabla\psi_3)$, $\bar{a}_2 = \nabla\psi_3 \times \nabla\psi_1 = \nabla \times (\psi_3 \nabla\psi_1)$, and $\bar{a}_3 = \nabla\psi_1 \times \nabla\psi_2 = \nabla \times (\psi_1 \nabla\psi_2)$ are introduced. The permittivity and permeability tensors in the physical space take the usual form

$$\begin{aligned} \varepsilon &= \frac{J^T \varepsilon' J}{\det(J)} = \frac{\sum_{i,j=1}^{3} \varepsilon'_{ij} \bar{a}_i \otimes \bar{a}_j}{\nabla\psi_1(\nabla\psi_2 \times \nabla\psi_3)}, \\ \mu &= \frac{J^T \mu' J}{\det(J)} = \frac{\sum_{i,j=1}^{3} \mu'_{ij} \bar{a}_i \otimes \bar{a}_j}{\nabla\psi_1(\nabla\psi_2 \times \nabla\psi_3)}, \end{aligned} \tag{4}$$

where $\det(J) = 1/\nabla\psi_1(\nabla\psi_2 \times \nabla\psi_3)$ and $\varepsilon'_{ij} = \bar{n}_i \varepsilon' \bar{n}_j$ are the components of the permittivity tensor ε' in the electromagnetic space. The inductions of the electric and magnetic fields easily follow up:

$$\bar{D} = \varepsilon\bar{E} = \left(\sum_{i,j=1}^{3} \varepsilon'_{ij} \bar{a}_i \otimes \bar{n}_j\right)\bar{E}', \quad \bar{B} = \mu\bar{H} = \left(\sum_{i,j=1}^{3} \mu'_{ij} \bar{a}_i \otimes \bar{n}_j\right)\bar{H}'. \tag{5}$$

The formulae demonstrated in this subsection is just a paraphrase of the well-known relations. Nevertheless, they have a different meaning now. We deal with the fields in different media, but not with the fields in the physical and electromagnetic space. So, $\bar{E}'$ and $\bar{H}'$ are an *arbitrary* pair of fields satisfying the Maxwell equations in the medium with ε' and μ'. In general, these fields and material parameters *do not coincide* with incident fields $\bar{E}^{(inc)}$, $\bar{H}^{(inc)}$ and material tensors $\varepsilon^{(0)}$, $\mu^{(0)}$ in the surrounding space. Potential functions ψ_i are not associated with a coordinate transformation. They are arbitrary functions, which introduce the *general solution* $\bar{E}$ and $\bar{H}$ of the Maxwell equations in the transformation media with ε, μ. These general solutions can be used now to construct TO devices using boundary conditions.

2.2. Boundary Conditions

2.2.1. Transparency

Continuity of the tangential components of the sum of incident ($\bar{E}^{(inc)}$ and $\bar{H}^{(inc)}$) and scattered ($r_E\bar{E}^{(inc)}$ and $r_H\bar{H}^{(inc)}$) fields, and transformed ($\bar{E}$ and $\bar{H}$) fields at the outer boundary of the transformation medium S results in

$$\bar{E}_t|_S = (1 + r_E)\bar{E}_t^{(inc)}|_S, \qquad \bar{H}_t|_S = (1 + r_H)\bar{H}_t^{(inc)}|_S, \tag{6}$$

where $r_{E,H}$ are the local reflection coefficients (matrices in general), the subscript t denotes the tangential components of the vectors.

Transparency means the absence of the scattered field ($r_{E,H} = 0$), i.e.,

$$\bar{E}_t|_S = \bar{E}_t^{(inc)}|_S, \qquad \bar{H}_t|_S = \bar{H}_t^{(inc)}|_S, \tag{7}$$

We refer a reader to Section 3.1 for the detailed discussion of the transparency condition (7).

2.2.2. Illusion

When one looks at object A, but sees object B, it is called an illusion [27–32]. It can be achieved using the advanced boundary conditions

$$\bar{E}_t|_S = \bar{E}_t^{(inc)}|_S + \bar{E}_t^{(sc)}|_S, \qquad \bar{H}_t|_S = \bar{H}_t^{(inc)}|_S + \bar{H}_t^{(sc)}|_S, \tag{8}$$

where the superscript (sc) means "scattering". To obtain the illusion application, one needs to put scattered fields $\bar{E}_t^{(r)}$ and $\bar{H}_t^{(r)}$ of object B at boundary S in expression (8). Then an observer will see exactly object B instead of A.

2.2.3. Absence of Penetration

The energy flux in the transformation medium does not penetrate through the boundary S, when

$$\mathrm{Re}\left[\bar{E}\times\bar{H}^*\right]_n|_S=0, \tag{9}$$

where the subscript n stands for the normal component of the vector with respect to surface S. Boundary condition (9) describes the inner boundary of an invisibility cloak.

It should be noted that Equation (9) is stricter than the condition for evanescent waves arising on the total internal reflection. The evanescent waves do not carry the energy as well, but it is not for any incident wave and not at every point of surface S. In spite of that the total energy flux is equal to zero for the evanescent waves, the local energy flux can exist for electromagnetic beams.

By substituting the fields in the form of expression (2) and assuming potential functions ψ_i being complex-valued, we derive the Poynting vector

$$\begin{aligned}\bar{S} &= (c/8\pi)\mathrm{Re}\Big[\left(E_1'H_2'^*-E_2'^*H_1'\right)(\nabla\psi_1\times\nabla\psi_2^*)\\ &\quad+\left(E_1'H_3'^*-E_3'^*H_1'\right)(\nabla\psi_1\times\nabla\psi_3^*)\\ &\quad+\left(E_2'H_3'^*-E_3'^*H_2'\right)(\nabla\psi_2\times\nabla\psi_3^*)\Big],\end{aligned} \tag{10}$$

where $E_i'=\bar{n}_i\bar{E}'$ and $H_i'=\bar{n}_i\bar{H}'$. This equation should hold true for arbitrary fields $\bar{E}'$ and $\bar{H}'$. Therefore, we get the following three equations:

$$(\nabla\psi_1\times\nabla\psi_2^*)_n|_S=0,\ \ (\nabla\psi_1\times\nabla\psi_3^*)_n|_S=0,\ \ (\nabla\psi_2\times\nabla\psi_3^*)_n|_S=0. \tag{11}$$

Specifically for real potential functions ψ_i we can recall the connection of potential functions with vectors $\bar{a}_i$. Then conditions (11) can be rewritten as $(\bar{a}_i)_n|_S=0$ $(i=1,2,3)$ or, according to Equation (5), as $\bar{D}_n|_S=0$ and $\bar{B}_n|_S=0$ [71].

It should be stressed that condition (9) should be distinguished from the ordinary total internal reflection. The latter also exhibits zero energy flux, but not locally (only as an integral characteristic over the interface). So, evanescent field penetrates into the medium from which reflection occurs. Conditions (11) provide that any incident field does not exist in the cavity.

2.2.4. Predetermined Electromagnetic Field

The continuity conditions at boundary S for fields $\bar{E}^{(in)}$ and $\bar{H}^{(in)}$ and fields in the transformation medium $\bar{E}$ and $\bar{H}$ reads as

$$\bar{E}_t|_S=\bar{E}_t^{(in)}|_S,\qquad \bar{H}_t|_S=\bar{H}_t^{(in)}|_S. \tag{12}$$

Equation (12) has a similar structure to Equation (7) for transparency. Boundary conditions (12) are applicable at the inner boundaries of concentrators, rotators, and TO lenses.

2.2.5. Junction of Transformation Media

When two transformation media border each other, the usual boundary conditions of continuity of the tangential components should hold:

$$\bar{E}_t^{(1)}|_S = \bar{E}_t^{(2)}|_S, \qquad \bar{H}_t^{(1)}|_S = \bar{H}_t^{(2)}|_S, \tag{13}$$

where S is the interface between media 1 and 2. Assuming that $\bar{E}^{(1)} = \sum_{i=1}^{3} \nabla\psi_i^{(1)}(\bar{n}_i\bar{E}')$ and $\bar{E}^{(2)} = \sum_{i=1}^{3} \nabla\psi_i^{(2)}(\bar{n}_i\bar{E}')$ (see Equation (2)), we can rewrite Equation (13) in terms of potential functions as

$$\nabla_t\psi_i^{(1)}|_S = \nabla_t\psi_i^{(2)}|_S, \tag{14}$$

where ∇_t stands for the tangential components of the nabla operator.

3. EXAMPLES

3.1. Transparency in General

Let's test our approach on some examples. The first trial is to impose transparency conditions on a certain space, e.g., on an infinitely long cylinder S in the three-dimensional physical space (see Figure 1). According to Equation (7) we claim continuity of the tangential fields, which can be rewritten in the form

$$\sum_{i=1}^{3} \nabla_t\psi_i\left(\bar{n}_i\bar{E}'\right)|_S = \sum_{i=1}^{3}\left((\bar{n}_i)_t(\bar{n}_i\bar{E}^{(inc)}\right)|_S, \tag{15}$$

where Equation (2) is applied. To provide transparency for arbitrary incident fields $\bar{E}^{(inc)}$ and $\bar{H}^{(inc)}$, the material tensors defined by potential functions ψ_i should not depend on the incident fields. This is possible, if fields $\bar{E}'$ and $\bar{H}'$ are expressed in terms of the incident fields. Equation (15) is split into two equations and we derive

$$\begin{aligned} \nabla_t\psi_i|_S &= \beta_i(\bar{r})(\bar{n}_i)_t|_S, \\ \beta_i(\bar{r})\left(\bar{n}_i\bar{E}'\right)|_S &= \left(\bar{n}_i\bar{E}^{(inc)}\right)|_S, \\ \beta_i(\bar{r})\left(\bar{n}_i\bar{H}'\right)|_S &= \left(\bar{n}_i\bar{H}^{(inc)}\right)|_S, \end{aligned} \tag{16}$$

where β_i ($i = 1, 2, 3$) are some scalar functions. If $\beta_i = 1$, one gets conventional TO expressions:

$$\begin{aligned} \bar{E}'(\psi_1, \psi_2, \psi_3)|_S &= \bar{E}^{(inc)}(x_1, x_2, x_3)|_S, \\ \bar{H}'(\psi_1, \psi_2, \psi_3)|_S &= \bar{H}^{(inc)}(x_1, x_2, x_3)|_S, \end{aligned} \tag{17}$$

that is equality of ψ_i and x_i at boundary S,

$$\psi_i|_S = x_i|_S. \tag{18}$$

Equation (18) means that potential functions ψ_i become the ordinary coordinates at boundary S and equations $\nabla_t \psi_i|_S = (\bar{n}_i)_t|_S$ are fulfilled identically. The example of an invisible region in an arbitrary ambient medium $\varepsilon^{(0)}$, $\mu^{(0)}$, is shown in Figure 1. A field mapping picture in Figure 1(b) is the same as for propagation of a plane wave in a homogeneous medium (Figure 1(a)).

Through the whole paper, commercial software COMSOL Multiphysics based on the Finite-Element Method (FEM) is used for simulations [78]. We specify the scattering boundary conditions or a perfect electric conductor for each of four sides of the numerical domain. Because of the numerical inaccuracies the plane wave front is distorted as one can observe in Figure 1. Nevertheless, the interaction of this distorted field with the transformation medium is small and the transparency phenomenon can be identified.

In Appendix C, the illustration of more general condition $\beta_i \neq 1$ is provided. In this case Equation (18) is violated, and full expressions for tensors ε and μ can differ from those of conventional TO.

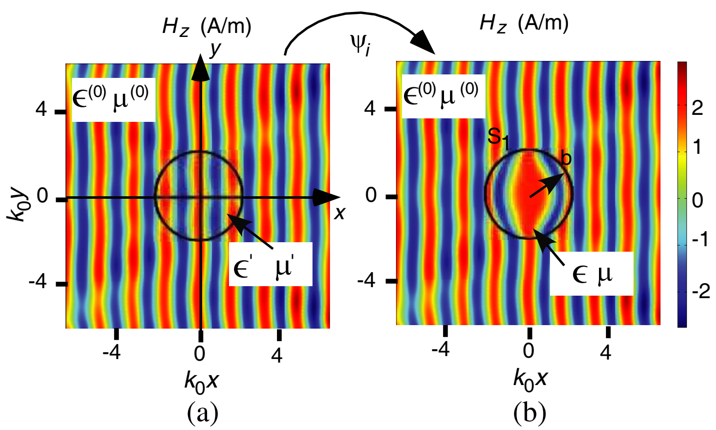


Figure 1. (a) Original (ε', μ') and (b) ψ_i-generated (ε, μ) media are invisible. In COMSOL simulation a plane wave is incident and the cylindrical transformation medium embedded into vacuum $\varepsilon^{(0)} = \mu^{(0)} = 1$ is characterized by the functions $f(r) = g(r) = r\exp(r - b)$ and $h = 1$ (definitions of f, g, and h are given in Section 3.2.2), $\varepsilon = \mu = (1+r)^{-1}\bar{e}_r \otimes \bar{e}_r + (1+r)\bar{e}_\varphi \otimes \bar{e}_\varphi + (1+r)\exp(2r-2b)\bar{e}_z \otimes \bar{e}_z$, where $\bar{e}_r$, $\bar{e}_\varphi$, and $\bar{e}_z$ are the basis vectors of the cylindrical coordinates.

3.2. Invisibility Cloak

An invisibility cloak is the combination of transparency at outer interface S_1 (Section 2.2.1) and absence of penetration through inner interface S_2 (Section 2.2.3):

$$\bar{E}_t|_{S_1} = \bar{E}_t^{(inc)}|_{S_1}, \quad \bar{H}_t|_{S_1} = \bar{H}_t^{(inc)}|_{S_1}, \quad \mathrm{Re}\left[\bar{E} \times \bar{H}^*\right]_n |_{S_2} = 0. \quad (19)$$

We further demonstrate the cloak in two well-studied examples.

3.2.1. Spherical Cloak $a \le r \le b$.

Transparency conditions (18) at the outer boundary $r = b$ requires $\psi_1(b, \theta, \varphi) = b \sin\theta \cos\varphi$, $\psi_2(b, \theta, \varphi) = b \sin\theta \sin\varphi$, and $\psi_3(b, \theta, \varphi) = b \cos\theta$. From the variety of forms for the potential functions we choose

$$\psi_1(\bar{r}) = f(r) \sin\theta \cos\varphi, \ \psi_2(\bar{r}) = g(r) \sin\theta \sin\varphi, \ \psi_3(\bar{r}) = h(r) \cos\theta, \ (20)$$

where $f(b) = g(b) = h(b) = b$. The energy flux will not penetrate through the inner boundary $r = a$, if Equation (11) holds true:

$$f(a)g(a) = 0, \qquad f(a)h(a) = 0, \qquad g(a)h(a) = 0. \qquad (21)$$

Thus, the invisibility cloak is realized if $f(a) = g(a) = 0$, $f(a) = h(a) = 0$, or $g(a) = h(a) = 0$. The material tensors can be calculated using Equation (4). The parameters of a conventional spherical cloak are recovered if $f = g = h$.

3.2.2. Cylindrical Cloak $a \le r \le b$

Potential functions

$$\psi_1(\bar{r}) = f(r) \cos\varphi, \qquad \psi_2(\bar{r}) = g(r) \sin\varphi, \qquad \psi_3(\bar{r}) = h(r) z \qquad (22)$$

meet transparency conditions $f(b) = g(b) = b$ and $h(b) = 1$. Then at the inner interface we derive

$$f(a)h(a) = 0, \qquad g(a)h(a) = 0. \qquad (23)$$

Invisibility cloaking appears for $f(a) = g(a) = 0$, or $h(a) = 0$, and the ordinary cylindrical cloak parameters are restored within the case $f = g$ and $h = 1$.

3.3. Two-shell Cloak

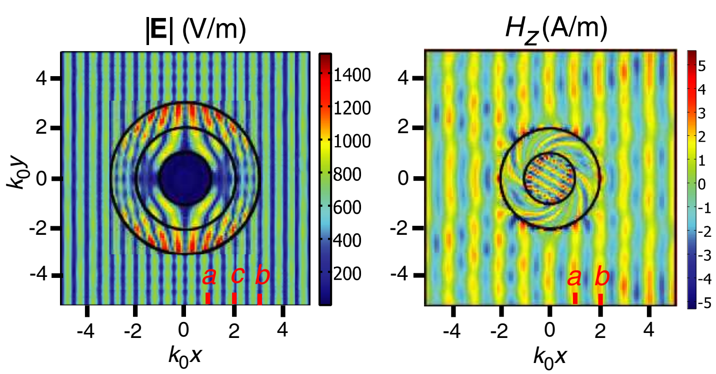
With this example we demonstrate both the invisibility cloak and connection of two transformation media (see Section 2.2.5). Let the external transformation medium occupies cylindrical layer $c \le r \le b$. It can be characterized by functions $f_1(r) = g_1(r) = r^2/b$ and $h = 1$, or material parameters $\varepsilon = \mu = 0.5\bar{e}_r \otimes \bar{e}_r + 2\bar{e}_\varphi \otimes \bar{e}_\varphi + 2(r^2/b^2)\bar{e}_z \otimes \bar{e}_z$. The external layer serves only for the transparency purposes. The internal transformation medium is in region $a < r \leq c$ and has $f_2(r) = (c^2/b)(r-a)/(c-a)$, which provides continuity of the potential functions at the interface as $f_2(c) = f_1(c) = c^2/b$ and absence of the energy flux through the inner interface via $f_2(a) = 0$. The dielectric permittivity and magnetic permeability corresponding to such potential functions are $\varepsilon = \mu = r^{-1}(r-a)\bar{e}_r \otimes \bar{e}_r + r(r-a)^{-1}\bar{e}_\varphi \otimes \bar{e}_\varphi + (c^4(r-a)/rb^2(c-a)^2)\bar{e}_z \otimes \bar{e}_z$. The described two-shell device is the invisibility cloak. Electric field $|\bar{E}|$ (Figure 2(a)) is discontinuous at the boundaries, but the cloak can be recognized due to absence of the scattered field and field in the inner cavity $r < a$.

3.4. Concentrator and Rotator

We consider a cylindrical concentrator-rotator in layer $a < r < b$ [79]. At the outer interface of the device we put the transparency conditions, i.e., the potentials can be chosen similarly to those of the cylindrical cloak:

$$\psi_1 = f(r)\cos(\varphi - \phi(r)), \qquad \psi_2 = f(r)\sin(\varphi - \phi(r)), \qquad \psi_3 = z. \quad (24)$$

Conditions of transparency at the outer interface are $f(b) = b$ and $\phi(b) = 0$. In the inner cavity incident fields should be rotated and amplified: $\bar{E}^{(in)}(r,\varphi,z) = A\bar{E}^{(inc)}(r,\varphi-\varphi_0,z)$ and $\bar{H}^{(in)}(r,\varphi,z) =$



(a) (b)

Figure 2. (a) Electric field $|\bar{E}|$ in the two-shell cloak. Plane waves from the left and from the right form the incident standing wave. (b) Magnetic field H_z for the concentrator-rotator application ($A = 3$, $\varphi_0 = \pi/3$). Incident plane wave goes from left to right.

$A\bar{H}^{(inc)}(r, \varphi - \varphi_0, z)$. So, the boundary conditions at the inner interface are similar to those at the outer interface, if we replace f with f/A and ϕ with $\phi - \varphi_0$. This means we get $f(a) = Aa$ and $\phi(a) = \varphi_0$. Now we can easily choose functions $f(r)$ and $\phi(r)$, e.g., as linear functions $f(r) = b(r-a)/(b-a) + Aa(b-r)/(b-a)$ and $\phi = \varphi_0(b-r)/(b-a)$. Then we derive

$$\varepsilon = \mu = \frac{p(r)}{(aA-b)r}\bar{e}_r \otimes \bar{e}_r + \frac{\varphi_0 p(r)}{(a-b)(aA-b)}(\bar{e}_r \otimes \bar{e}_\varphi + \bar{e}_\varphi \otimes \bar{e}_r) + \frac{r\left((a-b)^2(aA-b)^2 + p(r)^2\varphi_0^2\right)}{p(r)(a-b)^2(aA-b)}\bar{e}_\varphi \otimes \bar{e}_\varphi + \frac{(aA-b)p(r)}{(a-b)^2 r}\bar{e}_z \otimes \bar{e}_z. \quad (25)$$

where $p(r) = (aA-b)r - a(A-1)b$. The magnetic field in the concentrator-rotator is shown in Figure 2(b). One observes the simultaneous rotation and amplification of the field in the inner cavity. To selectively amplify either electric or magnetic field, one can tune the permittivity and permeability of the inner-cavity medium while keeping its refractive index unchanged.

In the concentrator/rotator application we are lucky to have the field inside the cavity in a similar form to the incident field. The conventional transformation optics well handles such cases. In general, fields $\bar{E}^{(inc)}$, $\bar{H}^{(inc)}$ and $\bar{E}^{(in)}$, $\bar{H}^{(in)}$ can be substantially different. In the following section we deal with the TO lensing allowing to get arbitrary field distributions transforming a plane wave.

3.5. Transformation-optics Lenses

The objective of transformation optics is to manipulate light in the desired manner. One of the most potentially-useful examples of such light manipulation is the design of a planar lens capable of creating a spatial beam with predefined properties. In terms of TO the transformation optics, the problem is to obtain the prescribed field mapping after the lens for a known incident field.

We start with the planar lens geometry as an infinite slab occupying space between $x = 0$ and $x = a$ in vacuum. The incident wave is not reflected at the first boundary and converted by the lens into fields $\bar{E}^{(out)}$ and $\bar{H}^{(out)}$. We apply the transparency boundary conditions at $x = 0$ (Section 2.2.1) and conditions of predetermined electromagnetic fields at $x = a$ (Section 2.2.4),

$$\begin{aligned} \bar{E}|_{x=0} &= \bar{E}^{(inc)}|_{x=0}, \qquad & \bar{E}|_{x=a} &= \bar{E}^{(out)}|_{x=a}, \\ \bar{H}|_{x=0} &= \bar{H}^{(inc)}|_{x=0}, \qquad & \bar{H}|_{x=a} &= \bar{H}^{(out)}|_{x=a}. \end{aligned} \quad (26)$$

Transparency conditions (18) are met automatically for potential functions

$$\psi_1(\bar{r}) = x + xf_1(\bar{r}), \quad \psi_2(\bar{r}) = y + xf_2(\bar{r}), \quad \psi_3(\bar{r}) = z + xf_3(\bar{r}). \quad (27)$$

Let us shape the fields in y-direction taking the potential functions in the form: $\psi_1 = x$, $\psi_2 = y + x\eta(y)$, and $\psi_3 = z + x\zeta(y)z$. Then for the y-polarized incident plane wave, the electric and magnetic fields in the transformation medium are

$$\begin{aligned} \bar{E}(x,y) &= \mathrm{e}^{\mathrm{i}k_0x}\left[\eta\bar{e}_x + \left(1 + x\frac{\partial\eta}{\partial y}\right)\bar{e}_y\right], \\ \bar{H}(x,y,z) &= \mathrm{e}^{\mathrm{i}k_0x}\left[z\zeta\bar{e}_x + xz\frac{\partial\zeta}{\partial y}\bar{e}_y + (1 + x\zeta)\bar{e}_z\right]. \end{aligned} \quad (28)$$

Requirement for z-independent fields ($z = 0$) reduces the magnetic field to $\bar{H}(x,y) = \mathrm{e}^{\mathrm{i}k_0x}(1 + x\zeta)\bar{e}_z$. In this case the permittivity and permeability tensors of the lens take the form

$$\varepsilon = \mu = \begin{pmatrix} \left(1 + x\dfrac{\partial\eta}{\partial y}\right)(1 + x\zeta) & -\eta(1 + x\zeta) & 0 \\ -\eta(1 + x\zeta) & \dfrac{1 + x\zeta}{1 + x\dfrac{\partial\eta}{\partial y}}(1 + \eta^2) & 0 \\ 0 & 0 & \dfrac{1 + x\dfrac{\partial\eta}{\partial y}}{1 + x\zeta} \end{pmatrix}. \quad (29)$$

Using the boundary conditions at the second interface ($x = a$) we derive the fields transmitted by the lens: $\bar{E}^{(out)}(a,y) = \mathrm{e}^{\mathrm{i}k_0a}(1 + a\frac{\partial\eta}{\partial y})\bar{e}_y$ and $\bar{H}^{(out)}(a,y) = \mathrm{e}^{\mathrm{i}k_0a}(1 + a\zeta)\bar{e}_z$, where $E_x^{(out)} = 0$ follows from the continuity of the normal component of the displacement vector. Since $\bar{E}^{(out)}$ and $\bar{H}^{(out)}$ are defined by different functions η and ζ, it is feasible to create the desired beam satisfying Maxwell's equations in the semi-space after the lens.

For instance, paraxial beams can be described as $E^{(out)} \approx H^{(out)} = \mathrm{e}^{\mathrm{i}k_0x}f(x,y)$, where function f is a solution of a diffraction equation. As a specific example, we consider generation of spatial Gaussian beam

$$E^{(out)} = A\frac{w_0}{w(x)}\mathrm{e}^{\mathrm{i}k_0x - \mathrm{i}\arctan(x/L) - y^2/w(x)^2}, \quad (30)$$

where $w(x) = w_0\sqrt{1 + x^2/L^2}$ is the beam width and $L = k_0w_0^2/2$. Equating this field at the interface with that expressed via functions η

and ζ we derive

$$\zeta = \frac{\partial \eta}{\partial y} = \left(A \frac{w_0}{w(a)} \mathrm{e}^{-\mathrm{i}\arctan(a/L) - y^2/w(a)^2} - 1 \right) / a,$$
$$\eta = \left(A \frac{\sqrt{\pi} w_0}{2} \mathrm{e}^{-\mathrm{i}\arctan(a/L)} \mathrm{erf}(y/w(a)) - y \right) / a. \tag{31}$$

Choosing $A = \mathrm{e}^{\mathrm{i}\arctan(a/L)}/w_0$ and $w_0 = 1$, we calculate wavenumber k_0, length L, and width $w(a)$ for simulations, the results of which are shown in Figure 3(a). Distributions of electric and magnetic fields are similar to those required by paraxial approximation. The output field is large until $|k_0 y| = 1$ (the point where the fields decay by e times). For greater values of y the transmitted field is rather small. As we designed the lens with transparent boundary at $x = 0$, we have only the incident field before the lens.

The transformation-optics lensing application does not suffer much from loss and dispersion in this case, because the incident wave can be highly monochromatic and we can choose optimal parameters of the transformation medium. Generation of another beam is demonstrated in Figure 3(b), where $E^{(out)}(a, y) = \mathrm{e}^{\mathrm{i}k_0 a}(1 + a\cos(y))$. The outgoing energy flux is redistributed, concentrating in periodically arranged maxima. This is unlike the case for the Gaussian beam, where energy is seen to escape out of the lens through the slab itself (see Figure 3(a)).

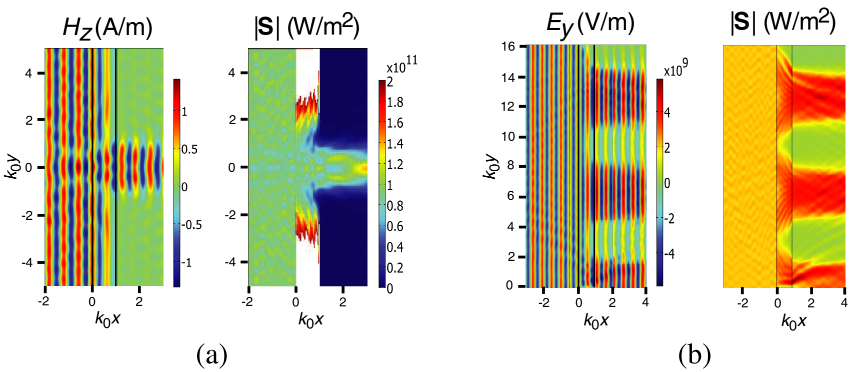


Figure 3. Transformation-optics lenses forming output (a) Gaussian beam and (b) cos-beam. The distributions of electric field E_y, magnetic field H_z and energy flux density $|\overline{S}|$ are depicted. The lens thickness is $k_0 a = 1$, a plane wave is incident from the left-hand side.

4. MATERIAL PARAMETER SINGULARITIES AND FIELD DISCONTINUITIES

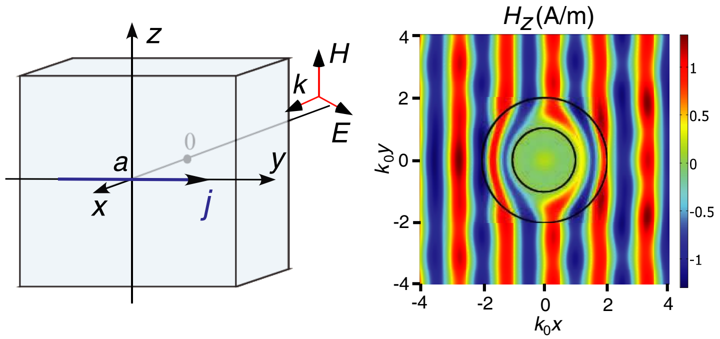
In the frames of the proposed field-based TO approach the notorious problem of TO singularities can be treated consistently. By singularities we mean mostly infinite values of material parameters. The explicit link between fields (potential functions) and material parameters offers a simple way to eliminate such infinite values. Below we demonstrate such approach on a couple of simple examples with plane and cylindrical geometries.

4.1. Invisible Curtain

As a good illustrative example we begin with a planar invisibility "curtain", that is a slab of thickness a infinite in y and z directions (Figure 4(a)). Such curtain being invisible will hide semi-space $x \geq a$ behind it from an external observer positioned in any point within $x < 0$ semi-space. In terms of the scattering problem, there are no both reflected and transmitted waves, energy of an incident wave must be transmitted outwards within the slab. Applying transparency condition (7) at interface $x = 0$, we choose potential functions

$$\psi_1(\bar{r}) = x, \qquad \psi_2(\bar{r}) = f(x)y, \qquad \psi_3(\bar{r}) = g(x)z, \tag{32}$$

where $f(0) = g(0) = 1$. The energy flux through interface $x = a$ (see Equation (11)) nullifies if $\bar{e}_x(\nabla\psi_2 \times \nabla\psi_3) = f(a)g(a) = 0$ with



(a) (b)

Figure 4. (a) Sketch of the planar cloak generating electric current $\bar{j}$ in y-direction to make magnetic field equal zero after the transformation slab. (b) Magnetic field H_z for the non-singular cylindrical cloak, when a plane wave is incident.

two remaining conditions hold identically. Simplifying the case we can choose one trivial function $g(x) = 1$, which requires $f(a) = 0$.

For y-polarized incident plane wave $\bar{E}^{(inc)} = \exp(\mathrm{i}k_0 x)\bar{e}_y$ and $\bar{H}^{(inc)} = \exp(\mathrm{i}k_0 x)\bar{e}_z$, the tangential component of transformed electric field $\bar{E} = \exp(\mathrm{i}k_0 x)(f\bar{e}_y + \frac{df}{dx}y\bar{e}_x)$ at interface $x = a$ equals zero. However, the tangential component of magnetic field $\bar{H} = \exp(\mathrm{i}k_0 x)\bar{e}_z$ does not. Since both electric and magnetic fields behind the slab should be zero, there is discontinuity of the magnetic field at the interface.

The physical reason of discontinuity can be revealed in the following way. Material tensors (4) of the curtain slab are

$$\varepsilon = \mu \; = \; f(\bar{e}_x \otimes \bar{e}_x + \bar{e}_z \otimes \bar{e}_z) + \frac{\left(\frac{df}{dx}y\right)^2 + 1}{f}\bar{e}_y \otimes \bar{e}_y \\ -y\frac{df}{dx}(\bar{e}_x \otimes \bar{e}_y + \bar{e}_y \otimes \bar{e}_x). \tag{33}$$

One of tensors components tends to infinity at the interface: $\varepsilon_{yy}(a, y, z) = \mu_{yy}(a, y, z) = \infty$. This infinite permittivity value reproduces a perfect electric conductor in the y-direction, while infinite permeability does not affect the z-polarized magnetic field. So y-directed electric current $\bar{j} = (\mathrm{i}\omega)/(4\pi)\exp(\mathrm{i}k_0 a)\bar{e}_y$ appears at the interface. It generates a z-directed magnetic field, which compensates tangential magnetic field $\exp(\mathrm{i}k_0 a)\bar{e}_z$ behind the cloaking slab.

The explicit link between fields (scalar function $f(x)$) and infinite material parameter ε_{yy} proposes the clear recipe to eliminate such singularities. Both electric and magnetic fields have to vanish at interface $x = a$. For example, taking $f(x) = g(x)$ results in $f(a) = g(a) = 0$, so the tangential components of both fields turn to zero. The material tensors no longer have infinite components:

$$\varepsilon = \mu \; = \; \left(y\frac{df}{dx}\bar{e}_y + z\frac{df}{dx}\bar{e}_z - f\bar{e}_x\right) \otimes \left(y\frac{df}{dx}\bar{e}_y + z\frac{df}{dx}\bar{e}_z - f\bar{e}_x\right) \\ +\bar{e}_z \otimes \bar{e}_z + \bar{e}_y \otimes \bar{e}_y. \tag{34}$$

These material tensors still possess the zero eigenvalues at interface $x = a$. This means the presence of infinite phase velocities. Nevertheless, we consider the nulls of the material tensors as a minor by-effect of the approach.

4.2. Cylindrical Cloak

A similar reasoning is valid for better-studied cylindrical invisibility cloaks. It is easy to check for a TM-polarized wave incident on a cylindrical cloak that functions $f = g$ and $h = 1$ (see Equation (22)) lead to the infinite azimuthal components of the material tensors and discontinuity of magnetic field H_z at the inner boundary $r = a$. Hence, the electric current appearing at this interface is directed azimuthally, along $\bar{e}_\varphi$. To avoid the singular values, we can try $f = g$ and $h \neq 1$. In this case, all components of the tensors are finite for the price that the material tensors are not diagonal in the cylindrical coordinates:

$$\varepsilon = \mu = \begin{pmatrix} \dfrac{fh}{r\dfrac{df}{dr}} & 0 & -\dfrac{zf\dfrac{dh}{dr}}{r\frac{df}{dr}} \\ 0 & \dfrac{r\dfrac{df}{dr}h}{f} & 0 \\ -\dfrac{zf\dfrac{dh}{dr}}{r\dfrac{df}{dr}} & 0 & \dfrac{f\left(\left(\dfrac{dh}{dr}\right)^2 z^2 + \left(\dfrac{df}{dr}\right)^2\right)}{rh\dfrac{df}{dr}} \end{pmatrix}. \tag{35}$$

A similar introduction of non-singular parameters has been recently made by conformal transformations [80]. It is evident that a nonsingular tensor is achievable when f and h simultaneously go to zero at $r = a$. So, in order to get non-singular parameters, we can replace Equation (9) with boundary conditions

$$\bar{E}_t|_{S_2} = 0, \qquad \bar{H}_t|_{S_2} = 0, \tag{36}$$

or in terms of the potential functions:

$$\nabla_t \psi_i|_{S_2} = 0. \tag{37}$$

One particular illustration of such cylindrical cloak is shown in Figure 4(b). To satisfy requirement (36) we choose the simple variant with $f(r) = bh(r)$, and consider an incident wave in the $z = 0$ cross-section of the cylinder. Then tensors ε and μ are diagonal and non-singular, and $f(r) = g(r) = bh(r) = b(r-a)/(b-a)$. The permittivity and permeability straightforwardly follow from Equation (35).

5. NON-MAGNETIC CLOAK

Typically, invisible cloaks are designed with equal dielectric permittivity and magnetic permeability tensors. However, a non-magnetic cloak is preferable in optics to mitigate material parameters requirements. For the sake of simplicity we consider that a TM-polarized incident wave with E_x, E_y, H_z field components propagates in the $z = 0$ cross-section of the singularity-free cloak (35).

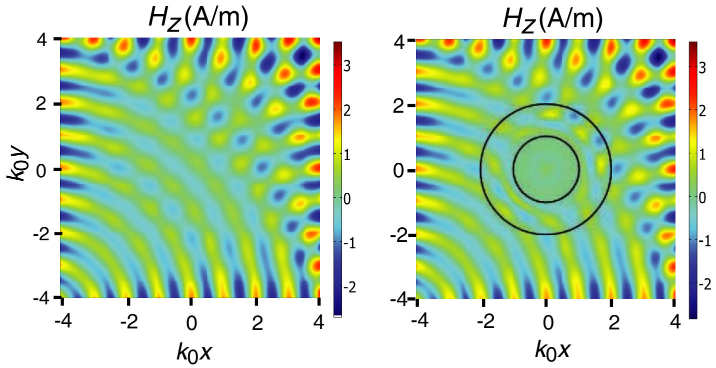


Figure 5. (a) The original TM-polarized field as superposition of two plane waves from left and from bottom. (b) Non-magnetic cloak.

The problem is to get the unit value of μ_{zz} component of the magnetic permeability tensor. This is feasible if we assume $h(r) = (\frac{df}{dr}f)/r$. Then the boundary conditions for the cloaking potential functions are satisfied if $f(a) = 0$, $f(b) = b$, and $\frac{df}{dr}|_{r=b} = 1$. Thus, the required components of the non-singular material tensors are

$$\varepsilon_{rr} = \frac{f^2}{r^2}, \qquad \varepsilon_{\varphi\varphi} = \left(\frac{df}{dr}\right)^2, \qquad \mu_{zz} = 1. \tag{38}$$

The proposed non-magnetic cloak is indeed functional, as demonstrated in Figure 5 with $f(r) = (r-a)[a(2b-a-r)/(b-a)^2+1]$ satisfying the required boundary conditions.

Note that the chosen combination of geometry and incident wave polarization allows ignoring other (non-z) components of the magnetic permeability tensor. For this reason, a non-magnetic cloak in the cylindrical geometry is possible only for the TM polarized incident wave.

6. CONCLUSION

In this paper, we have interpreted transformation optics from the point of view of the field rather than geometrical transformations. We keep the central idea of the invariance of the Maxwell equations with respect to coordinate transformations. However, we replace the coordinates with generalized potential functions, which can be complex and parameter (e.g., frequency) dependent. The potential functions serve to construct the general solutions of the Maxwell equations with the need of properly assigned field boundary conditions. For example, absence of fields inside the hidden region of a cloak is formulated in terms of nullifying the energy flux through the cloak inner interface. Considering several well-known examples (cylindrical cloaks, spherical cloaks, non-magnetic cloaks, concentrators, rotators) we have proved that both geometric and electrodynamic definitions of the TO devices are at the end equivalent. However, the field-based approach is superior in problems, where predetermined field configuration needs to be achieved. Among such problems are designing of transformation optics lenses capable of shaping a plane wave into beams with an arbitrary spatial profile. The case of a Gaussian beam has been demonstrated, and more complicated beams (e.g., Bessel or Airy) are contemplated for further studies.

We have also discussed the field structure of planar and cylindrical invisibility cloaks with singularities (infinite components) of the permittivity/permeability tensors. The explicit links between the potential functions and material parameters allow us to reveal that these singularities appear due to discontinuities in the tangential components of magnetic (electric) fields at the cloak boundaries. In presence of field discontinuity, material singularity creates a current compensating the fields inside the cloaked region. Therefore, a simple approach to avoid singularity has been proposed, by nullifying both electric and magnetic fields at the boundary.

ACKNOWLEDGMENT

The authors would like to dedicate this work as a tribute to the 100-year anniversary of Prof. Fedor I. Fedorov (1911–1994). Authored a number of pioneering works in diverse areas of theoretical physics [81–85], Prof. Fedorov predicted the lateral shift of polarized electromagnetic beams during total internal reflection (the Imbert-Fedorov effect) [86, 87], also known now as the spin-Hall effect of light [88], and proposed a novel vector parametrization of the Lorentz group [85]. The authors acknowledge Prof. Fedorov as their scientific forefather and wish to express their sincere gratitude to him.

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APPENDIX A. BI-ANISOTROPIC TRANSFORMATION MEDIA

We are looking for the relations between fields and material parameters in the Maxwell equations before and after transformation. Let quantities *before* transformation ["original" quantities, expressed in coordinates $\bar{r}' = (x_1', x_2', x_3')$ in the original (electromagnetic, virtual) space] be denoted by symbols with a prime. Likewise, let quantities *after* transformation ["transformed" quantities, expressed in coordinates $\bar{r} = (x_1, x_2, x_3)$ in the transformed (physical) space] be denoted by symbols without a prime. Using an orthonormal basis, we do not distinguish covariant and contravariant quantities in the three-dimensional Euclidean space [81–83].

Constitutive relations for a bi-anisotropic medium [82]

$$\bar{D}' = \varepsilon' \bar{E}' + \alpha' \bar{H}', \qquad \bar{B}' = \mu' \bar{H}' + \kappa' \bar{E}' \tag{A1}$$

connect monochromatic electric and magnetic fields $\bar{E}'$ and $\bar{H}'$ with inductions $\bar{D}'$ and $\bar{B}'$ via dielectric permittivity tensor ε', magnetic permeability tensor μ', and gyration pseudotensors α' and κ'. The notation for the contraction of a tensor and vector as $\varepsilon\bar{E}$ (in index form $(\varepsilon\bar{E})_i = \sum_{j=1}^{3} \varepsilon_{ij} E_j$) can be referred as Fedorov's form [81–83]. Such notation is used, e.g., in [15, 89].

Out of four Maxwell's equations, we consider one curl equation and one divergence equation. The two remaining equations are treated in the same way. Let us start with the curl equation (we adopt the Gaussian system of units)

$$\nabla' \times \bar{E}' = -\frac{1}{c} \frac{\partial \left(\mu' \bar{H}' + \kappa' \bar{E}'\right)}{\partial t} \tag{A2}$$

and find its transformed analogue. Using the link between coordinates $\bar{r} = \bar{r}(\bar{r}')$ we write straightforwardly

$$\nabla_i' = \frac{\partial}{\partial x_i'} = \frac{\partial x_j}{\partial x_i'} \frac{\partial}{\partial x_j}. \tag{A3}$$

Hereafter in this Appendix we assume the Einstein summation rule over repeated indices whenever they show up in vector and tensor components. The coefficients in front of derivatives $\nabla_j = \frac{\partial}{\partial x_j}$ form Jacobian matrix $J_{ij} = \frac{\partial x_j}{\partial x_i'}$. According to [90], the invariance holds only for proper transformations $\det(J) > 0$ (in the case of improper transformations $\det(J) < 0$, one should use the absolute value of the Jacobian). In the index-free notations, contraction of matrix J with vector operator ∇ can be written as

$$\nabla' = J\nabla. \tag{A4}$$

Introducing unit matrices $1 = JJ^{-1}$ before and after the curl operator, one gets to

$$\left(J^T\right)^{-1} J^T (J\nabla)^{\times}(JJ^{-1}\bar{E}') = -\frac{1}{c}\frac{\partial\left(\mu'\bar{H}' + \kappa'\bar{E}'\right)}{\partial t}, \tag{A5}$$

where J^T is the transposed Jacobian matrix, and $\bar{a}^{\times} = (J\nabla)^{\times}$ is the tensor dual to vector $\bar{a} = J\nabla$. The tensor dual to vector $\bar{a}$ is defined as $(\bar{a}^{\times})_{ik} = \varepsilon_{ijk}a_j$ (ε_{ijk} is the antisymmetric Levi-Civita's pseudotensor, which should be distinguished from permittivity ε_{ij}) [81–83]. The Jacobian matrix meets the identity relation

$$\left(\nabla' \times J\right)_{im} = \varepsilon_{ijk}\frac{\partial}{\partial x'_j}J_{km} = \varepsilon_{ijk}\frac{\partial^2 x_m}{\partial x'_j \partial x'_k} = 0, \tag{A6}$$

which permits us to take J out of the curl. We then present the curl Maxwell equation in the form

$$\left(J^T\right)^{-1} \hat{A}\left(J^{-1}\bar{E}'\right) = -\frac{1}{c}\frac{\partial\left(\mu'\bar{H}' + \kappa'\bar{E}'\right)}{\partial t}, \tag{A7}$$

where $\hat{A} = J^T(J\nabla)^{\times}J$ is the matrix differential operator (remember that $\nabla'^{\times}$ does not act on the Jacobian matrix on the right-hand side). We use a hat $\hat{}$ designation to stress that operator $\hat{A}$ is differential. Using the index form of $\hat{A}$ and mathematical relation $\varepsilon_{jmn}J_{ji}J_{ml}J_{nk} = (\det J)\varepsilon_{ilk}$, we derive

$$\hat{A}_{ik} = \varepsilon_{jmn}J_{ji}J_{ml}J_{nk}\nabla_l = (\det J)\varepsilon_{ilk}\nabla_l = (\det J)(\nabla^{\times})_{ik}. \tag{A8}$$

In formula $\varepsilon_{jmn}J_{ji}J_{ml}J_{nk} = (\det J)\varepsilon_{ilk}$, quantities $(\mathbf{u}_i)_j = J_{ji}$ can be treated as components of vectors $\mathbf{u}_i$. When two indices are equal, e.g., $i = l$, $\varepsilon_{iik} = 0$ and $\varepsilon_{jmn}J_{ji}J_{mi}J_{nk} = (\mathbf{u}_i \times \mathbf{u}_i)\mathbf{u}_k = 0$. If three indices are different, e.g., $i = 1$, $l = 2$, and $k = 3$, we arrive at the Jacobian determinant $\det J = \varepsilon_{jmn}J_{j1}J_{m2}J_{n3} = (\mathbf{u}_1 \times \mathbf{u}_2)\mathbf{u}_3$.

So, the Maxwell equation becomes

$$(\det J)\left(J^T\right)^{-1}\nabla^{\times}(J^{-1}\bar{E}') = -\frac{1}{c}\frac{\partial\left(\mu'\bar{H}' + \kappa'\bar{E}'\right)}{\partial t} \tag{A9}$$

or

$$\nabla \times \bar{E} = -\frac{1}{c}\frac{\partial(\mu\bar{H} + \kappa\bar{E})}{\partial t}, \tag{A10}$$

where the transformed fields and material parameters are

$$\bar{E} = J^{-1}\bar{E}', \quad \bar{H} = J^{-1}\bar{H}', \quad \mu = \frac{J^T\mu' J}{\det J}, \quad \kappa = \frac{J^T\kappa' J}{\det J}. \tag{A11}$$

From the second curl Maxwell equation, the transformation relations for the current and two remaining material tensors are derived:

$$\varepsilon = \frac{J^T \varepsilon' J}{\det J}, \qquad \alpha = \frac{J^T \alpha' J}{\det J}, \qquad \bar{j} = \frac{1}{\det J} J^T \bar{j}'. \tag{A12}$$

In the index form, the transformation rule can be presented in the form $\varepsilon_{ij} = (\det J)^{-1} J_{ki} \varepsilon'_{kl} J_{lj}$.

To transform the divergence equation

$$\nabla' \left(\varepsilon' \bar{E}' + \alpha' \bar{H}'\right) = 4\pi\rho' \tag{A13}$$

or

$$J\nabla \left((\det J) \left(J^T\right)^{-1} \left(\varepsilon \bar{E} + \alpha \bar{H}\right)\right) = 4\pi\rho' \tag{A14}$$

it is necessary to prove that

$$\nabla' \left((\det J) \left(J^T\right)^{-1} \right) = 0. \tag{A15}$$

To prove this relation we write the gradient of the determinant as

$$\nabla'_l \det J = \frac{\partial \det J}{\partial x'_l} = \varepsilon_{ijk} \left(\frac{\partial J_{i1}}{\partial x'_l} J_{j2} J_{k3} + J_{i1} \frac{\partial J_{j2}}{\partial x'_l} J_{k3} + J_{i1} J_{j2} \frac{\partial J_{k3}}{\partial x'_l} \right), \tag{A16}$$

and rearrange its first term as

$$\begin{aligned} \varepsilon_{ijk} \frac{\partial J_{i1}}{\partial x'_l} J_{j2} J_{k3} &= \varepsilon_{ijk} J_{im} J^{-1}_{mn} \frac{\partial J_{n1}}{\partial x'_l} J_{j2} J_{k3} \\ &= (\det J) \varepsilon_{m23} J^{-1}_{mn} \frac{\partial J_{n1}}{\partial x'_l} = (\det J) \tilde{J}^{-1}_{n1} \frac{\partial J^T_{1n}}{\partial x'_l}. \end{aligned} \tag{A17}$$

We can then use the identity $\frac{\partial J^T_{mn}}{\partial x'_l} = \frac{\partial^2 x_m}{\partial x'_l \partial x'_n} = \frac{\partial J^T_{ml}}{\partial x'_n}$ to arrive at

$$\nabla'_l \det J = (\det J) \tilde{J}^{-1}_{nm} \frac{\partial J^T_{mn}}{\partial x'_l} = (\det J) \left(J^T\right)^{-1}_{nm} \frac{\partial \left(J^T\right)_{ml}}{\partial x'_n}, \tag{A18}$$

Finally for the divergence we get

$$\begin{aligned} \left(\nabla' \left((\det J) \left(J^T\right)^{-1} \right)\right)_k &= \frac{\partial \det J}{\partial x'_l} \left(J^T\right)^{-1}_{lk} + \det J \frac{\partial \left(J^T\right)^{-1}_{lk}}{\partial x'_l} \\ &= \det J \left(\left(J^T\right)^{-1}_{nm} \frac{\partial J^T_{ml}}{\partial x'_n} \left(J^T\right)^{-1}_{lk} + \frac{\partial \left(J^T\right)^{-1}_{lk}}{\partial x'_l} \right) \\ &= \det J \left(\left(J^T\right)^{-1}_{nm} \frac{\partial J^T_{ml}}{\partial x'_n} + \frac{\partial \left(J^T\right)^{-1}_{nm}}{\partial x'_n} J^T_{ml} \right) \left(J^T\right)^{-1}_{lk} \\ &= (\det J) \frac{\partial \left(J^T\right)^{-1}_{nm} J^T_{ml}}{\partial x'_n} \left(J^T\right)^{-1}_{lk} = (\det J) \frac{\partial \delta_{nl}}{\partial x'_n} \left(J^T\right)^{-1}_{lk} = 0, \end{aligned} \tag{A19}$$

where δ_{nl} is the Kronecker delta.

Expression $(\det J)\left(J^T\right)^{-1}$ can then be taken out of the derivative ∇':

$$\begin{aligned} J\nabla\left((\det J)\left(J^T\right)^{-1}\varepsilon\bar{E}\right) &= J_{ij}\nabla_j\left((\det J)\left(\left(J^T\right)^{-1}\right)_{im}\left(\varepsilon\bar{E}\right)_m\right) \\ &= (\det J)\left(J^T\right)_{ji}\left(\left(J^T\right)^{-1}\right)_{im}\nabla_j\left(\left(\varepsilon\bar{E}\right)_m\right) \\ &= (\det J)\nabla_m\left((\varepsilon\bar{E})_m\right) = (\det J)\nabla\left(\varepsilon\bar{E}\right). \end{aligned} \tag{A20}$$

Thus the Gauss law in the transformed coordinates is

$$\nabla\left(\varepsilon\bar{E}+\alpha\bar{H}\right) = 4\pi\rho, \tag{A21}$$

where

$$\rho = \frac{\rho'}{\det J}. \tag{A22}$$

Gyration pseudotensors α' and κ' are transformed in the same way as permittivity and permeability tensors. The transformed medium is bi-anisotropic only if the original medium possesses nonzero α' and κ'.

APPENDIX B. INDEX-FREE FORM OF A JACOBIAN MATRIX

Assuming the same Cartesian basis $\bar{e}_i = \bar{n}_i$ for two coordinate systems $\bar{r} = \sum_{i=1}^{3} x_i\bar{e}_i$ and $\bar{r}' = \sum_{i=1}^{3} x'_i\bar{n}_i$, the Jacobian matrix $J_{ij} = \partial x_j/\partial x'_i$ can be presented in the index-free (coordinate-free) form:

$$J = \sum_{i,j=1}^{3} J_{ij}\bar{e}_i\otimes\bar{e}_j = \sum_{i=1}^{3}\bar{e}_i\frac{\partial}{\partial x'_i}\otimes\sum_{j=1}^{3}x_j\bar{e}_j = \nabla'\otimes\bar{r}. \tag{B1}$$

If we perform the inverse coordinate transformation, the Jacobian matrix takes a similar form: $J' = \nabla\otimes\bar{r}'$. When the direct and inverse transformations are made one by one, we will arrive at the trivial Jacobian matrix, i.e., $J'J = 1$. So, we conclude that $J' = J^{-1} = \nabla\otimes\bar{r}'$.

APPENDIX C. EXAMPLE OF GENERAL CONDITIONS OF THE TRANSPARENCY

In Section 3.1, we have introduced the general conditions of the transparency (16). In the special situation of $\beta_i = 1$ we have the ordinary Equation (18) posed in TO. However, if parameters of the surrounding medium $\varepsilon^{(0)}$ and $\mu^{(0)}$ do not coincide with ε' and μ', then $\beta_i \neq 1$ and incident fields $\bar{E}^{(inc)}$ and $\bar{H}^{(inc)}$ are no more fields $\bar{E}'$ and $\bar{H}'$. The case $\beta_i \neq 1$ is demonstrated below.

Let an incident wave in the medium $\varepsilon^{(0)}$, $\mu^{(0)}$ has the TM polarization and illuminates a cylindrical transformation medium (with the axis pointing in the y-direction). Then its fields can be represented as

$$\bar{E}^{(inc)}(x,z) = E_x\bar{n}_1 + E_z\bar{n}_3, \qquad \bar{H}^{(inc)}(x,z) = H_y\bar{n}_2. \tag{C1}$$

From the boundary conditions (16) we can derive three equations:

$$\begin{aligned}&\beta_1 E'_x\left(\bar{r}'\right)|_{S_1} = E_x^{(inc)}(\bar{r})|_{S_1}, \quad \beta_3 E'_z\left(\bar{r}'\right)|_{S_1} = E_z^{(inc)}(\bar{r})|_{S_1},\\ &\beta_2 H'_y\left(\bar{r}'\right)|_{S_1} = H_y^{(inc)}(\bar{r})|_{S_1}.\end{aligned} \tag{C2}$$

Due to the TM polarization coefficients β_1 and β_3 connect only electric fields, while β_2 connects magnetic fields. Magnetic fields $H_y^{(inc)}(\bar{r}) = H_y^{(inc)}(x,y,z)$ and $H'_y(\bar{r}') = H'_y(\psi_1,\psi_2,\psi_3)$ as a superposition of the plane waves result in

$$\begin{aligned}&\beta_2\int H'_y(k_x,k_z)\mathrm{e}^{\mathrm{i}n'(k_x\psi_1+k_z\psi_3)}dk_xdk_z|_{S_1}\\ &=\int H_y^{(inc)}(k_x,k_z)\mathrm{e}^{\mathrm{i}n^{(0)}(k_xx+k_zz)}dk_xdk_z|_{S_1},\end{aligned} \tag{C3}$$

where $n^{(0)} = \sqrt{\varepsilon^{(0)}\mu^{(0)}}$ and $n' = \sqrt{\varepsilon'\mu'}$ are refractive indices of the media. So we derive

$$\beta_2 H'_y(k_x,k_z) = H_y^{(inc)}(k_x,k_z), \quad n'\psi_1|_{S_1} = n^{(0)}x|_{S_1}, \quad n'\psi_3|_{S_1} = n^{(0)}z|_{S_1}. \tag{C4}$$

Performing similar calculus for incident electric field

$$\bar{E}^{(inc)}(x,z) = \frac{\gamma^{(0)}}{k_0}\int(\bar{n}_1k_z - \bar{n}_3k_x)H_y^{(inc)}(k_x,k_z)\mathrm{e}^{\mathrm{i}n^{(0)}(k_xx+k_zz)}dk_xdk_z, \tag{C5}$$

we have two more equations

$$\begin{aligned}\beta_1\gamma' H'_y(k_x,k_z) &= \gamma^{(0)}H_y^{(inc)}(k_x,k_z),\\ \beta_3\gamma' H'_y(k_x,k_z) &= \gamma^{(0)}H_y^{(inc)}(k_x,k_z),\end{aligned} \tag{C6}$$

where $\gamma^{(0)} = \sqrt{\mu^{(0)}/\varepsilon^{(0)}}$ and $\gamma' = \sqrt{\mu'/\varepsilon'}$ are impedances. The solution of the above equations gives $\beta_1 = \beta_3 = \beta_2\gamma^{(0)}/\gamma'$. To satisfy equations $\nabla_t\psi_i|_{S_1} = \beta_i(\bar{n}_i)_t|_{S_1}$ and Equation (C4) simultaneously, we specify $\beta_1 = \beta_3 = n^{(0)}/n'$ and, then, $\beta_2 = \gamma' n^{(0)}/\gamma^{(0)}n'$.

In contrary to Equation (18) coordinates x_i and potential functions ψ_i do not need to coincide at boundary S_1, as they are connected by means of expressions

$$\psi_1|_{S_1} = \frac{n^{(0)}}{n'}x|_{S_1}, \qquad \psi_2|_{S_1} = \frac{\varepsilon^{(0)}}{\varepsilon'}y|_{S_1}, \qquad \psi_3|_{S_1} = \frac{n^{(0)}}{n'}z|_{S_1}. \tag{C7}$$

The dielectric permittivity and magnetic permeability tensors restored from of expressions (C7)

$$\varepsilon = \frac{\varepsilon^{(0)}\left(\bar{a}_1 \otimes \bar{a}_1 + \bar{a}_3 \otimes \bar{a}_3\right) + \left(\mu^{(0)}/\mu'\right)\bar{a}_2 \otimes \bar{a}_2}{\nabla\psi_1(\nabla\psi_2 \times \nabla\psi_3)},$$
$$\mu = \frac{\left(\varepsilon^{(0)}\mu'/\varepsilon'\right)\left(\bar{a}_1 \otimes \bar{a}_1 + \bar{a}_3 \otimes \bar{a}_3\right) + \mu^{(0)}\bar{a}_2 \otimes \bar{a}_2}{\nabla\psi_1(\nabla\psi_2 \times \nabla\psi_3)} \tag{C8}$$

do not coincide with those following from Equation (18),

$$\varepsilon = \varepsilon^{(0)}\frac{\bar{a}_1 \otimes \bar{a}_1 + \bar{a}_3 \otimes \bar{a}_3 + \bar{a}_2 \otimes \bar{a}_2}{\nabla\psi_1(\nabla\psi_2 \times \nabla\psi_3)},$$
$$\mu = \mu^{(0)}\frac{\bar{a}_1 \otimes \bar{a}_1 + \bar{a}_3 \otimes \bar{a}_3 + \bar{a}_2 \otimes \bar{a}_2}{\nabla\psi_1(\nabla\psi_2 \times \nabla\psi_3)}. \tag{C9}$$

Nevertheless, the tensors components responsible for the propagation of the TM wave are the same in both cases.

REFERENCES

1. Dolin, L., "About the possibility of three-dimensional electromagnetic systems with inhomogeneous anisotropic filling," *Izvestiya Vuzov: Radiophysics*, Vol. 4, 964–967, 1961.
2. Ward, A. J. and J. B. Pendry, "Refraction and geometry in Maxwell's equations," *J. Mod. Opt.*, Vol. 43, 773–793, 1996.
3. Pendry, J. B., D. Schurig, and D. R. Smith, "Controlling electromagnetic fields," *Science*, Vol. 312, 1780–1783, 2006.
4. Leonhardt, U., "Optical conformal mapping," *Science*, Vol. 312, 1777–1780, 2006.
5. Schurig, D., J. J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr, and D. R. Smith, "Metamaterial electromagnetic cloak at microwave frequencies," *Science*, Vol. 314, 977–980, 2006.
6. Leonhardt, U. and T. G. Philbin, "Transformation optics and the geometry of light," *Prog. Opt.*, Vol. 53, 69–152, 2009.
7. Leonhardt, U. and T. G. Philbin, *Geometry and Light: The Science of Invisibility*, Dover, Mineola, 2010.
8. Cummer, S. A., B.-I. Popa, D. Schurig, D. R. Smith, and J. Pendry, "Full-wave simulations of electromagnetic cloaking structures," *Phys. Rev. E*, Vol. 74, 036621, 2006.
9. Schurig, D., J. B. Pendry, and D. R. Smith, "Calculation of material properties and ray tracing in transformation media," *Opt. Express*, Vol. 14, 9794–9804, 2006.

10. Gabrielli, L. H., J. Cardenas, C. B. Poitras, and M. Lipson, "Silicon nanostructure cloak operating at optical frequencies," *Nat. Photonics*, Vol. 3, 461–463, 2009.
11. Qiu, C.-W., L. Hu, B. Zhang, B.-I. Wu, S. G. Johnson, and J. D. Joannopoulos, "Spherical cloaking using nonlinear transformations for improved segmentation into concentric isotropic coatings," *Opt. Express*, Vol. 17, 13467–13478, 2009.
12. Kwon, D. and D. H. Werner, "Two-dimensional eccentric elliptic electromagnetic cloaks," *Appl. Phys. Lett.*, Vol. 92, 013505, 2008.
13. Yan, M., W. Yan, and M. Qiu, "Invisibility cloaking by coordinate transformation," *Prog. Opt.*, Vol. 52, 261–304, 2009.
14. Cai, W., U. K. Chettiar, A. V. Kildishev, and V. M. Shalaev, "Optical cloaking with metamaterials," *Nat. Photonics*, Vol. 1, 224–227, 2007.
15. Kildishev, A. V., W. Cai, U. K. Chettiar, and V. M. Shalaev, "Transformation optics: Approaching broadband electromagnetic cloaking," *New J. Phys.*, Vol. 10, 115029, 2008.
16. Kildishev, A. V. and V. M. Shalaev, "Engineering space for light via transformation optics," *Opt. Lett.*, Vol. 33, 43–45, 2008.
17. Nicolet, A., F. Zolla, and S. Guenneau, "Electromagnetic analysis of cylindrical cloaks of an arbitrary cross section," *Opt. Lett.*, Vol. 33, 1584–1586, 2008.
18. Greanleaf, A., Y. Kurilev, M. Lassas, and G. Uhlmann, "Invisibility and inverse problems," *Bulletin of the Americam Mathematical Society*, Vol. 46, 55–97, 2009.
19. Luo, Y., H. Chen, J. Zhang, L. Ran, and J. A. Kong, "Design and analytical fullwave validation of the invisibility cloaks, concentrators, and field rotators created with a general class of transformations," *Phys. Rev. B*, Vol. 77, 125127, 2008.
20. Zharova, N. A., I. V. Shadrivov, A. A. Zharov, and Y. S. Kivshar, "Ideal and nonideal invisibility cloaks," *Opt. Express*, Vol. 16, 21369–21374, 2008.
21. Valentine, J., J. Li, T. Zentgraf, G. Bartal, and X. Zhang, "An optical cloak made of dielectrics," *Nat. Mater.*, Vol. 8, 568–571, 2009.
22. Urzhumov, Y. A. and D. R. Smith, "Transformation optics with photonic band gap media," *Phys. Rev. Lett.*, Vol. 105, 163901, 2010.
23. Han, T., C. Qiu, and X. Tang, "An arbitrarily shaped cloak with nonsingular and homogeneous parameters designed using a twofold transformation," *J. Opt.*, Vol. 12, 095103, 2010.

24. Tuniz, A., B. T. Kuhlmey, P. Y. Chen, and S. C. Fleming, "Weaving the invisible thread: Design of an optically invisible metamaterial fibre," *Opt. Express*, Vol. 18, 18095–18105, 2010.
25. Rahm, M., D. Schurig, D. A. Roberts, S. A. Cummer, D. R. Smith, and J. B. Pendry, "Design of electromagnetic cloaks and concentrators using form-invariant coordinate transformations of Maxwell's equations," *Photon. Nanostruct.: Fundam. Applic.*, Vol. 6, 87, 2008.
26. Chen, H. and C. T. Chan, "Transformation media that rotate electromagnetic fields," *Appl. Phys. Lett.*, Vol. 90, 241105, 2007.
27. Lai, Y., H. Chen, Z.-Q. Zhang, and C. T. Chan, "Complementary media invisibility cloak that cloaks objects at a distance outside the cloaking shell," *Phys. Rev. Lett.*, Vol. 102, 093901, 2009.
28. Lai, Y., J. Ng, H. Chen, D. Z. Han, J. J. Xiao, Z.-Q. Zhang, and C. T. Chan, "Illusion optics: The optical transformation of an object into another object," *Phys. Rev. Lett.*, Vol. 102, 253902, 2009.
29. Li, C., X. Meng, X. Liu, F. Li, G. Fang, H. Chen, and C. T. Chan, "Experimental realization of a circuit-based broadband illusion-optics analogue," *Phys. Rev. Lett.*, Vol. 105, 233906, 2010.
30. Schultheiss, V. H., S. Batz, A. Szameit, F. Dreisow, S. Nolte, A. Tunnermann, S. Longhi, and U. Peschel, "Optics in curved space," *Phys. Rev. Lett.*, Vol. 105, 143901, 2010.
31. Luo, Y., L.-X. He, Y. Wang, H. L. W. Chan, and S.-Z. Zhu, "Changing the scattering of sheltered targets," *Phys. Rev. A*, Vol. 83, 043809, 2011.
32. Han, T., C.-W. Qiu, and X. Tang, "Distributed external cloak without embedded antiobjects," *Opt. Lett.*, Vol. 35, 2642–2644, 2010.
33. Jiang, W. X., T. J. Cui, H. F. Ma, X. Y. Zhou, and Q. Cheng, "Cylindrical-to-plane-wave conversion via embedded optical transformation," *Appl. Phys. Lett.*, Vol. 92, 261903, 2008.
34. Ward, A. J. and J. B. Pendry, "Calculating photonic Green's functions using a nonorthogonal finite-difference time-domain method," *Phys. Rev. B*, Vol. 58, 7252–7259, 1998.
35. Shyroki, D. M., "Note on transformation to general curvilinear coordinates for Maxwell's curl equations," 2003, Preprint, arXiv:physics/0307029v2.
36. Shyroki, D. M., "Squeezing of open boundaries by Maxwell-consistent real coordinate transformation," *IEEE Microwave and Wireless Components Letters*, Vol. 16, 576–578, 2006.

37. Shyroki, D. M., "Efficient Cartesian-grid-based modeling of rotationally symmetric bodies," *IEEE Transactions on Microwave Theory and Techniques*, Vol. 55, 1132–1138, 2007.
38. Shyroki, D. M., "Exact equivalent straight waveguide model for bent and twisted waveguides," *IEEE Transactions on Microwave Theory and Techniques*, Vol. 56, 414–419, 2008.
39. Smolyaninov, I. I., "Transformational optics of plasmonic metamaterials," *New J. Phys.*, Vol. 10, 115033, 2008.
40. Huidobro, P. A., M. L. Nesterov, L. Martin-Moreno, and F. J. Garca-Vidal, "Transformation optics for plasmonics," *Nano Lett.*, Vol. 10, 1985–1890, 2010.
41. Liu, Y., T. Zentgraf, G. Bartal, and X. Zhang, "Transformational plasmonics," *Nano Lett.*, Vol. 10, 1991–1997, 2010.
42. Kadic, M., S. Guenneau, and S. Enoch, "Transformational plasmonics: Cloak, concentrator and rotator," *Opt. Express*, Vol. 18, 12027–12032, 2010.
43. Aubry, A., D. Y. Lei, S. A. Maier, and J. B. Pendry, "Interaction between plasmonic nanoparticles revisited with transformation optics," *Phys. Rev. Lett.*, Vol. 105, 233901, 2010.
44. Cummer, S. A. and D. Schurig, "One path to acoustic cloaking," *New J. Phys.*, Vol. 9, 45, 2007.
45. Chen, H. and C. T. Chan, "Acoustic cloaking and transformation acoustics," *J. Phys. D*, Vol. 43, 113001, 2010.
46. Farhat, M., S. Enoch, S. Guenneau, and A. B. Movchan, "Broadband cylindrical acoustic cloak for linear surface waves in a fluid," *Phys. Rev. Lett.*, Vol. 101, 134501, 2008.
47. Popa, B.-I., L. Zigoneanu, and S. A. Cummer, "Experimental acoustic ground cloak in air," *Phys. Rev. Lett.*, Vol. 106, 253901, 2011.
48. Alitalo, P., F. Bongard, J.-F. Zurcher, J. Mosig, and S. Tretyakov, "Expermental verification of broadband cloaking using a volumetric cloak composed of periodically stacked cylindrical transmission-line networks," *Appl. Phys. Lett.*, Vol. 94, 014103, 2009.
49. Tretyakov, S., P. Alitalo, O. Luukkonen, and C. Simovski, "Broadband electromagnetic cloaking of long cylindrical objects," *Phys. Rev. Lett.*, Vol. 103, 103905, 2009.
50. Liu, R., C. Ji, J. J. Mock, J. Y. Chin, T. J. Cui, and D. R. Smith, "Broadband ground-plane cloak," *Science*, Vol. 323, 366–369, 2009.
51. Ma, H. F. and T. J. Cui, "Three-dimensional broadband ground-

plane cloak made of metamaterials," *Nat. Commun.*, Vol. 1, 21, 2010.

52. Chen, X., Y. Luo, J. Zhang, K. Jiang, J. B. Pendry, and S. Zhang, "Macroscopic invisibility cloaking of visible light," *Nat. Commun.*, Vol. 2, 176, 2011.
53. Zhang, B., Y. Luo, X. Liu, and G. Barbastathis, "Macroscopic invisible cloak for visible light," *Phys. Rev. Lett.*, Vol. 106, 033901, 2011.
54. Smolyaninov, I. I., V. N. Smolyaninova, A. V. Kildishev, and V. M. Shalaev, "Anisotropic metamaterials emulated by tapered waveguides: Application to optical cloaking," *Phys. Rev. Lett.*, Vol. 102, 213901, 2009.
55. Ergin, T., N. Stenger, P. Brenner, J. B. Pendry, and M. Wegener, "Three-dimensional invisibility cloak at optical wavelengths," *Science*, Vol. 328, 337, 2010.
56. Fischer, J., T. Ergin, and M. Wegener, "Three-dimensional polarization-independent visible-frequency carpet invisibility cloak," *Opt. Lett.*, Vol. 36, 2059–2061, 2011.
57. Collins, P. and J. McGuirk, "A novel methodology for deriving improved material parameter sets for simplified cylindrical cloaks," *J. Opt. A: Pure Appl.*, Vol. 11, 015104, 2009.
58. Jiang, W. X., T. J. Cui, X. M. Yang, Q. Cheng, R. Liu, and D. R. Smith, "Invisibility cloak without singularity," *Appl. Phys. Lett.*, Vol. 93, 194102, 2008.
59. Hu, J., X. Zhou, and G. Hu, "Nonsingular two dimensional cloak of arbitrary shape," *Appl. Phys. Lett.*, Vol. 95, 011107, 2009.
60. Yan, M., Z. Chao, and M. Qiu, "Cylindrical invisibility cloak with simplified material parameters is inherently visible," *Phys. Rev. Lett.*, Vol. 99, 233901, 2007.
61. Li, J. and J. B. Pendry, "Hiding under the carpet: A new strategy for cloaking," *Phys. Rev. Lett.*, Vol. 101, 203901, 2008.
62. Huang, L., D. Zhou, J. Wang, Z. Li, X. Chen, and W. Lu, "Generalized transformation for nonmagnetic invisibility cloak with minimized scattering," *J. Opt. Soc. Am. B*, Vol. 28, 922–928, 2011.
63. Han, T. and C.-W. Qiu, "Isotropic nonmagnetic flat cloaks degenerated from homogeneous anisotropic trapeziform cloaks," *Opt. Express*, Vol. 18, 13038–13043, 2010.
64. Chen, H., B.-I. Wu, B. Zhang, and J. A. Kong, "Electromagnetic wave interactions with a metamaterial cloak," *Phys. Rev. Lett.*, Vol. 99, 063903, 2007.

65. Novitsky, A., C.-W. Qiu, and S. Zouhdi, "Transformation-based spherical cloaks designed by an implicit transformation-independent method: Theory and optimization," *New J. Phys.*, Vol. 11, 113001, 2009.
66. Qiu, C.-W., A. Novitsky, H. Ma, and S. Qu, "Electromagnetic interaction of arbitrary radial-dependent anisotropic spheres and improved invisibility for nonlinear-transformation-based cloaks," *Phys. Rev. E*, Vol. 80, 016604, 2009.
67. Alu, A. and N. Engheta, "Achiving transparency with plasmonic and metamaterial coatings," *Phys. Rev. E*, Vol. 72, 016623, 2005.
68. Rainwater, D., A. Kerkhoff, K. Melin, J. C. Soric, G. Moreno, and A. Alu, "Experimental verification of three-dimensional plasmonic cloaking in free-space," *New J. Phys.*, Vol. 14, 013054, 2012.
69. Alu, A. and N. Engheta, "Cloaking a sensor," *Phys. Rev. Lett.*, Vol. 102, 233901, 2009.
70. Tretyakov, S. A., I. S. Nefedov, and P. Alitalo, "Generalized field-transforming metamaterials," *New J. Phys.*, Vol. 10, 115028, 2008.
71. Yaghjian, A. D. and S. Maci, "Alternative derivation of electromagnetic cloaks and concentrators," *New J. Phys.*, Vol. 10, 115022, 2008.
72. Novitsky, A. V., "Inverse problem in transformation optics," *J. Opt.*, Vol. 13, 035104, 2011.
73. Durnin, J., J. J. Miceli, Jr., and J. H. Eberly, "Diffraction-free beams," *Phys. Rev. Lett.*, Vol. 58, 1499–1504, 1987.
74. Siviloglou, G. A., J. Broky, A. Dogariu, and D. N. Christodoulides, "Observation of accelerating Airy beams," *Phys. Rev. Lett.*, Vol. 99, 213901, 2007.
75. McCall, M. W., A. Favaro, P. Kinsler, and A. Boardman, "A spacetime cloak, or a history editor," *J. Opt.*, Vol. 13, 024003, 2011.
76. Cummer, S. A. and R. T. Thompson, "Frequency conversion by exploiting time in transformation optics," *J. Opt.*, Vol. 13, 024007, 2011.
77. Fridman, M., A. Farsi, Y. Okawachi, and A. L. Gaeta, "Demonstration of temporal cloaking," *Nature*, Vol. 481, 62–65, 2012.
78. http://www.comsol.com/.
79. Zang, X. and C. Jiang, "A rotatable and amplifying optical transformation device," *J. Opt. Soc. Am. B*, Vol. 28, 1082–1087, 2011.
80. Perczel, J., C. Garcia-Meca, and U. Leonhardt, "Partial

transmutation of singularities in optical instruments," *J. Opt.*, Vol. 13, 075103, 2011.

81. Fedorov, F. I., *Optics of Anisotropic Media*, Izdatelstvo AN BSSR, Minsk, 1958.
82. Fedorov, F. I., *Theory of Gyrotropy*, Nauka i Tehnika, Minsk, 1976.
83. Fedorov, F. I. and V. V. Filippov, *Reflection and Transmission of Light by Transparent Crystals*, Nauka i Tehnika, Minsk, 1976.
84. Fedorov, F. I., *Theory of Elastic Waves in Crystals*, Plenum Press, New York, 1968.
85. Fedorov, F. I., *Lorentz Group*, Nauka, Moscow, 1979.
86. Fedorov, F. I., "To the theory of total reflection," *Doklady Akademii Nauk SSSR*, Vol. 105, 465–469, 1955.
87. Imbert, C., "Calculation and experimental proof of the transverse shift induced by total internal reflection of a circularly polarized light beam," *Phys. Rev. D*, Vol. 5, 787–796, 1972.
88. Onoda, M., S. Murakami, and N. Nagaosa, "Hall effect of light," *Phys. Rev. Lett.*, Vol. 93, 083901, 2004.
89. Serdyukov, A. N., I. V. Semchenko, S. A. Tretyakov, and A. Sihvola, *Electromagnetics of Bi-anisotropic Materials: Theory and Applications*, Gordon and Breach Science Publishers, Amsterdam, 2001.
90. Post, E. J., *Formal Structure of Electromagnetics*, Noth-Holland Publishing Company, Amsterdam, 1962.