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# COMPLEX POINT SOURCE FOR THE 3D LAPLACE OPERATOR

**M. J. González-Morales[1,*], R. Mahillo-Isla[1], C. Dehesa-Martínez[1], and E. Gago-Ribas[2]**

[1]Universidad de Valladolid, Depto. Teoría de la Señal y Comunicaciones e I.T., Campus Miguel Delibes s/n, Valladolid 47011, Spain

[2] Universidad de Oviedo, Spain

**Abstract**—The research about the so-called *complex beams*, localized solutions of the Helmholtz wave equation, lead to the problem of finding the sources of such solutions, which may be formally expressed as a Dirac delta function of a complex argument. To investigate about the meaning of the Dirac delta distribution of complex argument, the Green's function of the 3D Poisson problem with a point source localized at an imaginary position in free space is considered. The main physical features of the potential created by that source are described. The inverse problem consists in looking for the real source distribution which causes that potential. The sources appear on a disk in the real space. Their physical interpretation requires a regularization process based on including the border of the disk.

## 1. INTRODUCTION

*Complexification* of the real coordinates to obtain new solutions of partial differential equations is an old idea. It already appears in a letter by Appell about potential theory [1]. It is based on the invariance of the Laplace or Helmholtz operators under translations, which include complex shifts. Some authors proposed replacing a real point source with a complex placed one to obtain Gaussian beams as a paraxial approximation to certain solutions of the Helmholtz wave equation [2–5]. Radiation and scattering problems have been constructed within the framework of these solutions, with special

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* Corresponding author: Maria-Jesus González-Morales (gonmor@tel.uva.es).

attention to paraxial approximations as Gaussian beams [6–11]. A more general description of the exact *complex beams* and the *complex polar and spherical coordinates* resulting from the complex extension of the coordinates has been studied in [12–16]. The aforementioned papers implied that there are some sources which may be *formally* expressed as a Dirac delta function of a complex argument, but omitting its analysis and meaning. Nevertheless, the study of the solutions makes evident that there is a singularity on a *disk* in the real space, which must be the source's support. Kaiser deals with rigorous mathematic studies, for both complex spatial and time coordinates, specially for the Laplace equation in the $\mathbb{R}^n$ space [17, 18]. From the physical point of view, basically he proposes to cover the disk with a oblate spheroid (in $\mathbb{R}^3$) where some equivalent sources may be defined. Mahillo-Isla et al. [19], give a physical solution to the equivalent sources on the disk itself based on the evaluation of the field. Finally, Tagirdzhanov et al. consider in [20] the complexified Green's function of the Helmholtz equation and the field sources that create that solution. In [21], they deal with the singularity that appears in the rim of the disk by covering it by a circular torus.

This work deals with the analysis and the complete physical interpretation of the Dirac delta distribution in a specific problem. The starting point is the Green's function of the 3D Poisson problem with a point source localized at an imaginary position and provided that the potential vanishes at infinity. The resulting potential is a valid solution of the Laplace equation except on a singular surface which is the source's support. This solution was already studied by Gleiser and Pullin [22], in the context of Newtonian gravity and general relativity. The inverse problem consists in looking for the real source distribution which causes that potential. Physical meaning is found by separating the real and imaginary parts of the problem and their solutions. Both parts have a different physical insight. The real part of the potential is created by a surface charge density distributed on the disk which presents a non integrable singularity. To solve this difficulty we propose a *regularization* process which consists in taking into account the border of the disk, which contributes with an infinite charge in such a way that the asymptotic potential is finite as implied by the solution. With this prescription, a regularized source charge density is established. Finally it is checked that those charges certainly produce the given real part of the potential. The imaginary part of the potential is created by a dipole distribution on the same disk. It is checked that the associated dipolar moment surface density creates such a potential and no regularization process is necessary.

## 2. COMPLEXIFIED GREEN'S FUNCTION FOR THE 3D LAPLACE OPERATOR IN FREE SPACE

It is well known that a charge $q$ located at a position given by $\vec{r}_0$ produces at $\vec{r}$ a Coulomb potential

$$\Phi(\vec{r};\vec{r}_0) = \frac{q}{4\pi}\frac{1}{|\vec{r}-\vec{r}_0|}. \tag{1}$$

(Electric permittivity of the medium has been set to one.) This potential is the Green's function of the Poisson problem defined by

$$\begin{cases} \nabla^2\Phi = -q\,\delta^3\left(\vec{r}-\vec{r}_0\right), \\ \Phi\big|_{|\vec{r}-\vec{r}_0|\to\infty} \to 0. \end{cases} \tag{2}$$

where $\delta^3(\vec{r}-\vec{r}_0)$ is the 3D Dirac delta distribution [23].

Let $\vec{\mathbf{r}}_0$ be a complex position vector. (Concerning notation, boldface fonts are used to denote complex quantities.) It is always possible to choose some reference axes such that $OZ$ axis matches $\vec{\mathbf{r}}_0$ direction. Thus, the case with $\vec{\mathbf{r}}_0 = ib\hat{z}$ will be considered. The new Poisson equation becomes formally

$$\nabla^2\mathbf{\Phi} = -q\delta^3\left(\vec{r}-\vec{\mathbf{r}}_0\right). \tag{3}$$

The argument of the potential, $\mathbf{R} \doteq |\vec{r}-\vec{\mathbf{r}}_0| = |\vec{r}-ib\hat{z}|$, becomes complex and it is called *complex distance*. It may be written as

$$\mathbf{R} = \sqrt{x^2+y^2+(z-ib)^2} = \sqrt{x^2+y^2+z^2-b^2-2ibz}. \tag{4}$$

The appearance of a square root in (4) requires a branch cut selection. The branch usually chosen [12], is $\Re\{\mathbf{R}\} \geq \mathbf{0}$ and the branch cut becomes a disk, $\mathcal{D}$, in the real space that includes the circle $\mathcal{S} = \{z = 0,\, x^2+y^2 < b^2\}$ and its border, the circumference $\mathcal{C} = \{z = 0,\, \rho = \sqrt{x^2+y^2} = b\}$. Potential in (1) becomes also complex,

$$\mathbf{\Phi}\left(\vec{r};\vec{\mathbf{r}}_0\right) = \frac{q}{4\pi}\frac{1}{\mathbf{R}}. \tag{5}$$

The complex potential given by (5) fulfills Laplace equation, $\nabla^2\mathbf{\Phi} = 0$, except on the discontinuity defined by the branch cut [22]. It acquires physical meaning by taking individually its real and imaginary parts, which of course fulfill Laplace equation separately. Main physical features of the real part of the potential, $\Phi_R = \Re\{\mathbf{\Phi}\}$, and the imaginary part of the potential, $\Phi_I = \Im\{\mathbf{\Phi}\}$, are schematized in Appendix A.

## 3. COMPLEX POINT SOURCE IN THE REAL SPACE

### 3.1. Depiction of the Problem

The problem now consists in looking for the sources which cause the complex potential given by (5), thus, finding the second term in

$$\nabla^2 \mathbf{\Phi}(\vec{r}) = -f(\vec{r}). \tag{6}$$

Once the particular branch cut defined in Section 2 is chosen, the complex distance in (4), when $z \to 0$, may be written as

$$\mathbf{R} = -i\,\mathrm{sgn}(z)\sqrt{b^2 - x^2 - y^2 - z^2 + 2ibz}, \tag{7}$$

which is valid at both sides of the discontinuity and may be substituted in (5) in order to write the complex potential in terms of the observation points. There is an alternative and more convenient representation based on the Heaviside function, in order to separate the two semi-spaces $z > 0$ and $z < 0$, and their corresponding complex distance given by (7). Thus, when $z \to 0$ and $x^2 + y^2 < b^2$,

$$\mathbf{\Phi}(x, y, z) = \frac{qi}{4\pi} \frac{1}{\sqrt{b^2 - x^2 - y^2 - z^2 + 2ibz}} \mathrm{sgn}(z). \tag{8}$$

After the calculations resulting from (6), the source may be written as

$$f(\vec{r}) = -\frac{q}{2\pi}\left(\frac{i\delta'(z)}{\sqrt{b^2 - x^2 - y^2}} + \frac{b\delta(z)}{(b^2 - x^2 - y^2)^{3/2}}\right), \quad x^2 + y^2 < b^2. \tag{9}$$

The circumference, $\mathcal{C}$, which is excluded at this point, will be included and analyzed later. The real and imaginary parts may be separated, $\nabla^2\Phi_R = -f_R(\vec{r})$ and $\nabla^2\Phi_I = -f_I(\vec{r})$.

### 3.2. Real Part and Its Regularization

The source of the real part of the potential is given by

$$f_R(\vec{r}) = -\frac{qb}{2\pi} \frac{1}{\left(b^2 - x^2 - y^2\right)^{3/2}} \delta(z), \quad x^2 + y^2 < b^2, \tag{10}$$

which certainly defines a surface charge density on the disk, $\mathcal{D}$,

$$\sigma_s(x, y, z = 0) = -\frac{qb}{2\pi} \frac{1}{\left(b^2 - x^2 - y^2\right)^{3/2}}, \quad x^2 + y^2 \le b^2. \tag{11}$$

This result matches (A9) but presents a serious defect. The asymptotic expression of the potential, (A7), states that the total charge on the disk should be equal to $q$, instead, if the total charge is calculated by integration of (11) it turns out to be infinite. To solve this difficulty we propose a *regularization* process which consists in taking into account the border $\rho = b$. It must contribute with a charge, also infinite but with opposite sign in such a way that the whole charge be $q$. The most elementary regularization method consists in the substitution of the upper limit in the charge integral, $b$, by $pb$, being $p$ a non-dimensional parameter, $0 < p < 1$, which may approach to 1. Hence, the charge in the circle is now given by

$$Q_{\mathrm{reg},\mathcal{S}} = -\int_{\rho=0}^{pb} \frac{qb}{\left(b^2 - \rho^2\right)^{3/2}} \rho \, d\rho = q - \frac{q}{\sqrt{1-p^2}}, \quad 0 < p < 1. \tag{12}$$

This strongly suggests that the total regularized charge in the border must be

$$Q_{\mathrm{reg},\mathcal{C}} = \frac{q}{\sqrt{1-p^2}}, \quad 0 < p < 1, \tag{13}$$

uniformly distributed along the circumference to preserve the symmetry around $OZ$ axis.

Therefore, the regularized sources of the potential are

$$f_{R,\mathrm{reg}}(\rho, \varphi, z) = -\frac{qbp^2\delta(z)}{2\pi\left(b^2 - p^2\rho^2\right)^{3/2}} + \frac{q\delta(\rho - b)\delta(z)}{2\pi b\sqrt{1-p^2}}, \quad \rho \leq b. \tag{14}$$

Concerning the first term associated to the charge in the circle, notice that a change of variable has been made in the integral in (12) in such a way that if $\rho$ varied from 0 to $b$, the new variable has been scaled up to $pb$. It is to be understood that the limit when $p \to 1$ must be made subsequently. The second term represents the linear charge distribution by means of a volume density distribution, the integral thereof extended to the whole space certainly represents the regularized charge of the circumference.

### 3.3. Imaginary Part

On the other hand, the source of the imaginary part of the potential is a dipole distribution on the same disk,

$$f_I(\vec{r}) = -\frac{q}{2\pi} \frac{\delta'(z)}{\sqrt{b^2 - x^2 - y^2}}, \quad x^2 + y^2 < b^2, \tag{15}$$

amounting to a dipolar moment surface density of

$$\vec{\Pi}_s = \frac{q}{2\pi} \frac{1}{\sqrt{b^2 - x^2 - y^2}} \hat{z}, \quad x^2 + y^2 < b^2. \tag{16}$$

The total dipolar moment of the source distributed in the whole circle is

$$\int_{\mathcal{S}} \vec{\Pi}_s \cdot d\vec{S} = qb, \tag{17}$$

which is finite and agrees with the asymptotic result obtained in (A16). Hence, the circumference $\rho = b$ does not carry any dipole moment.

## 4. VALIDATION OF THE RESULTS

### 4.1. Real Part

To prove the validity of the regularization process, let us calculate the potential created by the charges in (14) to check that they produce the potential given by (A2) in the limit as $p \to 1$. Some representative observation points have been chosen.

#### *4.1.1. Potential Along* $\boldsymbol{OZ}$ *Axis*

Concerning the regularized potential created by the circumference, all the charge elements are at the same distance of the observation point, $d = \sqrt{b^2 + z^2}$, thus the regularized potential is

$$\Phi_{\text{reg},\mathcal{C}} = \frac{q}{\sqrt{1-p^2}} \frac{1}{4\pi\sqrt{b^2+z^2}}. \tag{18}$$

The regularized potential created by the circle is calculated from the first term in (14) and according to the well known expression

$$\Phi_{\text{reg},\mathcal{S}}(\vec{r}\,) = \int_V \frac{f\left(\vec{r'}\right) dV'}{4\pi\left|\vec{r} - \vec{r'}\right|}. \tag{19}$$

By using cylindrical coordinates, this volume integral can be worked out with the result

$$\Phi_{\text{reg},\mathcal{S}}(\vec{r}\,) = \frac{qp^2\,|z|}{4\pi\left(b^2+p^2z^2\right)} - \frac{qp^2\sqrt{z^2+b^2}}{4\pi\sqrt{1-p^2}\left(b^2+p^2z^2\right)}. \tag{20}$$

The total regularized potential is the sum of the results (18) and (20). Finally, the real potential is obtained by making the limit as $p \to 1$:

$$\Phi_R(z) = \lim_{p\to1} \Phi_{\text{reg}}(z) = \frac{q\,|z|}{4\pi\left(b^2+z^2\right)}, \tag{21}$$

which agrees with (A4).

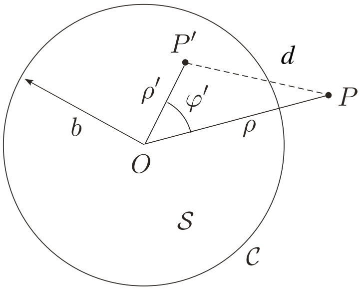


**Figure 1.** Calculation of the potential at a point $P$ located at $z = 0$ plane. $P'$ is any position of the charges on the disk, $\mathcal{D}$, which includes the circle $\mathcal{S} = \{z = 0,\, \rho < b\}$ and the circumference $\mathcal{C} = \{z = 0,\, \rho = b\}$.

### *4.1.2. Potential at the $z = 0$ Plane*

The regularized potential at a point $P$ located at the $z = 0$ plane may be obtained starting from

$$\Phi_{\text{reg}}(P) = \int_{\mathcal{C}} \frac{\lambda_l dl'}{4\pi d} + \int_{\mathcal{S}} \frac{\sigma_s dS'}{4\pi d}, \tag{22}$$

where $d$ is the distance from a source point, $P'$, to a observation point, $P$, as it is shown in Figure 1, and $\lambda_l$ and $\sigma_s$ are the charge densities at the circumference, $\mathcal{C}$, and the circle, $\mathcal{S}$, respectively.

Considering the linear charge density implicit in the second term of (14), and considering also that source point on that circumference with radius $b$ are at a distance $d = (\rho^2 + b^2 - 2b\rho\cos\varphi')^{1/2}$ from the observation points, the contribution of the circumference is

$$\Phi_{\text{reg},\mathcal{C}}(P) = \frac{q}{4\pi^2\sqrt{1-p^2}} \int_0^{\pi} \frac{d\varphi'}{\sqrt{\rho^2 + b^2 - 2b\rho\cos\varphi'}}. \tag{23}$$

After some calculations, this expression can be written as

$$\Phi_{\text{reg},\mathcal{C}}(\rho) = \frac{q}{2\pi^2\sqrt{1-p^2}(\rho+b)} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - m\sin^2\alpha}}, \tag{24}$$

which includes a complete elliptic integral of the first kind, $\mathrm{K}(m)$, being $m = 4b\rho/(b+\rho)^2$. Therefore, the regularized potential due to the circumference is

$$\Phi_{\text{reg},\mathcal{C}}(\rho) = \frac{q}{2\pi^2(b+\rho)\sqrt{1-p^2}} \mathrm{K}\Big(\frac{4b\rho}{(b+\rho)^2}\Big). \tag{25}$$

The regularized potential due to the circle is obtained from the second term of (22), where the surface charge density is found from the first term of (14), and is given by

$$\Phi_{\mathrm{reg},\mathcal{S}}(\rho) = -\frac{qp^2 b}{8\pi^2}\int_0^{2\pi}\int_0^{b}\frac{d\varphi'\,\rho'\,d\rho'}{\left(b^2 - p^2\rho'^2\right)^{3/2}\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\varphi'}} \tag{26}$$

Integration with $\varphi'$ is similar to the one above. The integral with respect to $\rho'$ must be numerically computed, as it has not an analytic expression. For this purpose, it is useful to normalize all lengths with respect to $b$. By calling $\bar{\rho}' = \rho'/b$ and $\bar{\rho} = \rho/b$, (26) may be finally written as

$$\Phi_{\mathrm{reg},\mathcal{S}}(\bar{\rho}) = -\frac{qp^2}{2\pi^2 b}\int_0^{1}\frac{\bar{\rho}'\,d\bar{\rho}'}{(1 - p^2\bar{\rho}'^2)^{3/2}\,(\bar{\rho} + \bar{\rho}')}\mathrm{K}\left(\frac{4\bar{\rho}\bar{\rho}'}{(\bar{\rho} + \bar{\rho}')^2}\right), \tag{27}$$

and, with the same normalization, (25) may be rewritten as

$$\Phi_{\mathrm{reg},\mathcal{C}}(\bar{\rho}) = \frac{q}{2\pi^2 b(1 + \bar{\rho})\sqrt{1 - p^2}}\mathrm{K}\left(\frac{4\bar{\rho}}{(1 + \bar{\rho})^2}\right). \tag{28}$$

To test how the regularization method behaves, Table 1 shows the potentials for points with $\bar{\rho} < 1$ (calculated with coefficient $q/(2\pi^2 b)$ set to unity) due to the circumference and the circle and the total potential for different $\bar{\rho}$ and regularization parameter $p$ values. As expected, the potential vanishes as $p \to 1$. At observation points with $\bar{\rho} > 1$ the calculated regularized total potential, $\Phi_{\mathrm{reg}}$, is an excellent approximation to the exact value given by (A2) as it is shown in Table 2.

**Table 1.** Numerical values of the potentials created by the circle, $\Phi_{\mathcal{S}}$, by the circumference, $\Phi_{\mathcal{C}}$, and total, $\Phi_{\mathcal{T}}$, for different observation points at $z = 0$ plane and for different values of the regularization parameter, $p$. The exact value of $\Phi_{\mathcal{T}}$ at these points is zero.

| $1-p$ | $\bar{\rho}=0.3$ | | | $\bar{\rho}=0.6$ | | |
|---|---|---|---|---|---|---|
| | $\Phi_{\mathcal{S}}$ | $\Phi_{\mathcal{C}}$ | $\Phi_{\mathcal{T}}$ | $\Phi_{\mathcal{S}}$ | $\Phi_{\mathcal{C}}$ | $\Phi_{\mathcal{T}}$ |
| $10^{-3}$ | −35.8907 | 35.9661 | 0.0754 | −39.0588 | 39.1578 | 0.0990 |
| $10^{-4}$ | −113.6852 | 113.7091 | 0.0239 | −123.7688 | 123.8001 | 0.0313 |
| $10^{-5}$ | −359.5640 | 359.5715 | 0.0075 | −391.4715 | 391.4814 | 0.0099 |
| $10^{-6}$ | −1137.0600 | 1137.0624 | 0.0024 | −1237.9671 | 1237.9702 | 0.0031 |
| $10^{-7}$ | −3595.7054 | 3595.7061 | 0.0008 | −3914.8036 | 3914.8046 | 0.0010 |

**Table 2.** Numerical values of the total potential calculated by the regularization method $\Phi_{\text{reg}}$ for different values of the regularization parameter, $p$, at different observation points at $z = 0$ plane. For reference, the exact values of the potential are also shown.

| | $\bar{\rho} = 1.5$ | $\bar{\rho} = 3.0$ | $\bar{\rho} = 10.0$ | $\bar{\rho} = 100.0$ |
|---|---|---|---|---|
| $1 - p$ | $\Phi_{\text{reg}}$ | $\Phi_{\text{reg}}$ | $\Phi_{\text{reg}}$ | $\Phi_{\text{reg}}$ |
| $10^{-3}$ | 1.3860 | 0.5539 | 0.1578 | 0.0157 |
| $10^{-4}$ | 1.3987 | 0.5549 | 0.1579 | 0.0157 |
| $10^{-5}$ | 1.4030 | 0.5552 | 0.1579 | 0.0157 |
| $10^{-6}$ | 1.4043 | 0.5553 | 0.1579 | 0.0157 |
| $\Phi_{\text{exact}}$ | 1.4050 | 0.5553 | 0.1579 | 0.0157 |

### 4.2. Imaginary Part

The potential created by the dipolar source distribution $f_I$ given by (15) is

$$\begin{aligned}\Phi_I(\rho, z) &= \int_{\mathcal{S}} \frac{\Pi_s z}{4\pi d^3} dS \\ &= \frac{qz}{8\pi^2} \int_0^b \int_0^{2\pi} \frac{\rho' d\rho' d\varphi'}{\left(b^2 - \rho'^2\right)^{1/2} \left(\rho^2 + z^2 + \rho'^2 - 2\rho\rho' \cos\varphi'\right)^{3/2}}. \end{aligned} \tag{29}$$

Integration with respect to $\varphi'$ may be analytically evaluated and written in terms of the complete elliptic integral of the second kind, $\mathrm{E}(m)$, with the result

$$\begin{aligned}\Phi_I(\rho, z) &= \frac{qz}{2\pi^2} \int_0^b \frac{\rho' d\rho'}{\left[(\rho - \rho')^2 + z^2\right] \sqrt{\left[(\rho + \rho')^2 + z^2\right] (b^2 - \rho'^2)}} \\ &\quad \times \mathrm{E}\left[\frac{4\rho\rho'}{(\rho + \rho')^2 + z^2}\right], \end{aligned} \tag{30}$$

which cannot be evaluated analytically. Nevertheless it has been numerically checked that it provides the same values as (A11). Neither correction nor regularization is necessary.

## 5. CONCLUSIONS

This paper deals with the analysis of the Dirac delta distribution of complex argument in an electrostatic problem. Its detailed study in the real space allows a deeper insight of the complex extension of the real coordinates to complex ones. It is shown that the extended Green's function is a valid solution of the Laplace equation except on a singular surface which is the source's support. Since the complex displacement along one axis has consequences in the orthogonal plane to it, a first interesting conclusion about 3D Dirac delta of complex argument is that it cannot be decomposed in 1D Delta functions as it is made in the real case, thus, $\delta^3(\vec{r}-ib\hat{z}) \neq \delta(x)\delta(y)\delta(z-ib)$. However, it may be interpreted in the real space as $\delta^3(\vec{r}-ib\hat{z}) = f_1(x,y)\delta(z)+if_2(x,y)\delta'(z)$. This paper carefully analyzes its meaning in the real space, always in the context of the particular proposed physical problem.

One of the most interesting aspects is that the result presents not only a singularity but also a non integrable term, which makes not possible its physical interpretation at a first instance. The problem is resolved by a regularization process which consists in taking into account the border of the disk, which contributes with an infinite charge in such a way that a physical prescription is fulfilled, namely, that the total charge must be $q$, as required by Gauss law. Some representative observation points are selected in order to validate the regularization process and the solution. In some simpler cases it has been analytically tested, in some other numerically.

As it may be inferred from (6), the obtained physical interpretation of the complex Dirac delta distribution depends on the particular operator analyzed. The extension of the present method to other problems, as Helmholtz equation with a complex point source and d'Alembert equation with a complex point time source, is currently in progress.

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## APPENDIX A. PHYSICAL DESCRIPTION OF THE POTENTIAL

The analysis of the potential given in (5) is based on the complex distance properties, which may be found in [12–16]. We point out two useful results concerning complex distance. First, expressions of

$\boldsymbol{R} = R' + iR''$ in terms of Cartesian coordinates,

$$R' = \frac{1}{\sqrt{2}} \sqrt{\sqrt{\left(x^2 + y^2 + z^2 - b^2\right)^2 + 4b^2 z^2} + x^2 + y^2 + z^2 - b^2},$$
$$R'' = -\frac{\text{sgn}(z)}{\sqrt{2}} \sqrt{\sqrt{\left(x^2 + y^2 + z^2 - b^2\right)^2 + 4b^2 z^2} - x^2 - y^2 - z^2 + b^2}. \tag{A1}$$

On the other hand, if the complex distance defined by a source shifted to $ib$ is given by $\mathbf{R}$ in (4), the complex distance defined by a source shifted to $-ib$ will be $\mathbf{R}^*$.

### A.1. Analysis of the Real Part

The real part of the potential given by (5) may be written as,

$$\Phi_R(\vec{r}; \vec{\boldsymbol{r}}_0) = \Re\{\boldsymbol{\Phi}\} = \frac{1}{2}\left[\boldsymbol{\Phi} + \boldsymbol{\Phi}^*\right] = \frac{1}{2}\frac{q}{4\pi}\left[\frac{1}{\boldsymbol{R}} + \frac{1}{\boldsymbol{R}^*}\right]. \tag{A2}$$

This expression shows that $\Phi_R$ is obtained by the summation of two complex potentials created by a charge $q/2$ located at $\vec{\mathbf{r}}_0 = ib\hat{z}$ and an identical second charge located at $\vec{\mathbf{r}}_0 = -ib\hat{z}$. Potential $\Phi_R$ may be written in terms of the observation point coordinates by substituting (A1) in (A2),

$$\Phi_R(\vec{r}; \vec{\mathbf{r}}_0) = \frac{q}{8\pi} \frac{\mathbf{R} + \mathbf{R}^*}{\mathbf{R}\mathbf{R}^*} = \frac{q}{8\pi} \frac{\sqrt{2}\sqrt{\sqrt{\left(r^2 - b^2\right)^2 + 4b^2 z^2} + r^2 - b^2}}{\sqrt{\left(r^2 - b^2\right)^2 + 4b^2 z^2}} \tag{A3}$$

where $r^2 = x^2 + y^2 + z^2$ is the squared distance from the observation point to the origin of the reference system. Although complex distance in (4) is singular at the disk $\mathcal{D}$, special circumstance occurs for the potential $\Phi_R$. Inside the circle $\mathcal{S}$ it fulfills that $\mathbf{R} + \mathbf{R}^* = 0$ meanwhile $\mathbf{R}\mathbf{R}^* \neq 0$ consequently $\Phi_R = 0$ at these points. The singularity reduces to the border of the circle, $\mathcal{C}$, where $\mathbf{R} + \mathbf{R}^*$ and $\mathbf{R}\mathbf{R}^*$ are zero and the potential is not defined.

Main qualitative features of $\Phi_R$ behaviour are shown in Figure A1 Let us analyze some particular expressions.

- Potential along $OZ$ axis. When $r = z$ and $\rho = 0$,

$$\Phi_R(\rho = 0, z) = \frac{q}{4\pi} \frac{|z|}{z^2 + b^2}. \tag{A4}$$

Notice that potential vanishes at $z = 0$, then it increases up to a maximum value $\Phi_{R,\max} = q/(8\pi b)$ at $z = \pm b$ and, finally monotonically decreases to zero. Far from the origin, when $|z| >> b$,

$$\Phi_R(\rho = 0, z) \sim \frac{q}{4\pi\,|z|}. \tag{A5}$$

which is the potential created by a point charge $q$.

- Potential at $z = 0$ plane. Thus $r = \rho$. Two cases may be distinguished. Points on the circle $\mathcal{S}$, $\Phi_R(\rho, z = 0) = 0$ as it was argued above. Points outside the circumference $\mathcal{C}$, that is $\rho > b$:

$$\Phi_R(\rho, z = 0) = \frac{q}{4\pi\sqrt{\rho^2 - b^2}}. \tag{A6}$$

Notice that, as $\rho \to b$, $\Phi_R \to \infty$.

- Asymptotic expression. When $r >> b$, the potential becomes

$$\Phi_R \sim \frac{q}{4\pi r}, \tag{A7}$$

which is the potential created by a point charge, $q$, at a distance $r$ from the observation point.

The electric field intensity may be obtained from (A3). Special interest has the field on $\mathcal{S}$, where the potential vanishes. The electric field has only $z$ component. At the side $z = 0^+$ it is given by

$$E_z = -\frac{qb}{4\pi}\frac{1}{(b^2 - \rho^2)^{3/2}}. \tag{A8}$$

This allows obtaining the surface charge density on the circle,

$$\sigma_s = 2E_z = -\frac{qb}{2\pi}\frac{1}{(b^2 - \rho^2)^{3/2}}. \tag{A9}$$

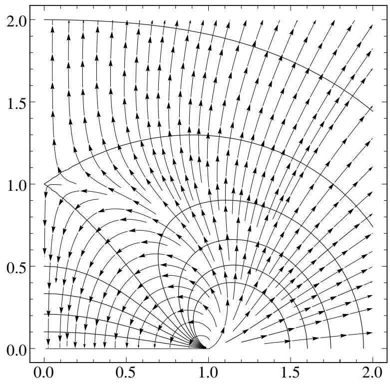


**Figure A1.** Some equipotential surfaces of potential $\Phi_R$ and field lines are plotted in the $(\rho, z)$ plane with coordinates normalized by $b$. Potential vanishes inside the circle $\{\rho < 1, z = 0\}$. For symmetry reasons, only the $z \geq 0$ part is plotted.

## A.2. Analysis of the Imaginary Part

The imaginary part of the complex potential given by (5) may be analyzed in a similar way as the real part, leading to

$$\Phi_I(\vec{r};\vec{\mathbf{r}}_0)=\Im\{\mathbf{\Phi}\}=\frac{1}{2i}\left[\mathbf{\Phi}-\mathbf{\Phi}^*\right]=-\frac{q}{4\pi}\frac{R''}{\mathbf{R}\mathbf{R}^*}. \tag{A10}$$

This potential may be seen as the summation of two complex potentials created by two charges $-iq/2$ and $iq/2$ located at imaginary positions $\vec{\mathbf{r}}_0 = ib\hat{z}$ and $\vec{\mathbf{r}}_0 = -ib\hat{z}$, respectively. Potential given by (A10) may be written in terms of the observation point coordinates by substituting (A1) in (A10),

$$\Phi_I(\rho,z)=\frac{q}{4\sqrt{2}\pi}\mathrm{sgn}(z)\frac{\sqrt{\sqrt{\left(\rho^2+z^2-b^2\right)^2+4b^2z^2}-\rho^2-z^2+b^2}}{\sqrt{\left(\rho^2+z^2-b^2\right)^2+4b^2z^2}}. \tag{A11}$$

Potential cancels at the points where

$$\sqrt{\left(\rho^2+z^2-b^2\right)^2+4b^2z^2}=\rho^2+z^2-b^2. \tag{A12}$$

If both terms are squared, it is found that this condition fulfills only if $z = 0$ and $r^2 > b^2$, thus, when $\rho > b$. This describes plane $z = 0$ outside the circumference $\rho = b$. Notice that denominator of (A11) only cancels in the circumference $\mathcal{C}$. The potential at the circle $\{z = 0,\ \rho < b\}$ is discontinuous, as it may be found from (A11) making $z$ close enough to zero at both sides of the discontinuity. Finally the potential at $z = 0$ plane is given by

$$\Phi_I(\rho,z)=\begin{cases}\dfrac{q}{4\pi}\dfrac{\mathrm{sgn}(z)}{\sqrt{b^2-\rho^2}}, & \text{when} \quad z\to 0 \quad \text{and} \quad \rho<b,\\ 0, & \text{when} \quad z\to 0 \quad \text{and} \quad \rho>b.\end{cases} \tag{A13}$$

Far away from the source, the potential (A11) may be asymptotically approximated by substituting

$$\sqrt{\left(r^2-b^2\right)^2+4b^2z^2}\simeq r^2 \tag{A14}$$

and

$$\sqrt{\sqrt{\left(r^2-b^2\right)^2+4b^2z^2}-r^2+b^2}\simeq\frac{\sqrt{2}b\,|z|}{r} \tag{A15}$$

leading to,

$$\Phi_I\simeq\frac{qb\cos\theta}{4\pi r^2}, \tag{A16}$$

being $r, \theta$ spherical coordinates. This is the familiar expression for the potential created by a point dipole with dipolar moment $\vec{p} = qb\hat{z}$.

## REFERENCES

1. Appell, P. E., "Quelques remarques sur la théorie des potentiels multiformes," *Mathematische Annalen*, Vol. 30, 155–156, 1887.
2. Arnaud, J. A. and H. Kogelnik, "Gaussian light beams with general astigmatism," *Applied Optics*, Vol. 8, No. 8, 1687–1694, 1969.
3. Izmest'ev, A. A., "Wave fields of beam type and spatial quantization of the angular momentum," *Theoretical and Mathematical Physics*, Vol. 7, No. 3, 591–599, 1971.
4. Deschamps, G. A., "Gaussian beam as a bundle of complex rays," *Electron. Lett.*, Vol. 7, No. 23, 684–685, 1971.
5. Felsen, L. B., "Complex-source-point solutions of the field equations and their relations to the propagation and scattering of Gaussian beams," *Symp. Math.*, Vol. 18, 39–56, 1976.
6. Lumori, M. L., "Gaussian beam modeling of SAR enhancement in paraxial and non-paraxial regions of biological tissues," *Progress In Electromagnetics Research M*, Vol. 11, 1–12, 2010.
7. Varaulta, S., S. Boliolib, and J. Sokoloffc, "Scattering by an array of rods using the gaussian beam formalism coupled to the scattering matrix method," *Journal of Electromagnetic Waves and Applications*, Vol. 25, No. 8–9, 1131–1145, 2011.
8. Menga, H. F., W. B. Doub, and J. L. Zhangc, "Generation of mathieu beams in millimeter wave band using diffractive elements," *Journal of Electromagnetic Waves and Applications*, Vol. 25, No. 16, 2296–2307, 2011.
9. Zhang, T.-L., Z.-H. Yan, F.-F. Fan, and B. Li, "Design of a ku-band compact corrugated horn with high gaussian beam efficiency," *Journal of Electromagnetic Waves and Applications*, Vol. 25, No. 1, 123–129, 2011.
10. Lim, S.-H., J.-H. Han, S.-Y. Kim, and N.-H. Myung, "Azimuth beam pattern synthesis for airborne SAR system optimization," *Progress In Electromagnetics Research*, Vol. 106, 295–309, 2010.
11. Mokhtari, A. and A. A. Shishegar, "Rigorous 3D vectorial gaussian beam modeling of demultiplexing performance of virtually-imaged-phased-arrays," *Progress In Electromagnetics Research M*, Vol. 13, 1–16, 2010.
12. Gago-Ribas, E., M. J. González-Morales, and C. Dehesa-Martínez, "Analytical parametrization of a 2D real propagation space in terms of complex electromagnetic beams," *IEICE Trans. on Electronics*, Vol. E80-C, No. 11, 1434–1439, 1997.

13. González-Morales, M. J., C. Dehesa-Martínez, and E. Gago-Ribas, "About complex extensions and their application in electromagnetics," *Springer Proceedings in Physics*, Vol. 104, 81–86, Springer, 2006.
14. Mahillo-Isla, R. and M. J. González-Morales, "Plane wave spectrum of 2D complex beams," *Journal of Electromagnetic Waves and Applications*, Vol. 23, Nos. 8–9, 1123–1131, 2009.
15. González-Morales, M. J., R. Mahillo-Isla, E. Gago-Ribas, and C. Dehesa-Martínez, "3D complex beams in the spatial and the spectral domains," *Journal of Electromagnetic Waves and Applications*, Vol. 24, Nos. 8–9, 1103–1112, 2010.
16. González-Morales, M. J., R. Mahillo-Isla, E. Gago-Ribas, and C. Dehesa-Martínez, "Complex polar coordinates in electromagnetics," *Journal of Electromagnetic Waves and Applications*, Vol. 25, Nos. 2–3, 389–398, 2011.
17. Kaiser, G., "Complex-distance potential theory, wave equations, and physical wavelets," *Mathematical Methods in the Applied Sciences, Special Issue on Clifford Analysis in Applications*, F. Sommen and W. Sproessig, Editors, Vol. 25, 1577–1588, 2002.
18. Kaiser, G., "Physical wavelets and their sources: Real physics in complex spacetime," *Topical Review, Journal of Physics A: Mathematical and General*, Vol. 36, No. 30, 291–338, 2003.
19. Mahillo-Isla, R., M. J. González-Morales, and C. Dehesa-Martínez, "Regularization of complex beams," *12th International Conference on Mathematical Methods in EM Theory*, 242–244, 2008.
20. Tagirdzhanov, A. M., A. S. Blagovestchenskii, and A. P. Kiselev "Complex source: Singularities in real space," *Progress In Electromagnetics Research Symposium Proceedings*, 1527–1530, Moscow, 2009.
21. Tagirdzhanov, A. M., A. S. Blagovestchenskii, and A. P. Kiselev "Complex source wavefields: Sources in real space," *J. Phys. A: Math. Theor.*, Vol. 44, 2011.
22. Gleiser, R. J. and J. A. Pullin "Appell rings in general relativity," *Class. Quantum Grav.* 977–985, 1988.
23. Gel'fand, I. B. and G. E. Shilov, *Generalized Functions*, Academic Press, Inc., 1964.