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THE *REAL-VALUED* TIME-DOMAIN *TE*-MODES IN LOSSY WAVEGUIDES

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Abstract—The time-domain studies of the modal fields in a lossy waveguide are executed. The waveguide has a perfectly conducting surface. Its cross section domain is bounded by a singly-connected contour of rather arbitrary but enough smooth form. Possible waveguide losses are modeled by a conductive medium which fills the waveguide volume. Standard formulation of the boundary-value problem for the system of Maxwell's equations with time derivative is given and rearranged to the transverse-longitudinal decompositions. Hilbert space of the real-valued functions of coordinates and time is chosen as a space of solutions. Complete set of the TE-time-domain modal waves is established and studied in detail. A continuity equation for the conserved energetic quantities for the time-domain modal waves propagating in the lossy waveguide is established. Instant velocity of transportation of the modal flux energy is found out as a function of time for any waveguide cross section. Fundamental solution to the problem is obtained in accordance with the causality principle. Exact explicit solutions are obtained and illustrated by graphical examples.

1. INTRODUCTION

Classical waveguide theory started from the pioneering Lord Rayleigh's article [1]. Since then, during more than a century, almost all theoretical studies were based on the time-harmonic field concept which was put forward in [1]. A lot of useful results were obtained within the framework of the time-harmonic field concept and were

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successfully applied to numerous problems in physics and technology, see [2–5] among many other outstanding publications.

In the time-domain studies, the Maxwell's equations with time derivative were altered (in one way or another) to the Klein-Gordon equation (KGE) well known earlier in quantum physics [6]. Probably, Schelkunoff was the first who altered the time-domain waveguide problem to solving a generalized telegraphist's equation which is reducible, eventually, to the KGE as well [7, 8]. To the best of our knowledge, Gabriel was the first who derived the KGE directly from the Maxwell's equations with time derivative in [9]. More or less regular studies of the time-domain waveguide problems were started at the end of 80s, see [10–19]. One can find many other essential results in the publications referenced there.

In the time-harmonic theory and in the time-domain studies, as well, appropriate class of *complex-valued* quadratically integrable functions is specified as a space of solutions. Definition of that space is usually made by introducing an inner product of the field vectors with applying *complex conjugation* to one of the multipliers. Hence, the energy flux and the stored energy densities can be obtained with automatic *averaging* these quantities over a period of variations of the fields in time. Similar problems arise in any time-domain approach if the space of solutions belongs to the complex-valued functions.

The goal of this article is solving the time-domain waveguide problem in a class of the real-valued functions. Therefore, this approach enables to study various dynamic processes, which are pertinent to the electromagnetic fields and also to their energetic characteristics. The article is composed as follows.

In Section 2, standard formulation of the time-domain problem is presented. In Section 3, the system of Maxwell's equations with time derivative is transformed by the transverse-longitudinal decompositions to a form convenient for analysis just in the time domain. In Section 4, a complete set of the TE-time-domain modes is derived. Every modal field component consists of two factors. One factor is a vector function of transverse waveguide coordinates, $\mathbf{r}$. The other one is a scalar function dependent on axial coordinate, z, and time, t. The vector functions of $\mathbf{r}$-variables, taken jointly, originate a modal basis in the waveguide cross section. The basis elements are derived with needed physical dimensions, namely: (*volt per meter*) and (*ampere per meter*) for the electric and magnetic fields, respectively. Meanwhile, the scalar factors dependent on (z, t) are dimensionless. They have the physical sense of the modal amplitudes. Present theory is intended for *real-valued* solutions. In Section 5, conservation of energy law is considered. A continuity equation is derived for the conserved energetic field quantities averaged over the waveguide cross section. Instant velocity of transportation of the modal field energy as a function of (z, t) is obtained. In Section 6, evolutionary equations for the modal amplitudes are obtained. In Section 7, verification of present theory is performed via applying that to analysis of the time-harmonic modes in the class of real-valued functions. The time-harmonic mode properties, which were established by the classical theory, are corroborated by our theory. However, the real-valued solutions disclose also some new properties, which were inaccessible for the classical approach. In particular, it turns out that propagation of time-harmonic modal fields, as such, is accompanied with a dynamic *energetic wave* process. In Section 8, fundamental solution for the time-domain modal waveguide wave is obtained in compliance with the causality principle. In Section 9, possible extension of the theory and applications of the results are discussed.

2. FORMULATION OF THE PROBLEM

2.1. Description of the Waveguide and Notations

The waveguide under study has a surface with the properties of the perfect electric conductor. The waveguide cross-section domain, S, is bounded by a closed singly-connected contour, L. The shape and size of the contour, L, are invariable along the waveguide axis, Oz. The shape of L may be rather arbitrary provided that none of the possible its *inner* angles (i.e., measured *within* S) exceed π. In particular, the standard waveguides with rectangular cross section satisfy this requirement. Introduce over the contour a right-handed triplet $\{\mathbf{z}, \mathbf{l}, \mathbf{n}\}$ of the mutually orthogonal unit vectors supposing that $\mathbf{z} \times \mathbf{l} = \mathbf{n}$. The vector $\mathbf{z}$ is oriented along the axis, Oz, the vector $\mathbf{l}$ is tangential to the contour, L, and $\mathbf{n}$ is the outer normal to the domain S. Denote a point of observation within the waveguide by a three-component vector $\mathbf{R}$ and an observation time by t.

The waveguide is filled by a lossy medium. For the sake of simplicity, suppose that the relative permittivity and permeability, ε and μ, of the medium are equal to 1 both, but its linear conductivity, σ, may be distinct from *zero*. The case $\sigma = 0$ corresponds to the *hollow* waveguide studied in [18]. The current density, $\mathcal{J}$, which is induced by the waveguide field, $\mathcal{E}$, in the conducting medium is specified by the Ohm's law as $\mathcal{J} = \sigma\,\mathcal{E}$.

2.2. The Set of Equations

We have to solve the curl Maxwell's equations

$$\nabla\times\mathcal{E}\left(\mathbf{R},t\right)=-\mu_0\partial_t\mathcal{H}\left(\mathbf{R},t\right),\quad \nabla\times\mathcal{H}\left(\mathbf{R},t\right)=\epsilon_0\partial_t\mathcal{E}\left(\mathbf{R},t\right)+\sigma\mathcal{E}\left(\mathbf{R},t\right) \tag{1}$$

where the calligraphic letters represent the time-dependent field vectors, $\mathcal{E}$ and $\mathcal{H}$, which specify the electric and magnetic-field strengths. Vector Equation (1) should be solved simultaneously with the divergent equations, i.e.,

$$\epsilon_0\,\nabla\cdot\mathcal{E}\left(\mathbf{R},t\right)=\rho\left(\mathbf{R},t\right)\quad\text{and}\quad\nabla\cdot\mathcal{H}\left(\mathbf{R},t\right)=0 \tag{2}$$

where ρ is the electric charge density induced by the field $\mathcal{E}$ in the medium. Making use of the continuity equation, that is,

$$\nabla\cdot\mathcal{J}\left(\mathbf{R}\cdot t\right)=-\partial_t\rho\left(\mathbf{R}\cdot t\right), \tag{3}$$

one can find a relation between ρ and $\mathcal{E}$ as

$$\rho\left(\mathbf{R}\cdot t\right)=-\sigma\int_0^t\nabla\cdot\mathcal{E}\left(\mathbf{R},t'\right)\,dt'. \tag{4}$$

The field vectors should satisfy the boundary conditions as

$$\mathbf{n}\cdot\mathcal{H}|_L=0,\qquad \mathbf{l}\cdot\mathcal{E}|_L=0,\quad\text{and}\quad\mathbf{z}\cdot\mathcal{E}|_L=0 \tag{5}$$

over the perfectly conducting waveguide surface.

The Maxwell's Equation (1) belong to the *hyperbolic* type of the partial differential equations (PDE). Consequently, they can be supplemented with the *initial conditions* given at a fixed instant (say, $t=0$) and/or with an additional boundary condition given at a fixed axial coordinate (say, at $z=z_0$) in accordance with physical content of a problem under study. Besides, the problem should be solved in compliance with the causality principle.

The physical postulate that the electromagnetic field energy is always finite requires to find out a desirable solution to problem (1)–(5) in a class of integrable vector functions of coordinates and time.

2.3. Remark

The free-space permeability and permittivity constants, μ_0 and ϵ_0 in (1), carry SI units, respectively, as $4\pi\times10^{-7}\,\mathrm{Hm}^{-1}$ *(henry per meter)* and $8.854187817\times10^{-12}\,\mathrm{Fm}^{-1}$ (*farad per meter*). The constant σ in (1) is linear conductivity measurable in Sm^{-1} (*siemens per meter*). The electric current and charge densities, $\mathcal{J}$ and ρ in (3), carry SI units, respectively, as Am^{-2} (*ampere per meter*^{2}) and Cm^{-3} (*coulomb per meter*^{3}). The electromagnetic field vector quantities $\mathcal{E}$ and $\mathcal{H}$ *sought for* are measurable in Vm^{-1} (*volt per meter*) and Am^{-1} (*ampere per meter*), respectively.

3. TRANSVERSE-LONGITUDINAL DECOMPOSITIONS

The position vector, $\mathbf{R}$, and the operator nabla, ∇, are presentable as

$$\mathbf{R} = \mathbf{r} + \mathbf{z}\, z \quad \text{and} \quad \nabla = \nabla_\perp + \mathbf{z}\, \partial_z \tag{6}$$

where $\mathbf{r}$ and $\nabla_\perp$ are the projections of $\mathbf{R}$ and ∇, respectively, on the cross-section domain, S. The field vectors, $\mathcal{E}$ and $\mathcal{H}$, are presentable analogously as

$$\mathcal{E}(\mathbf{R}, t) = \mathbf{E}\,(\mathbf{R}, t) + \mathbf{z}\, E_z\,(\mathbf{R}, t) \text{ and } \mathcal{H}(\mathbf{R}, t) = \mathbf{H}\,(\mathbf{R}, t) + \mathbf{z}\, H_z(\mathbf{R}, t) \tag{7}$$

where the two-component vectors, $\mathbf{E}$ and $\mathbf{H}$, are the projections of the vectors $\mathcal{E}$ and $\mathcal{H}$ on the domain S, respectively. Notice that the argument $(\mathbf{R}, t)$ of the three-component vector functions, $\mathcal{E}$ and $\mathcal{H}$, and their projections in (7) is equivalent to the argument $(\mathbf{r}, z, t)$.

Projecting of the Curl Equations (1): Denote for a while the field vectors, $\mathcal{E}$ and $\mathcal{H}$ each, as a three-component vector function, $\mathcal{F} = \mathbf{F} + \mathbf{z}\, F_z$, dependent on $(\mathbf{r}, z, t)$. Formally, the curl vector, $\nabla \times \mathcal{F}$, is

$$[\nabla \times \mathcal{F}] = (\nabla_\perp + \mathbf{z}\, \partial_z) \times (\mathbf{F} + \mathbf{z}\, F_z) = [\nabla_\perp \times \mathbf{F}] + [\nabla_\perp \times \mathbf{z} F_z] + \partial_z\, [\mathbf{z} \times \mathbf{F}] \tag{8}$$

where the transverse operator nabla, $\nabla_\perp$, acts only on the *transverse* variables, $\mathbf{r}$, in the argument $(\mathbf{r}, z, t)$ of the functions $\mathbf{F}$ and F_z both. Simple manipulations with projecting the vector (8) yield the transverse and longitudinal parts, $[\nabla \times \mathcal{F}]_\perp$ and $[\nabla \times \mathcal{F}]_z$, respectively, of the curl vector as

$$(\nabla \times \mathcal{F})_\perp = [\nabla_\perp \times \mathbf{z}\, F_z] + \partial_z\, [\mathbf{z} \times \mathbf{F}] \text{ and } (\nabla \times \mathcal{F})_z = \nabla_\perp \cdot [\mathbf{F} \times \mathbf{z}] \tag{9}$$

Applying formulas (9) to the first equation from (1) results in one two-component vector equation, (10a), and another scalar one, (10b) namely:

$$\begin{cases} [\nabla_\perp E_z \times \mathbf{z}\,] + \partial_z\, [\mathbf{z} \times \mathbf{E}] = -\mu_0 \partial_t \mathbf{H} & \text{(10a)} \\ \nabla_\perp \cdot [\mathbf{z} \times \mathbf{E}] = \mu_0 \partial_t H_z & \text{(10b)} \end{cases}$$

where the identity $[\nabla_\perp \times \mathbf{z}\, E_z] = [\nabla_\perp E_z \times \mathbf{z}]$ was used in passing. The same manipulations with the second equation from (1) yield a similar pair as

$$\begin{cases} [\nabla_\perp H_z \times \mathbf{z}] + \partial_z\, [\mathbf{z} \times \mathbf{H}] = \epsilon_0 \partial_t \mathbf{E} + \sigma\, \mathbf{E} & \text{(11a)} \\ \nabla_\perp \cdot [\mathbf{H} \times \mathbf{z}] = \epsilon_0 \partial_t E_z + \sigma\, E_z & \text{(11b)} \end{cases}$$

Notice that the set Equations (10)–(11) is equivalent to Equation (1).

3.1. The TE- and TM-modal Field Problems

Compose two problems from the Maxwell's Equations (10)–(11), (2)–(4) and the boundary conditions (5) in the following way. All equations, in which the H_z-component participates, we refer to the set (12). In the position (12a) the vector equation (11a) is placed. Before placing, we multiplied that vectorially from the left-hand side by $\mathbf{z}\times$. In the position (12b), the scalar equation (10b) stands. In the position (12c), the second divergent equation from (2) stands in the appropriate form. The positions (12d) and (12e) are occupied by the boundary conditions taken from (5).

$$\begin{cases} \nabla_\perp H_z = \partial_z \mathbf{H} + \epsilon_0 \partial_t \left[\mathbf{z} \times \mathbf{E}\right] + \sigma \left[\mathbf{z} \times \mathbf{E}\right] & \text{(12a)} \\ \mu_0 \partial_t H_z = \nabla_\perp \cdot \left[\mathbf{z} \times \mathbf{E}\right] & \text{(12b)} \\ \partial_z H_z = -\nabla_\perp \cdot \mathbf{H} & \text{(12c)} \\ \mathbf{n} \cdot \mathbf{H}|_L = 0 & \text{(12d)} \\ \mathbf{l} \cdot \mathbf{E}|_L = 0 & \text{(12e)} \end{cases}$$

We take now the vector Equation (10a), multiply that by $\mathbf{z}\times$ and place the result in the position (13a). In the position (13b), herein, the Equation (11b) stands. In the position (13c), an evident combination of the first divergent equation from (2) and the definition (4) for ρ is placed. In the lines (12d), (12e) and (12f), the boundary conditions (5) are repeated.

$$\begin{cases} \nabla_\perp E_z = \partial_z \mathbf{E} + \mu_0 \partial_t \left[\mathbf{H} \times \mathbf{z}\right] & \text{(13a)} \\ \epsilon_0 \partial_t E_z + \sigma\, E_z = \nabla_\perp \cdot \left[\mathbf{H} \times \mathbf{z}\right] & \text{(13b)} \\ \epsilon_0 \partial_z E_z + \sigma \displaystyle\int_0^t \partial_z E_z dt' = -\epsilon_0 \nabla_\perp \cdot \mathbf{E} + \int_0^t \nabla_\perp \cdot \mathbf{E}\, dt' & \text{(13c)} \\ E_z|_L = 0 & \text{(13d)} \\ \mathbf{l} \cdot \mathbf{E}|_L = 0 & \text{(13e)} \\ \mathbf{n} \cdot \mathbf{H}|_L = 0 & \text{(13f)} \end{cases}$$

Notice that the problems (12) and (13), taken jointly, are completely equivalent to the Equations (1)–(5) from the original formulation of the problem.

Similar transverse-longitudinal decompositions were used previously in study of the *time-harmonic* fields with dependence on time *given in advance* as $\exp(i\omega t)$; take for instance [5]. One can consider the sets (12) and (13) as an extension of those decompositions to a form available for the time-domain analysis.

In this article, we solve the problem (12) for a set of the modal fields like

$$\mathcal{E} = \mathbf{E}'(\mathbf{r}, z, t) \equiv \mathcal{E}', \quad \text{and} \quad \mathcal{H} = \mathbf{H}'(\mathbf{r}, z, t) + \mathbf{z}\, H_z(\mathbf{r}, z, t) \equiv \mathcal{H}' \quad (14)$$

each of which has E_z component equal to zero. In the waveguide theory, it is adopted to name the fields like (14) as *transverse* (with respect to Oz-axis) *electric* ones, or the TE-fields, shortly[†].

Proceed to solving the problem (12) in the class of *real-valued* quadratically integrable vector functions of coordinate and time. One can receive evidence in what follows that the real-valued solutions are much simpler for physical analysis, especially in respect of the energetic field properties.

4. COMPLETE SET OF THE *TE*-TIME-DOMAIN MODES

4.1. Derivation of a Vectorial Modal Basis

The boundary condition (d) from (12) suggests taking the transverse field component composed as

$$\mathbf{H}'(\mathbf{r}, z, t) = I'(z, t)\left[\mu_0^{-\frac{1}{2}}\nabla_\perp\Psi(\mathbf{r})\right] \tag{16}$$

where the potential, Ψ, and the amplitude factor, I', are unknown yet and should be found out further on. The physical constant, $\mu_0^{-\frac{1}{2}}$, was inserted in (16) heuristically in order to provide the field $\mathbf{H}'$ with the required physical dimension (*ampere per meter*) hereinafter. Substitution of the field (16) to that boundary condition yields

$$\mathbf{n}\cdot\nabla_\perp\Psi(\mathbf{r})\,|_L \equiv \partial_\mathbf{n}\Psi(\mathbf{r})\,|_L = 0. \tag{17}$$

If the boundary condition $\mathbf{n}\cdot\mathbf{H}'|_L = 0$ holds, the boundary condition (12e) should hold automatically. To this aim, the field $\mathbf{E}'$ can be taken as

$$\mathbf{E}'(\mathbf{r}, z, t) = V'(z, t)\left[\epsilon_0^{-\frac{1}{2}}\nabla_\perp\Psi(\mathbf{r})\times\mathbf{z}\right] \tag{18}$$

where V' is one more amplitude factor which should be found out hereinafter. The factor $\epsilon_0^{-\frac{1}{2}}$, which is introduced in (18) heuristically, will provide the field $\mathbf{E}'$ with required physical dimension (*volt per meter*) in the end. Notice that identity $[\nabla_\perp\Psi(\mathbf{r})\times\mathbf{z}]\cdot\mathbf{l} = [\mathbf{z}\times\mathbf{l}]\cdot\nabla_\perp\Psi(\mathbf{r}) = \mathbf{n}\cdot\nabla_\perp\Psi(\mathbf{r})$ holds. This results in the boundary condition (17) again.

[†] Problem (13) yields the set of TM-time-domain modal fields as

$$\mathcal{H} = \mathbf{H}''(\mathbf{r}, z, t) \equiv \mathcal{H}'', \text{ and } \mathcal{E} = \mathbf{E}''(\mathbf{r}, z, t) + \mathbf{z}\,E_z(\mathbf{r}, z, t) \equiv \mathcal{E}''. \tag{15}$$

This problem will be solved and analyzed elsewhere because of lack of space herein.

Introduce into consideration H_z-component in the form of

$$H_z(\mathbf{r}, z, t) = A(z, t)\left[\mu_0^{-\frac{1}{2}}\Psi(\mathbf{r})\right] \tag{19}$$

where A is the last amplitude factor unknown yet. Substitution of the field components (16), (18) (19) to the Equations (12b) and (12c) yields

$$\begin{aligned} \partial_{ct}A(z,t)\left[\Psi(\mathbf{r})\right] &= -V'(z,t)\left[-\nabla_\perp^2\Psi(\mathbf{r})\right] \\ \partial_z A(z,t)\left[\Psi(\mathbf{r})\right] &= I'(z,t)\left[-\nabla_\perp^2\Psi(\mathbf{r})\right] \end{aligned} \tag{20}$$

where $\partial_{ct} = \sqrt{\epsilon_0\mu_0}\partial_t$, identity $[\mathbf{z}\times[\nabla_\perp\Psi \times \mathbf{z}]] = \nabla_\perp\Psi$ was used in passing.

The boundary condition (17) itself along with presence of the factor $\left[-\nabla_\perp^2\Psi\right]$ in Equations (20) both induces to take into consideration the Neumann boundary eigenvalue problem for the transverse Laplacian, $\nabla_\perp^2$, formulated as

$$\nabla_\perp^2\Psi_n(\mathbf{r}) + \nu_n^2\Psi_n(\mathbf{r}) = 0 \quad \text{and} \quad \mathbf{n}\cdot\nabla_\perp\Psi_n(\mathbf{r})|_L = 0 \tag{21}$$

where $\nu_n^2 > 0$ are the eigenvalues, $n = 1, 2, \ldots$, and Ψ_n are the appropriate eigensolutions. Besides, number $\nu_0^2 = 0$ is also eigenvalue of the operator $\nabla_\perp^2$ which has an eigensolution, $\Psi_0(\mathbf{r})$, distinct from zero in the general case. We shall take into account this fact at the end of the TE-field analysis.

The differential equation and the boundary condition in (21) are *homogeneous,* i.e., their right-hand sides are equal to *zero*. Hence, the solution, $\Psi_n(\mathbf{r})$, can be factorized as

$$\boxed{\Psi_n(\mathbf{r}) = \mathcal{N}_n'\psi_n(\mathbf{r})} \tag{22}$$

where $\mathcal{N}_n'$ is a normalization constant and $\psi_n(\mathbf{r})$ is a *dimensionless* solution to the same Neumann problem, namely:

$$\boxed{\nabla_\perp^2\psi_n(\mathbf{r}) + \nu_n^2\psi_n(\mathbf{r}) = 0 \quad \text{and} \quad \mathbf{n}\cdot\nabla_\perp\psi_n(\mathbf{r})|_L = 0.} \tag{23}$$

Keeping in mind physical content of the problem under study, introduce a *normalization condition* in a special form as

$$\boxed{(\mathcal{N}_n')^2\left(\nu_n^2/S\right)\int_S|\psi_n(\mathbf{r})|^2\,ds = N} \tag{24}$$

where S is the cross-section domain, N is *newton* (the physical dimension of force). Evidently, $\mathcal{N}_n'$, has the same physical dimension as the product $\nu_n^{-1}N^{\frac{1}{2}}$. Notice that the Helmholtz equation in (23) yields the quantity ν_n with dimension m^{-1} (*inverse meter*) as far as the Laplacian, $\nabla_\perp^2$, has dimension m^{-2} in SI units.

The set of eigensolutions $\{\psi_n(\mathbf{r})\}_{n=0}^{\infty}$ is orthonormal and complete in appropriate Hilbert space what follows from Sturm-Liouville theorems. In other words, this set originates an orthonormal *basis* in that space of quadratically integrable functions.

The set of *normalized* solutions, $\Psi_n(\mathbf{r})$, generates the set of TE-time-domain waveguide modes via choosing the eigenfunctions, Ψ_n, as the potential Ψ introduced above. Substitution of Ψ_n as Ψ and $\left[-\nabla_\perp^2 \Psi_n\right]$ as $\nu_n^2 \Psi_n$ to (20) yields

$$V_n'(z,t) = -\partial_{ct}\, h_n(z,t) \ \text{ and } I_n'(z,t) = \partial_z\, h_n(z,t) \qquad (25)$$

for the eigenvalues $\nu_n^2 \neq 0$. It is convenient to scale the amplitude factor, A_n, as

$$A_n(z,t) = \nu_n^2\, h_n(z,t) \qquad (26)$$

where a new unknown function, h_n, is introduced instead of A_n. Substitutions of Ψ_n as Ψ, and $\nu_n^2 h_n$ as A, to formulas (16), (18), (19) result in

$$\begin{aligned}\mathbf{E}_n'(\mathbf{r},z,t) &= \mathcal{V}_n'(z,t)\left[\epsilon_0^{-\frac{1}{2}}\,\nu_n\,\nabla_\perp \Psi_n(\mathbf{r}) \times \mathbf{z}\right]\\ \mathbf{H}_n'(\mathbf{r},z,t) &= \mathcal{I}_n'(z,t)\left[\mu_0^{-\frac{1}{2}}\,\nu_n\,\nabla_\perp \Psi_n(\mathbf{r})\right]\\ H_{z\,n}(\mathbf{r},z,t) &= h_n(z,t)\ \left[\mu_0^{-\frac{1}{2}}\,\nu_n^2\,\Psi_n(\mathbf{r})\right]\end{aligned} \qquad (27)$$

where the modal amplitudes, $\mathcal{V}_n'$ and $\mathcal{I}_n'$, are specified via partial derivatives (25) of the function $h_n(z,t)$ scaled by a factor, ν_n^{-1}, what yields

$$\boxed{\mathcal{V}_n'(z,t) = -\partial_{\nu_n ct}\, h_n(z,t) \quad \text{and} \quad \mathcal{I}_n'(z,t) = \partial_{\nu_n z}\, h_n(z,t)\,.} \qquad (28)$$

Notice that only one function, h_n, remains unknown herein. One can consider as already known all functions of the transverse coordinates selected by square brackets in (27). They are specified by the formula (22), the Neumann problem (23), and normalization condition (24). The example that follows illustrates how to use this definition in practice.

Example 1 *Consider a standard waveguide with rectangular cross section specified in the Cartesian coordinates,* (x, y), *as* $0 \le x \le a$ *and* $0 \le y \le b$. *Separation of the* x*- and* y*-variables in the Neumann problem* (23) *yields the potential,* $\psi_n(\mathbf{r}) \equiv \psi_n(x,y)$, *as*

$$\psi_n(x,y) \equiv \psi_{p,\,q}(x,y) = \cos(\pi p\, x/a)\cos(\pi q\, y/b)\,, \qquad (29)$$

which corresponds to the eigenvalue distinct from zero,

$$\nu_n^2 \equiv \nu_{p,\,q}^2 = \pi^2\left[(p/a)^2 + (q/b)^2\right], \qquad (30)$$

where the subscript parameter, n, is a doublet, (p, q), composed of the integers, $p = 0, 1, 2, \ldots$ and $q = 0, 1, 2, \ldots$, provided that $p + q \neq 0$. Condition (24) *yields the normalization constant, $\mathcal{N}_n'$, as*

$$\mathcal{N}_n' = 2\,\nu_n^{-1} N^{\frac{1}{2}} \Rrightarrow \mathcal{N}_{p,q}' = 2\,\nu_{p,q}^{-1} N^{\frac{1}{2}}\,. \tag{31}$$

The subscript parameter, $({}_n) \equiv ({}_{p,q})$, identifies the modal type.

4.2. Complete Set of the *TE*-modal Waves

Equation (27) suggests introducing a set of vector functions of variable $\mathbf{r}$ as

$$\begin{aligned}\mathbb{E}_n'(\mathbf{r}) &= \epsilon_0^{-\frac{1}{2}} \mathcal{N}_n'\,\nu_n\,\nabla_\perp \psi_n(\mathbf{r}) \times \mathbf{z}\\ \mathbb{H}_n'(\mathbf{r}) &= \mu_0^{-\frac{1}{2}} \mathcal{N}_n'\,\nu_n\,\nabla_\perp \psi_n(\mathbf{r})\\ \mathbb{H}_{z\,n}(\mathbf{r}) &= \mathbf{z}\left[\mu_0^{-\frac{1}{2}} \mathcal{N}_n'\,\nu_n^2\,\psi_n(\mathbf{r})\right]\end{aligned} \tag{32}$$

where $\nu_n^2 \neq 0$, $\mathbb{E}_n'$ and $\mathbb{H}_n'$ are the two-component transverse vectors, $\mathbb{H}_{z\,n}$ is the one-component vector oriented along Oz-axis. The set of vectors (32) originates a *modal basis* in the waveguide *cross-section* domain, S.

Remark. We have already provided the basis elements (32) with required physical dimensions due to successfully chosen normalization condition (24) ‡. The basis elements $\mathbb{H}_n'$ and $\mathbb{H}_{z\,n}$ have dimension Am^{-1} (*ampere per meter*) and the basis element $\mathbb{E}_n'$ has dimension Vm^{-1} (*volt per meter*).

Thus, the TE-modal fields, $\mathcal{E}_n'$ and $\mathcal{H}_n'$, can be written as

$$\begin{aligned}\mathcal{E}_n'(\mathbf{r},z,t) &= \mathcal{V}_n'(z,t)\,\mathbb{E}_n'(\mathbf{r}) + \mathbf{z}\,0\\ \mathcal{H}_n'(\mathbf{r},z,t) &= \mathcal{I}_n'(z,t)\,\mathbb{H}_n'(\mathbf{r}) + h_n(z,t)\,\mathbb{H}_{z\,n}(\mathbf{r})\end{aligned} \tag{33}$$

where the amplitude h_n remains unknown yet, only. The other amplitudes, $\mathcal{V}_n'$ and $\mathcal{I}_n'$, are expressed via h_n by formulas (28). The modal amplitude h_n should be found out hereinafter as a *dimensionless* quantity. As it follows from Equation (28), the other modal amplitudes, $\mathcal{V}_n'$ and $\mathcal{I}_n'$, will be dimensionless, as well. Besides, the eigenvalue $\nu_n^2 = 0$ from (23) generates one more modal field as

$$\mathcal{E}_0'(\mathbf{r},z,t) = \mathbf{0} \text{ and } \mathcal{H}_0'(\mathbf{r},z,t) = \mathbf{z}\,C\,\mathrm{Am}^{-1} \tag{34}$$

where C is a constant. That is, the modal field (34) is static magnetic.

Introduce a six-component vector, $\mathcal{X}_n'$, composed of the field vectors (33) as $\mathcal{X}_n' = \mathrm{col}(\mathcal{E}_n', \mathcal{H}_n')$ where col means "column". Choose

‡ Constant $\mu_0^{-1/2} N^{1/2}$ has dimension A (ampere) and $\epsilon_0^{-1/2} N^{1/2}$ has dimension V (volt).

a Hilbert space as the space of solutions. Remind that the set $\{\mathcal{E}'_n, \mathcal{H}'_n\}_{n=0}^{\infty}$ consists of the *real-valued* vector elements. Specify that functional space by inner product

$$\left\langle \mathcal{X}'_n, \mathcal{X}'_m \right\rangle = \frac{1}{S} \int_S \left(\epsilon_0 \mathcal{E}'_n \cdot \mathcal{E}'_m + \mu_0 \mathcal{H}'_n \cdot \mathcal{H}'_m \right) ds \tag{35}$$

where $\mathcal{X}'_n$ and $\mathcal{X}'_m$ is a pair of arbitrary real-valued vectors from that set, the free-space constants, ϵ_0 and μ_0, play role of the weighting coefficients.

If $m \neq n$, then $\langle \mathcal{X}'_n, \mathcal{X}'_m \rangle = 0$ due to orthogonality of the eigensolutions to the Neumann problem (23). Physically, that implies orthogonality of the modal fields (33) in the space of solutions. Completeness of the set $\{\mathcal{X}'_n\}_{n=0}^{\infty}$ follows from the Sturm-Liouville's theorems of mathematical physics. If $m = n$, then product (35) yields the modal field energy, W_n, stored at an instant t in the cross-section domain, S, located at a fixed coordinate z.

5. ENERGETIC CHARACTERISTICS OF THE MODAL WAVES

A linkage between the modal fields (33) and their energetic characteristics can be established by applying the Poynting's theorem to Maxwell's Equation (1). Supposing application of that theorem in its integral form later, introduce a control volume, V, bounded by two consecutive waveguide cross sections located at coordinates z and $z+\delta z$ where $\mathbf{r} \in S$ and $\mathbf{r} \in L$. Standard manipulations with the Maxwell's equations result in

$$\frac{1}{V} \oint_{\Sigma} \mathbf{N}_{\Sigma} \cdot \mathcal{P}'_n(\mathbf{r}, z, t)\, d\Sigma = -\frac{1}{V} \int_V \left[\partial_t \left(\epsilon_0 \mathcal{E}'^2_n + \mu_0 \mathcal{H}'^2_n \right) /2 + \sigma \mathcal{E}'^2_n \right] dv \tag{36}$$

where Σ is the surface surrounding V, $\mathbf{N}_\Sigma$ is the outward normal to Σ, $\mathcal{P}'_n = \mathcal{E}'_n \times \mathcal{H}'_n$ is the Poynting's vector, $V = S\delta z$. Notice that $\mathbf{N}_\Sigma \cdot \mathcal{P}'_n = 0$ if $\mathbf{r} \in L$ due to the boundary conditions (5). Substitution of the fields (33) to (36) and integration over the domain S yields

$$\frac{c\nu_n^2 N}{\delta z} \left[\mathbf{z} \cdot \mathcal{P}'_n|_{z+\delta z} - \mathbf{z} \cdot \mathcal{P}'_n|_z \right] = -\frac{\nu_n^2 N}{\delta z} \int_z^{z+\delta z} \left[\partial_t \mathcal{W}'_n(z,t) + 2\varrho \mathcal{V}'^2_n(z,t) \right] dz \tag{37}$$

where the normalization condition (24) is used and notations are adopted as

$$\mathcal{W}'_n(z,t) = \left[\mathcal{V}'^2_n(z,t) + \mathcal{I}'^2_n(z,t) + h_n^2(z,t) \right] /2, \quad \varrho = \frac{\sigma}{2} \sqrt{\mu_0/\epsilon_0} \geq 0. \tag{38}$$

In the limiting case, when $\delta z \to 0$, the quantity $\mathbf{z} \cdot \mathcal{P}'_n|_{z+\delta z}$ in (37) can be approximated by a Taylor series expansion as

$$\mathbf{z} \cdot \mathcal{P}'_n|_{z+\delta z} \approxeq \mathbf{z} \cdot \mathcal{P}'_n|_z + \delta z\, \partial_z \left(\mathbf{z} \cdot \mathcal{P}'_n|_z\right) + \frac{1}{2} (\delta z)^2\; \partial_z^2 \left(\mathbf{z} \cdot \mathcal{P}'_n|_z\right) \ldots . \quad (39)$$

Neglecting by the small quantities of order δz^2 and higher in (39) yields

$$\begin{gathered} \frac{1}{\delta z} \left\{\mathbf{z} \cdot \mathcal{P}'_n|_{z+\delta z} - \mathbf{z} \cdot \mathcal{P}'_n|_z\right\} \approxeq \partial_z \left(\mathbf{z} \cdot \mathcal{P}'_n|_z\right) \equiv \partial_z \mathcal{P}'_{z\,n} \\ \mathcal{P}'_{z\,n}(z,t) = \mathcal{V}'_n(z,t)\, \mathcal{I}'_n(z,t) \end{gathered} \quad (40)$$

where $\mathcal{P}'_{z\,n}$ is z-component of the Poynting vector of the modal field (33) averaged over the waveguide cross section, S. Finally, this result yields a *law of conservation* for the energetic field characteristics as

$$\partial_z \mathcal{P}'_{z\,n}(z,t) + \partial_{ct} \mathcal{W}'_n(z,t) + 2\varrho \mathcal{V}'^2_n(z,t) = 0 \quad (41)$$

where the mean-value theorem was applied for estimation of the integral in (37).

Physically, $P_{z\,n}$ specifies directional along Oz-axis flux of the modal field energy density averaged over the cross-section domain, S. Indeed, the factor $[c\,\nu_n^2 N]$ in (37), where c is the light speed, has physical dimension Wm^{-2} (*watt per meter*^{2}). The quantity $\mathcal{W}'_n$ specifies the modal field energy density stored in the waveguide cross section and averaged over S. The factor $[\nu_n^2 N]$ in (37) has physical dimension Jm^{-3} (*joule per meter*^{3}). The last term in (41), $\varrho \mathcal{V}'^2_n$, has physical sense of the work done by the induced electric current, $\sigma \mathcal{E}'_n$, under action of the electric field, $\mathcal{E}'_n$. That work is converted into a *heat energy.*

Mathematically, Equation (41) specifies the *local* properties of the modal fields in the space solutions. The quadric characteristics, $\mathcal{W}'_n$ and $\mathcal{P}'_{z\,n}$, as such, specify the *global* field properties in that space. Physically, (41) is a *continuity equation* for the conserved energetic field quantities.

In the general case, the energy flux vector, $\mathbf{U}$, has been discovered by Umov as the product of a velocity vector, $\mathbf{v}$, and an energy, W, see [21]. This definition describes energy flux in liquids, elastic media, etc.. Besides, that concept is also available for the flux of electromagnetic energy. Poynting and Heaviside independently co-invented the electromagnetic energy flux, $\mathcal{P}$, as the cross-product of the electric and magnetic field strengths, $\mathcal{E} \times \mathcal{H}$, see in [22] as

$$\mathbf{U} = \mathbf{v}\, W \quad \text{and} \quad \mathcal{P} = \mathcal{E} \times \mathcal{H}\,. \quad (42)$$

Let us take $\mathbf{U} \equiv \mathcal{P}$ as a particular case. Then formula $\mathbf{v} = \mathcal{P}/W$ specifies an *instant* velocity, $\mathbf{v}$, of transportation of the flux of electromagnetic field energy.

Apply this formula the modal flux density, $\mathcal{P}'_{z\,n}$, averaged over the domain S, and to the modal field energy density, $\mathcal{W}'_n$, stored in the same cross-section domain and averaged analogously. That yields a value of the *instant* velocity of transportation of the flux of modal field energy along the waveguide as

$$v\,(z,t)\,/c = 2\mathcal{V}'_n\,(z,t)\,\mathcal{I}'_n\,(z,t)\,/\,\left[\mathcal{V}'^2_n\,(z,t) + \mathcal{I}'^2_n\,(z,t) + h^2_n\,(z,t)\right] \quad (43)$$

where c is the speed of light.

6. EVOLUTIONARY EQUATION FOR $h_n\,(z,t)$

Finally, derive a governing equation for the amplitude $h_n\,(z,t)$ in (33). That equation can be obtained by substitution of needed field components from the set (33) to Equation (12a). Simple manipulations after result in

$$\boxed{\partial^2_{\nu_n c\,t} h_n + 2\beta_n\,\partial_{\nu_n c\,t} h_n - \partial^2_{\nu_n z} h_n + h_n = 0} \quad (44)$$

where c is the light speed, $\beta_n = \varrho/\nu_n$ is a dimensionless lossy parameter because the quantities ϱ and ν_n, both, have the same physical dimension, m^{-1}.

Mathematicians call all differential equations, which involve partial (or/and ordinary) time derivative, as the *evolutionary* ones, see link [20]. Evolution Equation (44) is known under names *Klein-Gordon equation* (*KGE*) [6], telegraph equation, generalized wave equation [8].

Time derivative of the first order can be eliminated from Equation (44) by applying a substitution for the function $h_n\,(z,t)$ sought for in the form of

$$h_n\,(z,t) = e^{-\varrho\,ct}\tilde{h}_n\,(z,t)\,. \quad (45)$$

This yields governing equation for a new unknown function, $\tilde{h}_n\,(z,t)$, as

$$\partial^2_{\nu_n c\,t}\tilde{h}_n\,(z,t) - \partial^2_{\nu_n z}\tilde{h}_n\,(z,t) + \left(1-\beta^2_n\right)\,\tilde{h}_n\,(z,t) = 0\,. \quad (46)$$

Notice that conductivity, σ, appears as a parameter via $\beta_n = \varrho/\nu_n$ in the Equations (44) and (46), but that is absent in the modal basis (32).

7. THE *REAL-VALUED* TIME-HARMONIC *TE*-MODES

In accordance with the Bohr correspondence principle, the time-domain theory has to exhibit the time-harmonic solutions as a particular case. Verify that it is so and demonstrate in passing that the time-domain theory is capable of disclose *new properties* of the time-harmonic waveguide waves.

Inasmuch the time-harmonic waves freely propagate in waveguides, i.e., $-\infty < t < +\infty$, the initial conditions and the causality principle are superfluous in the frequency-domain analysis. Klein-Gordon Equation (46) has, as well, two linearly independent solutions in elementary functions as

$$\tilde{h}_n(z,t) = a_n \sin(\omega t - \gamma_n z) + b_n \cos(\omega t - \gamma_n z) \tag{47}$$

where ω is a given frequency parameter, γ_n is a propagation constant sought for, a_n and b_n are free numerical parameters. The sin- and cos-solutions in (47) can be unified if we introduce new notations as

$$c_n = \sqrt{a_n^2 + b_n^2}, \quad a_n/c_n = \cos\varphi, \quad b_n/c_n = \sin\varphi. \tag{48}$$

Standard combination with the trigonometric functions in (47) results in

$$\tilde{h}_n(z,t) = c_n \sin(\omega t - \gamma_n z + \varphi) \tag{49}$$

where $\varphi = \sin^{-1}(b_n/c_n) = \cos^{-1}(a_n/c_n)$. Solutions (47) and (49) are equivalent. One can put the factor c_n in (49) as $c_n = 1$ without loss of generality.

Substitution of (49) to (46) yields the propagation constant, γ_n, as

$$\gamma_n = \pm\frac{1}{c}\sqrt{\omega^2 - (\nu_n c)^2\left(1 - \beta_n^2\right)} \tag{50}$$

where c is the free-space light speed. In the doublet, $(\pm)$, the upper sign corresponds to the wave propagation lengthway Oz-axis; the lower one corresponds to the opposite direction. The standard procedure applied to the phase, $\vartheta(z,t) = \omega t - \gamma_n z + \varphi$, in (49) results in the phase velocity, v_{ph}, as

$$v_{ph} = \pm c\frac{\omega}{\sqrt{\omega^2 - (\nu_n c)^2\left(1 - \beta_n^2\right)}}. \tag{51}$$

Condition $\gamma_n = 0$ yields an equation for the frequency parameter, ω. The solutions to this equation generate a set of the *cut-off frequencies,* ω_n, i.e.,

$$\left\{\omega_n = \nu_n c\sqrt{1 - \beta_n^2}\right\}_{n=1}^{\infty}. \tag{52}$$

If $\sigma = 0$, then $\beta_n = 0$. Hence, the quantities $\nu_n c$ (denoted as $\nu_n c \equiv \omega_n'$ henceforward) are the cut-off frequencies of the TE-modes propagating in the *hollow lossless* waveguide. Notice, that the lossy parameter, $0 < \beta_n < 1$, downgrades the cut-off frequency levels for the lossy waveguides. The quantities $\nu_n = \omega_n'/c$ have the physical sense of the cut-off wave numbers for the *hollow lossless* waveguides. Respectively, values $\omega_n/c = \nu_n \sqrt{1-\beta_n^2}$, $n = 1, 2, \ldots$ are the cut-off wave numbers for the *lossy* waveguides.

Introduce a dimensionless time, τ, and a dimensionless coordinate, ξ, via scaling the real coordinate, z, and time, t, as follows:

$$\xi = \nu_n z = 2\pi\, z/\lambda_n' \text{ and } \tau = \omega_n'\, t = 2\pi\, t/T_n' \tag{53}$$

where $\lambda_n' = 2\pi/\nu_n$ is the cut-off wavelength of the TE-modes ($n = 1, 2, \ldots$) in the lossless waveguide and $T_n' = \lambda_n'/c$ is the period of oscillations corresponding to the cut-off frequency, ω_n'. In terms of these notations, the dimensionless amplitudes of the TE-modal fields (33) can be written as follows:

$$\begin{aligned} \mathcal{V}_n'(\xi,\tau) &= -e^{-\beta_n \tau}\sqrt{\varpi^2+\beta_n^2}\cos(\vartheta+\delta) \\ \mathcal{I}_n'(\xi,\tau) &= -e^{-\beta_n \tau}\sqrt{\varpi^2-1+\beta_n^2}\cos(\vartheta) \\ h_n(\xi,\tau) &= e^{-\beta_n \tau}\sin(\vartheta) \end{aligned} \tag{54}$$

where $0 \le \beta_n \le 1$, $\varpi = \omega/\omega_n'$, ω is a given frequency, and

$$\begin{aligned} \delta &= \sin^{-1}\left(\beta_n/\sqrt{\varpi^2+\beta_n^2}\right) = \cos^{-1}\left(\varpi/\sqrt{\varpi^2+\beta_n^2}\right) \\ \vartheta &= \omega t - \gamma_n z + \varphi \equiv \varpi\tau - \xi\sqrt{\varpi^2-1+\beta_n^2} + \varphi. \end{aligned} \tag{55}$$

7.1. Special Value of the Lossy Parameter ϱ

In Equation (46), one can put $\beta_n = 1$ as a possible particular case. This supposition specifies the value of the lossy parameter, ϱ, as $\varrho = \nu_n$. Substitution of $\beta_n = 1$ to (46) turns that KGE into one-dimensional *wave equation* for $\tilde{h}_n \equiv \tilde{h}_n^w$, i.e.,

$$\partial_\tau^2 \tilde{h}_n^w(\xi,\tau) - \partial_\xi^2 \tilde{h}_n^w(\xi,\tau) = 0. \tag{56}$$

This equation supports two linearly independent solutions as

$$\tilde{h}_n^w(\xi,\tau) = w_-(\tau-\xi) + w_+(\tau+\xi) \tag{57}$$

where w_- and w_+ are *arbitrary* functions twice differentiable by their arguments. Solution w_- represents a *waveform* traveling along the waveguide, the other one, w_+, corresponds to the waveform propagating in opposite direction.

For example, take the first solution, $w_-(\tau-\xi)$, in the form of (49) with $\beta_n = 1$ and $c_n = 1$. Then substitute that to formula (45) and rewrite the result in real time, t, and coordinate z (for clearness) as

$$h_n^w(z,t) = e^{-\omega_n' t}\sin(\omega t - z\,\omega/c + \varphi) \tag{58}$$

where $\omega'_n = \nu_n c$ is the cut-off frequency. This solution satisfies to the *KGE* (44) with $\beta_n = 1$. Formulas (28) supply with the other modal amplitudes as

$$\begin{aligned}\mathcal{V}_n^w &= -e^{-\omega'_n t}\left[\varpi\cos\left(\omega t - z\,\omega/c + \varphi\right) - \sin\left(\omega t - z\,\omega/c + \varphi\right)\right]\\ \mathcal{I}_n^w &= -e^{-\omega'_n t}\,\varpi\cos\left(\omega t - z\,\omega/c + \varphi\right).\end{aligned} \tag{59}$$

7.2. Dynamics of the Energetic Modal Characteristics

Explicit time-domain solution (54) enables us to calculate the dimensionless amplitude, $\mathcal{P}'_{z\,n}$, of the modal energy flux density what results in

$$\begin{aligned}\mathcal{P}'_{z\,n}(\xi,\tau) &= e^{-2\beta_n\tau}\frac{1}{2}\sqrt{\varpi^2 - 1 + \beta_n^2}\left[\varpi + \ddddot{\mathcal{P}}'_{z\,n}(\vartheta)\right]\\ \ddddot{\mathcal{P}}'_{z\,n}(\vartheta) &= \varpi\cos(2\vartheta) - \beta_n\sin(2\vartheta)\end{aligned} \tag{60}$$

where $\ddddot{\mathcal{P}}'_{zn}$ is a dynamic part dependent on time and z periodically, see ϑ in (55). The dimensionless amplitude, $\mathcal{W}'_n$, of the stored energy density is

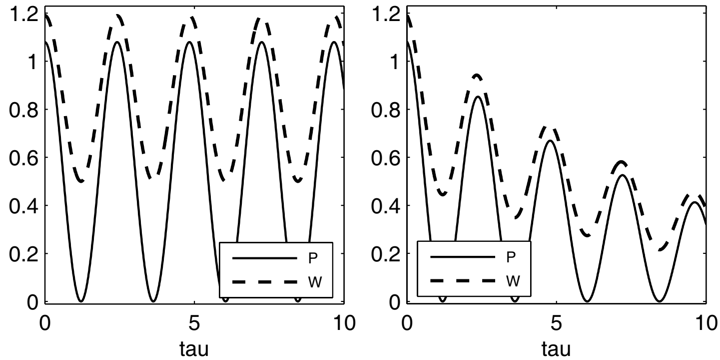
$$\begin{aligned}\mathcal{W}'_n(\xi,\tau) &= e^{-2\beta_n\tau}\frac{1}{2}\left[\varpi^2 + \beta_n^2 + \ddddot{\mathcal{W}}'_n(\vartheta)\right]\\ \ddddot{\mathcal{W}}'_n(\vartheta) &= \left(\varpi^2 - 1\right)\cos(2\vartheta) - \varpi\beta_n\sin(2\vartheta)\end{aligned} \tag{61}$$

where $\ddddot{\mathcal{W}}'_n$ is a part periodically dependent on time and z.

In Fig. 1, dependence on time, τ, of the energy flux density, $\mathcal{P}'_{z\,n}(0,\tau)$, and the energy density, $\mathcal{W}'_n(0,\tau)$, are exhibited for a fixed position, $\xi = 0$, of the waveguide cross section. In Fig. 1(a), the graphical results correspond to the value $\beta_n = 0$ of the lossy parameter. In other words, this is the case of the lossless hollow waveguide. In Fig. 1(b), the results correspond to the lossy waveguide where $\beta_n = 0.05$.

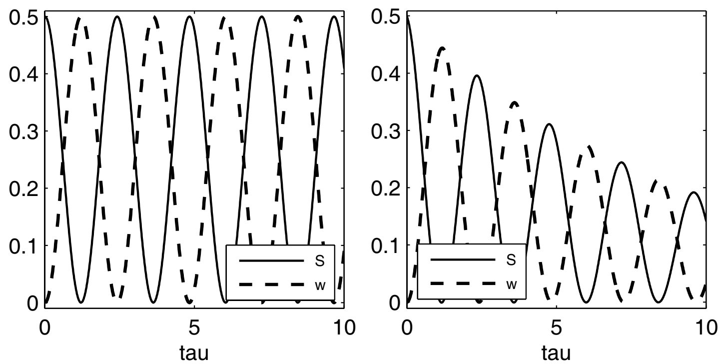
Periodical time-dependence of $\mathcal{W}'_n(0,\tau)$ (in Fig. 1(a)) suggests that a *wave process* of exchange by energy between some constituent parts of the modal field should accompany the phenomenon of the field propagation. Glancing at the field composition (33) one can conclude that the energy of $\mathcal{E}'_n$-field is accumulated completely in the transverse field component and equal to $\mathcal{V}'^2_n/2$. The energy of $\mathcal{H}'_n$-field should be distributed between its transverse and longitudinal components as $\mathcal{I}'^2_n/2$ and $w'_n = h_n^2/2$, respectively. One can expect that the value $\mathcal{V}'^2_n/2$ prevails over the value $\mathcal{I}'^2_n/2$, generally speaking. This supposition suggests to introduce a new energetic quantity, $\mathcal{S}'_n(\xi,\tau)$, specified in (62) and named next as a "surplus" of energy stored in the transverse field components

$$\mathcal{S}'_n(\xi,\tau) = \left(\mathcal{V}'^2_n - \mathcal{I}'^2_n\right)/2 \quad \text{and} \quad w'_n(\xi,\tau) = h_n^2/2. \tag{62}$$



(a) (b)

Figure 1. Time dependence of the energy flow density, $\mathcal{P}'_{z\,n}(\xi,\tau)$, and the energy density, $\mathcal{W}'_n(\xi,\tau)$ for $\varpi = 1.3$, $\xi = 0$ and (a) $\beta_n = 0$, (b) $\beta_n = 0.05$.



(a) (b)

Figure 2. Wave process of exchange by energy between the surplus of energy, $S'_n(\xi,\tau)$, stored in the transverse field components and the energy, $\omega'_n(\xi,\tau)$, stored in the longitudinal field component; $\varpi = 1.3$, $\xi = 0$, and (a) $\beta_n = 0$, (b) $\beta_n = 0.05$.

Figure 2(a) corroborates that above-mentioned supposition about the *energetic wave process* is true in the case of lossless hollow cavity. Moreover, similar wave process occurs for the lossy waveguide as that confirms the data in Fig. 2(b).

Instant velocity, v'_n, as a function of the variables (ξ,τ) is

$$v'_n\left(\xi,\tau\right)/c = \mathcal{P}'_{z\,n}\left(\xi,\tau\right)/\mathcal{W}'_n\left(\xi,\tau\right) \tag{63}$$

accordingly to definition (43).

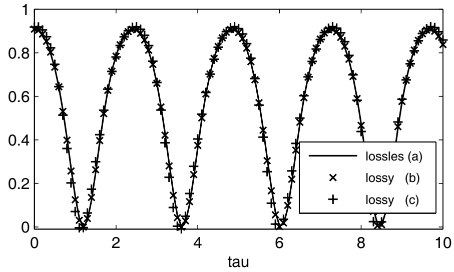


Figure 3. Normalized by c instant velocity of transportation energy, v'/c, for $\varpi = 1.3$, $\xi = 0$ and (a) $\beta_n = 0$; (b) $\beta_n = 0.05$; (c) $\beta_n = 0.2$.

In the case of $\beta_n = 0$, formula (63) can be rearranged in the form of

$$v'_n/c = \frac{\varpi\sqrt{\varpi^2-1}\,\{1+\cos[2\vartheta(\xi,\tau)]\}}{\varpi^2+(\varpi^2-1)\cos[2\vartheta(\xi,\tau)]} \tag{64}$$

via substitution of (60) and (61) to (63). This result enables to establish possible *extrema values* of the velocity of transportation of the modal field energy. *Minimal* value, $v_n^{\min} = 0$, can be achieved by the reasons of either $\varpi^2 = 1$ (i.e., $\omega = \omega'_n$) or $\cos[2\vartheta(\xi,\tau)] = -1$. The latter is supplied with the energetic wave process which goes in parallel to the field propagation. *Maximal* value is

$$v_n^{\max}/c = 2\varpi\sqrt{\varpi^2-1}/\left(2\varpi^2-1\right) \tag{65}$$

under condition $\cos[2\vartheta(\xi,\tau)] = 1$ in (64). It is appropriate to notice that $v_n^{\max}$ never exceeds the light speed, c.

In Fig. 3, time-dependence of $v'_n(0,\tau)/c$ is presented for the lossless waveguide ($\beta_n = 0$) and for the lossy ones.

Averaged value of $v'_n(\xi,\tau)/c$ over time-interval $0 \le t \le T$ is defined as

$$\bar{v}'_n/c = \frac{1}{T}\int_\theta^{T+\theta}\frac{\mathcal{P}'_{z\,n}(\xi,\tau)}{\mathcal{W}'_n(\xi,\tau)}dt = \frac{1}{T}\int_\theta^{T+\theta}\frac{\sqrt{\varpi^2-1+\beta_n^2}\,\left[\varpi+\ddddot{\mathcal{P}}'_{z\,n}(\vartheta)\right]}{\varpi^2+\beta_n^2+\ddddot{\mathcal{W}}'_n(\vartheta)}dt \tag{66}$$

where $T = 2\pi/\omega$, θ is a constant, for ϑ as $\vartheta(z,t)$ see (55). As far as

$$\int_\theta^{T+\theta}\ddddot{\mathcal{P}}'_{z\,n}(\vartheta(z,t))\,dt = \int_\theta^{T+\theta}\ddddot{\mathcal{W}}'_n(\vartheta(z,t))\,dt = 0, \tag{67}$$

integration by formula (66) results in

$$\bar{v}'_n/c = \frac{\varpi\sqrt{\varpi^2-1+\beta_n^2}}{\varpi^2+\beta_n^2} \equiv \frac{\omega\sqrt{\omega^2-\omega_n'^2(1-\beta_n^2)}}{\omega^2+\omega_n'^2\beta_n^2} \tag{68}$$

where $\varpi = \omega/\omega'_n$, ω is a given frequency, $\omega'_n = \nu_n c$ is the cut-off frequency.

If $\beta_n = 0$ then $\bar{v}'_n = c\sqrt{\omega^2 - \omega'^2_n}/\omega < c$ what coincides with the classical result. If $\beta_n = 1$ then $\bar{v}'_n = c\,\omega^2/\left(\omega^2 + \omega'^2_n\right) < c$ for solution (58).

8. FUNDAMENTAL SOLUTION

Equations (45) and (46) have so-called *fundamental* solution [8] as

$$\mathfrak{F}_n(\xi,\tau) = \frac{1}{2} H(\tau - |\xi|)\, e^{-\beta_n \tau} J_0\left(\alpha_n \sqrt{\tau^2 - \xi^2}\right) \tag{69}$$

where $\alpha_n = \sqrt{1-\beta_n^2} \le 1$, $0 \le \beta_n \le 1$, $J_0(\cdot)$ is the Bessel function of *zero* order, and $H(\tau - |\xi|)$ is the Heaviside unit step function§. The first factor, $H(\tau - |\xi|)$, in (69) *symbolizes* correspondence of the solution to the causality principle. Substitution of the function $\mathfrak{F}_n$ to formulas (28) results in the modal amplitudes of the transverse field components denoted as

$$\begin{aligned} \mathcal{V}_n^{\mathfrak{F}}(\xi,\tau) &= 0.5H(\tau - |\xi|)\, e^{-\beta_n \tau}\left[\alpha_n J_1(\alpha_n \ell)\,(\tau/\ell) + \beta_n J_0(\alpha_n \ell)\right] \\ \mathcal{I}_n^{\mathfrak{F}}(\xi,\tau) &= 0.5H(\tau - |\xi|)\, e^{-\beta_n \tau}\left[\alpha_n J_1(\alpha_n \ell)\,(\xi/\ell)\right] \end{aligned} \tag{70}$$

where $\ell = \sqrt{\tau^2 - \xi^2}$. In the case $\beta_n = 1$, the solutions (69) and (70) are

$$\mathfrak{F}_n = \tfrac{1}{2} H(\tau - |\xi|)\, e^{-\tau}, \quad \mathcal{I}_n^{\mathfrak{F}} = 0, \quad \mathcal{V}_n^{\mathfrak{F}} = \tfrac{1}{2} H(\tau - |\xi|)\, e^{-\tau}. \tag{71}$$

In Fig. 4, time dependence of the modal amplitudes, $\mathfrak{F}_n$ and $\mathcal{V}_n^{\mathfrak{F}}$, are exhibited for the lossless waveguide provided that $z = 0$ and $\beta_n = 0$. As far as $\mathcal{I}_n^{\mathfrak{F}}(\xi,\tau)$ is proportional to ξ, the amplitude $\mathcal{I}_n^{\mathfrak{F}}(0,\tau) = 0$.

§ $H(\tau - |\xi|) = 1$ if $\tau \ge |\xi|$ and $H(\tau - |\xi|) = 0$ if $\tau < |\xi|$.

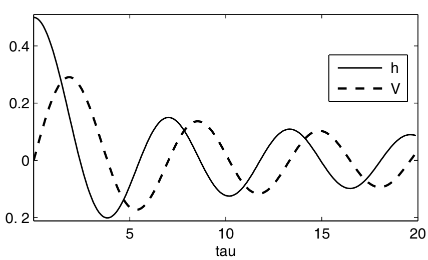


Figure 4. Time-dependence of the amplitudes $h_n \equiv \Im_n$ and $V_n^{\Im}$ when $\xi = 0$, $\beta_n = 0$.

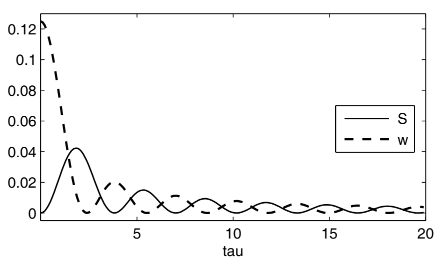


Figure 5. Time-dependence of the surplus of energy, $S_n^{\Im}$, and the energy, $\omega_n^{\Im}$, stored in the longitudinal component $h_n \equiv \Im_n$ when $\xi = 0$, $\beta_n = 0$.

Fig. 5 illustrates the energetic wave process for fundamental solution. The partners in this wave process are specified by formulas

$$\mathcal{S}_n^{\mathfrak{F}}(\xi,\tau) = \left(\mathcal{V}_n^{\mathfrak{F}2} - \mathcal{I}_n^{\mathfrak{F}2}\right)/2 \quad \text{and} \quad w_n^{\mathfrak{F}}(\xi,\tau) = \mathfrak{F}_n^2/2\,; \quad \tau \geq \xi. \tag{72}$$

9. DISCUSSION

In our statement of the problem, we supposed that the waveguide is filled with a medium which has the relative permittivity and permeability, ε and μ, equal to 1 (the same as for the free space). The constant conductivity, $\sigma \geq 0$, was introduced heuristically for modeling possible losses in the waveguide.

In realistic situations, waveguides can be filled with a dielectric medium. In this case, the second equation in (1) should be replaced by

$$\nabla\times\mathcal{H}(\mathbf{R},t) = \partial_t\mathcal{D}(\mathbf{R},t) + \sigma\mathcal{E}(\mathbf{R},t) \tag{73}$$

where $\mathcal{D}(\mathcal{E}) = \epsilon_0\mathcal{E} + \mathcal{P}(\mathcal{E})$, $\mathcal{P}(\mathcal{E})$ is the polarization vector induced by the field $\mathcal{E}$ applied to the material, $\mathcal{D}$ is the electric flux density. Dependence $\mathcal{D}(\mathcal{E})$ is known as the material *constitutive relation.* The latter is specified completely if the constitutive relation $\mathcal{P}(\mathcal{E})$ is given somehow.

Relationship between $\mathcal{P}$ and $\mathcal{E}$ should be *dynamic* in the time-domain theory. That can be established by solving appropriate *motion equation* for the polarization vector, $\mathcal{P}$. There are two wide classes of dielectric materials each of which has own physical mechanism of polarization. Atoms and molecules of so-called *Lorentz media* have a distorting mechanism of polarization. Relationship between $\mathcal{P}$ and $\mathcal{E}$ is describable by Newton's motion equation as

$$\frac{d^2}{dt^2}\mathcal{P}(\mathbf{R},t) + 2\gamma_0\frac{d}{dt}\mathcal{P}(\mathbf{R},t) + \omega_0^2\,\mathcal{P}(\mathbf{R},t) = \epsilon_0 N q_e^2/(m_e\epsilon_0)\,\mathcal{E}(\mathbf{R},t)\,. \tag{74}$$

where γ_0 and ω_0 are the material constants, m_e and q_e are the mass an unsigned charge of electron, N is the number density of the polarizable atoms and/or molecules per unit of volume. In the static case, time derivative, $\frac{d}{dt}$, turns into *zero* the first two terms in (74). Hence, that equation yields constitutive relation $\mathcal{P} = \epsilon_0\chi_s\mathcal{E}$ where $\chi_s = Nq_e^2/\left(\epsilon_0 m_e\omega_0^2\right)$ is the *static* dimensionless susceptibility. In this case, constitutive relation (73) yields $\mathcal{D} = \epsilon_0\varepsilon\mathcal{E}$, where $\varepsilon = 1 + \chi_s$, what corresponds to an *instantaneous* polarization, physically.

Physical mechanism of so-called *Debye media* is orientational. Relationship between $\mathcal{P}$ and $\mathcal{E}$ is describable by Debye equation as

$$\frac{d}{dt}\mathcal{P}(\mathbf{R},t) + (1/\tau_0)\,\mathcal{P}(\mathbf{R},t) = \epsilon_0\,(\chi/\tau_0)\,\mathcal{E}(\mathbf{R},t) \tag{75}$$

where $\tau_0 = 2\gamma_0/\omega_0^2$ is a relaxation time. Molecules H_2O, N_2, O_2, O_3, C, CO, SO_2, HCl, etc. compose polar dielectrics. The human and animal tissues involve water (H_2O) in high doses. The tissues should be interpreted as polar dielectrics in biological studies when interacting with electromagnetic radiation.

Combination of the Maxwell's equations with ∂_t and the dynamic constitutive relations (74) or (75), where the time derivative plays essential role, is prospective for studies of dielectrics with temporal dispersion. One can find realization of this idea for solving a cavity problem in [23]. Expansion of this idea on a wide class of the waveguide problems can be made, as well.

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