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SCATTERING OF ELECTROMAGNETIC PLANE WAVE BY A CIRCULAR DISK WITH SURFACE IMPEDANCE

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Abstract—In this investigation, scattering from a circular disk with surface impedance has been studied rigorously. The method of analysis is Kobayashi Potential (KP). The mathematical formulation yields the dual integral equations (DIEs). These DIEs are solved by using the discontinuous properties of Weber-Schafheitlin's integral. After applying the boundary conditions and projection, the resulting expressions, finally, reduce to matrix equations for expansion coefficients. The matrix elements are in the form of infinite integrals with single variable. These are then used to compute the values of expansion coefficients. The far field patterns of the scattered wave are computed for different incident angles and surface impedances for both E- and H-polarizations. To verify the results, we have computed the solution based on the physical optics approximation. The agreement between them is fairly good.

1. INTRODUCTION

The circular disk is a canonical scatterer in the field of electromagnetics and has been a subject of investigation since long time. It has a wide range of application in radars, and antennas, etc.. Electromagnetic field problems are generally defined by Maxwell equations and boundary conditions. The surfaces with large conductivity can be approximated with surface impedance boundary condition. The impedance boundary condition relates the electric and magnetic field components tangential to the boundary through a surface impedance factor linearly. The use of surface impedance boundary condition (SIBC) in problems where electromagnetic wave penetration is low,

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reduces the complexity of the problem to solve [1]. Even such surfaces can be synthesized which follow SIBC [2]. The concept of SIBC is not new and was introduced by Shchukin [3] and Leontovich [4] in 1940s [5]. A variety of methods may be used to analyze the present problem [6–28].

In this paper, we have formulated the problem first time by applying the KP [27, 28] method to study the scattering from the disk with surface impedance. The KP method is like eigen function expansion and also is similar to the Method of Moments (MoM) [26] in its spectral domain, but the formulation is different. The MoM is based on an integral equation, whereas the KP method has dual integral equations. In addition, the characteristic functions used in the KP method satisfy a proper edge condition as well as the required boundary conditions. The KP method has already been successfully applied to perfectly conducting circular disk [20–22].

In formulation of the problem, first we introduced two longitudinal components of the vector potentials of electric and magnetic types to express the scattered field in the form of Fourier-Hankel transform. By applying the boundary conditions, we derived the dual integral equations (DIE) for the tangential components of the electric and magnetic fields. The equations may be written in the form of the vector Hankel transform given by Chew and Kong [29–32]. The expressions for the field are expanded in terms of a set of the functions with expansion coefficients. These functions are constructed by applying the discontinuous properties of the Weber-Schafheitlin's integrals [33–35] and it is readily shown that these functions satisfy the required edge conditions [36–38] as well as boundary conditions. By using the projection, the problem reduces to the matrix equations for the expansion coefficients of the electromagnetic fields. The matrix elements are given in the form of an infinite integrals which converge for all indices. Numerical computation is carried out to obtain the far field patterns and the results are compared with those obtained through physical optics method.

2. STATEMENT OF THE PROBLEM AND EXPRESSIONS FOR INCIDENT WAVE

The geometry of the problem and the associated coordinates are described in Fig. 1, where the radius of the disk is a and thickness of the conducting plane is assumed to be negligibly small. Two kinds of incident plane wave are possible and these are expressed by

$$\mathbf{E^i}=\left(E_2\hat{\theta}+E_1\hat{\phi}\right)\exp\left[jk\Phi^i(r)\right], \ \mathbf{H^i}=Y_0\left(-E_2\hat{\phi}+E_1\hat{\theta}\right)\exp\left[jk\Phi^i(r)\right]. \quad (1)$$

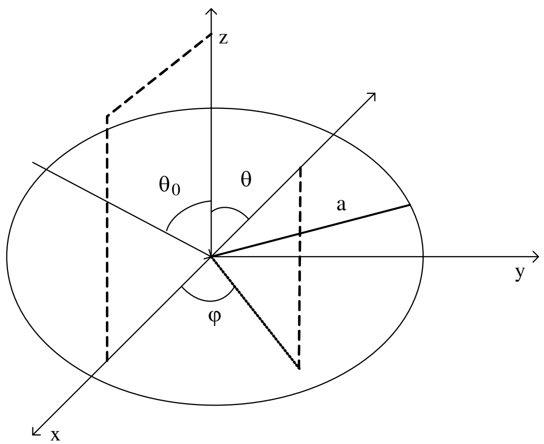

Figure 1. Scattering of a plane wave by a circular disk with surface impedance.

where

$$\hat{\theta} = \cos\theta_0 \cos\phi_0 \hat{x} + \cos\theta_0 \sin\phi_0 \hat{y} - \sin\theta_0 \hat{z},$$
$$\hat{\phi} = -\sin\phi_0 \hat{x} + \cos\phi_0 \hat{y}, \tag{2a}$$
$$\Phi^i(r) = x\sin\theta_0\cos\phi_0 + y\sin\theta_0\sin\phi_0 + z\cos\theta_0. \tag{2b}$$

where (θ_0, ϕ_0) are the angles of incidence, and $Y_0 = \frac{1}{Z_0} = \sqrt{\frac{\epsilon_0}{\mu_0}}$ is the free space intrinsic admittance. Since a disk has rotational symmetry with respect to z-axis, we can assume without loss of generality that the plane of incidence lies in xz-plane ($\phi_0 = 0$). We may split field into two kinds of polarization namely E-polarization specified by E_1 and H-polarization specified by E_2, and discuss both cases simultaneously.

The electric and magnetic vector potentials are defined by $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{D} = -\nabla \times \mathbf{F}$, respectively. Therefore, the z-components of the vector potentials F_z and A_z for the incident and reflected waves are obtained as follows:

$$A_z^i = \frac{\mu_0 Y_0 E_2}{jk\sin\theta_0} \exp[jkx\sin\theta_0 + jkz\cos\theta_0],$$
$$F_z^i = \frac{\epsilon_0 E_1}{jk\sin\theta_0} \exp[jkx\sin\theta_0 + jkz\cos\theta_0]. \tag{3a}$$
$$A_z^r = \frac{\mu_0 Y_0 E_2}{jk\sin\theta_0} \exp[jkx\sin\theta_0 - jkz\cos\theta_0],$$
$$F_z^r = -\frac{\epsilon_0 E_1}{jk\sin\theta_0} \exp[jkx\sin\theta_0 - jkz\cos\theta_0]. \tag{3b}$$

We express a plane wave given above by the cylindrical coordinates to facilitate imposing the boundary conditions. These are obtained by using the formulas of wave transformation given by

$$\begin{aligned}\exp[jk\rho\sin\theta_0\cos\phi] &= \sum_{m=-\infty}^{\infty} j^m J_m(k\rho\sin\theta_0)\exp(-jm\phi)\\ &= \sum_{m=0}^{\infty}\epsilon_m j^m J_m(k\rho\sin\theta_0)\cos m\phi \qquad (4)\end{aligned}$$

where ϵ_m is Neumann's constant given by $\epsilon_m = 1$ for $m = 0$ and $\epsilon_m = 2$ for $m \geq 1$, and $x = \rho\cos\phi$, $y = \rho\sin\phi$.

2.1. E-wave (Magnetic Field Is Perpendicular to the Plane of Incidence)

The incident electromagnetic plane wave over the $z = 0$ plane are given by

$$H_\rho^i = Y_0 E_\phi^i \cos\theta_0 = -jY_0 E_1\cos\theta_0\sum_{m=0}^{\infty}\epsilon_m j^m J_m{}'(k\rho\sin\theta_0)\cos m\phi \qquad (5a)$$

$$H_\phi^i = -Y_0 E_\rho^i \cos\theta_0 = jY_0 E_1\cos\theta_0\sum_{m=0}^{\infty}\epsilon_m j^m \frac{m}{k\rho\sin\theta_0} J_m(k\rho\sin\theta_0)\sin m\phi \qquad (5b)$$

where $J_x(x)$ and $J'_m(x)$ are the Bessel function of the first kind and its derivative with respect to the argument.

2.2. H-wave (Electric Field Is Perpendicular to the Plane of Incidence)

In this case, the incident waves corresponding to Equations (5) are given by

$$H_\rho^i = jY_0 E_2\sum_{m=0}^{\infty}\epsilon_m j^m \frac{m}{k\rho\sin\theta_0} J_m(k\rho\sin\theta_0)\sin m\phi, \quad E_\phi^i = Z_0\cos\theta_0 H_\rho^i \qquad (6a)$$

$$H_\phi^i = jY_0 E_2\sum_{m=0}^{\infty}\epsilon_m j^m J'_m(k\rho\sin\theta_0)\cos m\phi, \quad E_\rho^i = -Z_0\cos\theta_0 H_\phi^i \qquad (6b)$$

3. THE EXPRESSIONS FOR THE FIELDS SCATTERED BY A DISK

We now discuss about our analytical method for predicting the field scattered by an impedance disk on the plane $z = 0$.

3.1. Spectrum Functions of the Fields on the Disk

We assume the vector potential corresponding to the diffracted field is expressed in the form

$$A_z^{d\pm}(\rho,\phi,z) = \mu_0 a\kappa Y_0 \sum_{m=0}^{\infty}\int_0^{\infty}\left[\tilde{f}_{cm}^{\pm}(\xi)\cos m\phi+\tilde{f}_{sm}^{\pm}(\xi)\sin m\phi\right] J_m(\rho_a\xi)\exp\left[\mp\sqrt{\xi^2-\kappa^2}z_a\right]\xi^{-1}d\xi \tag{7a}$$

$$F_z^{d\pm}(\rho,\phi,z) = \epsilon_0 a \sum_{m=0}^{\infty}\int_0^{\infty}\left[\tilde{g}_{cm}^{\pm}(\xi)\cos m\phi+\tilde{g}_{sm}^{\pm}(\xi)\sin m\phi\right] J_m(\rho_a\xi)\exp\left[\mp\sqrt{\xi^2-\kappa^2}z_a\right]\xi^{-1}d\xi \tag{7b}$$

where the upper and lower signs refer to the region $z>0$ and $z<0$, respectively, and $\rho_a=\frac{\rho}{a}$ and $z_a=\frac{z}{a}$ are the normalized variables with respect to the radius a of the disk. In the above equations $\widetilde{f}(\xi)$ and $\widetilde{g}(\xi)$ are the unknown spectrum functions and they are to be determined so that they satisfy all the required boundary conditions. Equations (7a) and (7b) are of the form of the Hankel transform for $z=0$. First we consider the surface field at the plane $z=0$ to derive the dual integral equations associated with them. By using the relation between the vector potentials and the electromagnetic field, the tangential components of the electric field and the magnetic field become

$$\begin{aligned}\begin{bmatrix}E_\rho^{d+}(\rho,\phi,0)\\ E_\rho^{d-}(\rho,\phi,0)\end{bmatrix} &= \sum_{m=0}^{\infty}\left[\begin{bmatrix}E_{\rho c,m}^{+}(\rho_a)\\ E_{\rho c,m}^{-}(\rho_a)\end{bmatrix}\cos m\phi+\begin{bmatrix}E_{\rho s,m}^{+}(\rho_a)\\ E_{\rho s,m}^{-}(\rho_a)\end{bmatrix}\sin m\phi\right]\\
&= j\sum_{m=0}^{\infty}\int_0^{\infty}\sqrt{\xi^2-\kappa^2}\left[\begin{bmatrix}\widetilde{f}_{c,m}^{+}(\xi)\\ \widetilde{f}_{c,m}^{-}(\xi)\end{bmatrix}\cos m\phi+\begin{bmatrix}\widetilde{f}_{s,m}^{+}(\xi)\\ \widetilde{f}_{s,m}^{-}(\xi)\end{bmatrix}\sin m\phi\right]J_m'(\xi\rho_a)d\xi\\
&\quad -\sum_{m=0}^{\infty}\int_0^{\infty}\left[-\begin{bmatrix}\widetilde{g}_{c,m}^{+}(\xi)\\ \widetilde{g}_{c,m}^{-}(\xi)\end{bmatrix}\sin m\phi+\begin{bmatrix}\widetilde{g}_{s,m}^{+}(\xi)\\ \widetilde{g}_{s,m}^{-}(\xi)\end{bmatrix}\cos m\phi\right]\frac{m}{\xi\rho_a}J_m(\xi\rho_a)d\xi\end{aligned} \tag{8a}$$

$$\begin{aligned}\begin{bmatrix}E_\phi^{d+}(\rho,\phi,0)\\ E_\phi^{d-}(\rho,\phi,0)\end{bmatrix} &= \sum_{m=0}^{\infty}\left[\begin{bmatrix}E_{\phi c,m}^{+}(\rho_a)\\ E_{\phi c,m}^{-}(\rho_a)\end{bmatrix}\cos m\phi+\begin{bmatrix}E_{\phi s,m}^{+}(\rho_a)\\ E_{\phi s,m}^{-}(\rho_a)\end{bmatrix}\sin m\phi\right]\\
&= j\sum_{m=0}^{\infty}\int_0^{\infty}\sqrt{\xi^2-\kappa^2}\left[-\begin{bmatrix}\widetilde{f}_{c,m}^{+}(\xi)\\ \widetilde{f}_{c,m}^{-}(\xi)\end{bmatrix}\sin m\phi+\begin{bmatrix}\widetilde{f}_{s,m}^{+}(\xi)\\ \widetilde{f}_{s,m}^{-}(\xi)\end{bmatrix}\cos m\phi\right]\frac{m}{\xi\rho_a}\\
&\quad J_m(\xi\rho_a)d\xi+\sum_{m=0}^{\infty}\int_0^{\infty}\left[\begin{bmatrix}\widetilde{g}_{c,m}^{+}(\xi)\\ \widetilde{g}_{c,m}^{-}(\xi)\end{bmatrix}\cos m\phi+\begin{bmatrix}\widetilde{g}_{s,m}^{+}(\xi)\\ \widetilde{g}_{s,m}^{-}(\xi)\end{bmatrix}\sin m\phi\right]J_m'(\xi\rho_a)d\xi\end{aligned} \tag{8b}$$

$$\begin{bmatrix} H_\rho^{d+}(\rho,\phi,0) \\ H_\rho^{d-}(\rho,\phi,0) \end{bmatrix} = \sum_{m=0}^{\infty} \left[\begin{bmatrix} H_{\rho c,m}^{+}(\rho_a) \\ H_{\rho c,m}^{-}(\rho_a) \end{bmatrix} \cos m\phi + \begin{bmatrix} H_{\rho s,m}^{+}(\rho_a) \\ H_{\rho s,m}^{-}(\rho_a) \end{bmatrix} \sin m\phi \right]$$

$$= \kappa Y_0 \sum_{m=0}^{\infty} \int_0^\infty \left[\begin{bmatrix} -\tilde{f}_{c,m}^{+}(\xi) \\ \tilde{f}_{c,m}^{-}(\xi) \end{bmatrix} \sin m\phi + \begin{bmatrix} \tilde{f}_{s,m}^{+}(\xi) \\ -\tilde{f}_{s,m}^{-}(\xi) \end{bmatrix} \cos m\phi \right] \frac{m}{\xi \rho_a} J_m(\xi\rho_a) d\xi$$

$$+ j\frac{Y_0}{\kappa} \sum_{m=0}^{\infty} \int_0^\infty \sqrt{\xi^2-\kappa^2} \left[\begin{bmatrix} \tilde{g}_{c,m}^{+}(\xi) \\ -\tilde{g}_{c,m}^{-}(\xi) \end{bmatrix} \cos m\phi + \begin{bmatrix} \tilde{g}_{s,m}^{+}(\xi) \\ -\tilde{g}_{s,m}^{-}(\xi) \end{bmatrix} \sin m\phi \right] J_m'(\xi\rho_a) d\xi \quad (9a)$$

$$\begin{bmatrix} H_\phi^{d+}(\rho,\phi,0) \\ H_\phi^{d-}(\rho,\phi,0) \end{bmatrix} = \sum_{m=0}^{\infty} \left[\begin{bmatrix} H_{\phi c,m}^{+}(\rho_a) \\ H_{\phi c,m}^{-}(\rho_a) \end{bmatrix} \cos m\phi + \begin{bmatrix} H_{\phi s,m}^{+}(\rho_a) \\ H_{\phi s,m}^{-}(\rho_a) \end{bmatrix} \sin m\phi \right]$$

$$= \kappa Y_0 \sum_{m=0}^{\infty} \int_0^\infty \left[\begin{bmatrix} -\tilde{f}_{c,m}^{+}(\xi) \\ \tilde{f}_{c,m}^{-}(\xi) \end{bmatrix} \cos m\phi + \begin{bmatrix} -\tilde{f}_{s,m}^{+}(\xi) \\ \tilde{f}_{s,m}^{-}(\xi) \end{bmatrix} \sin m\phi \right] J_m'(\xi\rho_a) d\xi$$

$$+ j\frac{Y_0}{\kappa} \sum_{m=0}^{\infty} \int_0^\infty \sqrt{\xi^2-\kappa^2} \left[\begin{bmatrix} -\tilde{g}_{c,m}^{+}(\xi) \\ \tilde{g}_{c,m}^{-}(\xi) \end{bmatrix} \sin m\phi \right.$$

$$\left. + \begin{bmatrix} \tilde{g}_{s,m}^{+}(\xi) \\ -\tilde{g}_{s,m}^{-}(\xi) \end{bmatrix} \cos m\phi \right] \frac{m}{\xi\rho_a} J_m(\xi\rho_a) d\xi \quad (9b)$$

The required boundary conditions for the problem under investigation are given by (1) The tangential components of electric and magnetic fields are continuous on the plane $z = 0$ for $\rho_a \geq 1$ (2) $E_\rho^+ = -Z_s^+ H_\phi^+$, $E_\rho^- = Z_s^- H_\phi^-$, $E_\phi^+ = Z_s^+ H_\rho^+$, $E_\phi^- = -Z_s^- H_\rho^-$ for $\rho_a \leq 1$ where Z_s^+ and Z_s^- are assumed to be surface impedances of upper and lower surfaces respectively.

The boundary condition (1) gives

$$\begin{bmatrix} E_{\rho c,m}^{d+}(\rho_a) - E_{\rho c,m}^{d-}(\rho_a) \\ E_{\phi s,m}^{d+}(\rho_a) - E_{\phi s,m}^{d-}(\rho_a) \end{bmatrix} = \int_0^\infty \left[H^-(\xi\rho_a) \right] \begin{bmatrix} j\sqrt{\xi^2-\kappa^2}\left[\tilde{f}_{cm}^{+}(\xi) + \tilde{f}_{cm}^{-}(\xi)\right]\xi^{-1} \\ \left[\tilde{g}_{sm}^{+}(\xi) - \tilde{g}_{sm}^{-}(\xi)\right]\xi^{-1} \end{bmatrix} \xi d\xi = 0$$

$$= \int_0^\infty \left[H^-(\xi\rho_a) \right] \begin{bmatrix} \tilde{E}_{\rho c,m}(\xi) \\ \tilde{E}_{\phi s,m}(\xi) \end{bmatrix} \xi d\xi = 0, \qquad \rho_a \geq 1 \quad (10a)$$

$$\begin{bmatrix} E_{\rho s,m}^{d+}(\rho_a) - E_{\rho s,m}^{d-}(\rho_a) \\ E_{\phi c,m}^{d+}(\rho_a) - E_{\phi c,m}^{d-}(\rho_a) \end{bmatrix} = \int_0^\infty \left[H^+(\xi\rho_a) \right] \begin{bmatrix} j\sqrt{\xi^2-\kappa^2}\left[\tilde{f}_{sm}^{+}(\xi) + \tilde{f}_{sm}^{-}(\xi)\right]\xi^{-1} \\ \left[\tilde{g}_{cm}^{+}(\xi) - \tilde{g}_{cm}^{-}(\xi)\right]\xi^{-1} \end{bmatrix} \xi d\xi = 0$$

$$= \int_0^\infty \left[H^+(\xi\rho_a) \right] \begin{bmatrix} \tilde{E}_{\rho s,m}(\xi) \\ \tilde{E}_{\phi c,m}(\xi) \end{bmatrix} \xi d\xi = 0, \qquad \rho_a \geq 1 \quad (10b)$$

$$\begin{bmatrix} H^{d+}_{\rho c,m}(\rho_a) - H^{d-}_{\rho c,m}(\rho_a) \\ H^{d+}_{\phi s,m}(\rho_a) - H^{d-}_{\phi s,m}(\rho_a) \end{bmatrix}$$
$$= Y_0 \int_0^\infty \left[H^-(\xi\rho_a)\right] \begin{bmatrix} j\sqrt{\xi^2-\kappa^2}\left[\widetilde{g}^+_{cm}(\xi)+\widetilde{g}^-_{cm}(\xi)\right](\kappa\xi)^{-1} \\ -\kappa\left[\widetilde{f}^+_{sm}(\xi)-\widetilde{f}^-_{sm}(\xi)\right]\xi^{-1} \end{bmatrix} \xi d\xi = 0$$
$$= \int_0^\infty \left[H^-(\xi\rho_a)\right] \begin{bmatrix} \widetilde{H}_{\rho c,m}(\xi) \\ \widetilde{H}_{\phi s,m}(\xi) \end{bmatrix} \xi d\xi = 0, \qquad \rho_a \geq 1 \tag{10c}$$

$$\begin{bmatrix} H^{d+}_{\rho s,m}(\rho_a) - H^{d-}_{\rho s,m}(\rho_a) \\ H^{d+}_{\phi c,m}(\rho_a) - H^{d-}_{\phi c,m}(\rho_a) \end{bmatrix}$$
$$= Y_0 \int_0^\infty \left[H^+(\xi\rho_a)\right] \begin{bmatrix} j\sqrt{\xi^2-\kappa^2}\left[\widetilde{g}^+_{sm}(\xi)+\widetilde{g}^-_{sm}(\xi)\right](\kappa\xi)-1 \\ -\kappa\left[\widetilde{f}^+_{cm}(\xi)-\widetilde{f}^-_{cm}(\xi)\right]\xi^{-1} \end{bmatrix} \xi d\xi = 0$$
$$= \int_0^\infty \left[H^+(\xi\rho_a)\right] \begin{bmatrix} \widetilde{H}_{\rho s,m}(\xi) \\ \widetilde{H}_{\phi c,m}(\xi) \end{bmatrix} \xi d\xi = 0, \qquad \rho_a \geq 1 \tag{10d}$$

where the kernel matrices $\left[H^+(\xi\rho_a)\right]$ and $\left[H^-(\xi\rho_a)\right]$ are given by

$$\left[H^\pm(\xi\rho_a)\right] = \begin{bmatrix} J'_m(\xi\rho_a) & \pm\frac{m}{\xi\rho_a}J_m(\xi\rho_a) \\ \pm\frac{m}{\xi\rho_a}J_m(\xi\rho_a) & J'_m(\xi\rho_a) \end{bmatrix} \tag{11}$$

The boundary condition (2) gives

$$\begin{bmatrix} E^{t+}_{\rho c,m}(\rho_a) \\ E^{t+}_{\phi s,m}(\rho_a) \end{bmatrix} = \mp Z^+_s \begin{bmatrix} H^{t+}_{\phi c,m}(\rho_a) \\ H^{t+}_{\rho s,m}(\rho_a) \end{bmatrix} = 0,$$
$$\begin{bmatrix} E^{t+}_{\rho s,m}(\rho_a) \\ E^{t+}_{\phi c,m}(\rho_a) \end{bmatrix} = \mp Z^+_s \begin{bmatrix} H^{t+}_{\phi s,m}(\rho_a) \\ H^{t+}_{\rho c,m}(\rho_a) \end{bmatrix} = 0, \qquad \rho_a \leq 1 \tag{12a}$$
$$\begin{bmatrix} E^{t-}_{\rho c,m}(\rho_a) \\ E^{t-}_{\phi s,m}(\rho_a) \end{bmatrix} = \pm Z^-_s \begin{bmatrix} H^{t-}_{\phi c,m}(\rho_a) \\ H^{t-}_{\rho s,m}(\rho_a) \end{bmatrix} = 0,$$
$$\begin{bmatrix} E^{t-}_{\rho s,m}(\rho_a) \\ E^{t-}_{\phi c,m}(\rho_a) \end{bmatrix} = \pm Z^-_s \begin{bmatrix} H^{t-}_{\phi s,m}(\rho_a) \\ H^{t-}_{\rho c,m}(\rho_a) \end{bmatrix} = 0, \qquad \rho_a \leq 1 \tag{12b}$$

where

$$\begin{bmatrix} H^{t\pm}_{\rho c,m}(\rho_a) \\ H^{t\pm}_{\phi s,m}(\rho_a) \end{bmatrix} = \begin{bmatrix} H^{d\pm}_{\rho c,m}(\rho_a) \\ H^{d\pm}_{\phi s,m}(\rho_a) \end{bmatrix} + \begin{bmatrix} H^{i}_{\rho c,m}(\rho_a) \\ H^{i}_{\phi s,m}(\rho_a) \end{bmatrix},$$
$$\begin{bmatrix} H^{t\pm}_{\rho s,m}(\rho_a) \\ H^{t\pm}_{\phi c,m}(\rho_a) \end{bmatrix} = \begin{bmatrix} H^{d\pm}_{\rho s,m}(\rho_a) \\ H^{d\pm}_{\phi c,m}(\rho_a) \end{bmatrix} + \begin{bmatrix} H^{i}_{\rho s,m}(\rho_a) \\ H^{i}_{\phi c,m}(\rho_a) \end{bmatrix} \tag{12c}$$
$$\begin{bmatrix} E^{t\pm}_{\rho c,m}(\rho_a) \\ E^{t\pm}_{\phi s,m}(\rho_a) \end{bmatrix} = \begin{bmatrix} E^{d\pm}_{\rho c,m}(\rho_a) \\ E^{d\pm}_{\phi s,m}(\rho_a) \end{bmatrix} + \begin{bmatrix} E^{i}_{\rho c,m}(\rho_a) \\ E^{i}_{\phi s,m}(\rho_a) \end{bmatrix},$$
$$\begin{bmatrix} E^{t\pm}_{\rho s,m}(\rho_a) \\ E^{t\pm}_{\phi c,m}(\rho_a) \end{bmatrix} = \begin{bmatrix} E^{d\pm}_{\rho s,m}(\rho_a) \\ E^{d\pm}_{\phi c,m}(\rho_a) \end{bmatrix} + \begin{bmatrix} E^{i}_{\rho s,m}(\rho_a) \\ E^{i}_{\phi c,m}(\rho_a) \end{bmatrix} \tag{12d}$$

In the above equations, $H^i_{\rho c,m}$, $H^i_{\rho s,m}$ and $E^i_{\rho c,m}$, $E^i_{\rho s,m}$ denote the $\cos m\phi$ and $\sin m\phi$ parts of the incident wave H^i_ρ and E^i_ρ, respectively, and same is true for $H^i_{\phi c,m}$ and $H^i_{\phi s,m}$ and $E^i_{\phi c,m}$ and $E^i_{\phi s,m}$. The expressions for these factors are given by

$$\begin{aligned} H^i_{\rho c,m}(\rho_a) &= -jY_0E_1\cos\theta_0\epsilon_m j^m J'_m(\kappa\rho_a\sin\theta_0), \\ H^i_{\rho s,m}(\rho_a) &= jY_0E_2\epsilon_m j^m \frac{m}{\kappa\rho_a\sin\theta_0}J_m(\kappa\rho_a\sin\theta_0) \end{aligned} \tag{12e}$$

$$\begin{aligned} H^i_{\phi c,m}(\rho_a) &= jY_0E_2\epsilon_m j^m J'_m(\kappa\rho_a\sin\theta_0), \\ H^i_{\phi s,m}(\rho_a) &= jY_0E_1\cos\theta_0\epsilon_m j^m \frac{m}{\kappa\rho_a\sin\theta_0}J_m(\kappa\rho_a\sin\theta_0) \end{aligned} \tag{12f}$$

$$\begin{aligned} E^i_{\rho c,m}(\rho_a) &= -jE_2\cos\theta_0\epsilon_m j^m J'_m(\kappa\rho_a\sin\theta_0), \\ E^i_{\rho s,m}(\rho_a) &= -jE_1\epsilon_m j^m \frac{m}{\kappa\rho_a\sin\theta_0}J_m(\kappa\rho_a\sin\theta_0) \end{aligned} \tag{12g}$$

$$\begin{aligned} E^i_{\phi c,m}(\rho_a) &= -jE_1\epsilon_m j^m J'_m(\kappa\rho_a\sin\theta_0) \\ E^i_{\phi s,m}(\rho_a) &= jE_2\cos\theta_0\epsilon_m j^m \frac{m}{\kappa\rho_a\sin\theta_0}J_m(\kappa\rho_a\sin\theta_0) \end{aligned} \tag{12h}$$

Equations (10) are the dual integral equations to determine the spectrum functions $\widetilde{f}_m(\xi)$ and $\widetilde{g}_m(\xi)$. The solution of the equations must satisfy the Maxwell equations and edge conditions. Such functions can be found by taking into account the discontinuous properties of the Weber-Schafheitlin's integrals. Thus we can set

$$E^+_{\rho c,m}(\rho_a) - E^-_{\rho c,m}(\rho_a) = \sum_{n=0}^{\infty}\Big[-A^E_{mn}F^+_{mn}(\rho_a) + B^E_{mn}G^-_{mn}(\rho_a)\Big], \tag{13a}$$

$$E^+_{\rho s,m}(\rho_a) - E^-_{\rho s,m}(\rho_a) = \sum_{n=0}^{\infty}\Big[C^E_{mn}F^+_{mn}(\rho_a) + D^E_{mn}G^-_{mn}(\rho_a)\Big], \tag{13b}$$

$$E^+_{\phi s,m}(\rho_a) - E^-_{\phi s,m}(\rho_a) = \sum_{n=0}^{\infty}\Big[A^E_{mn}F^-_{mn}(\rho_a) - B_{mn}G^+_{mn}(\rho_a)\Big], \tag{13c}$$

$$E^+_{\phi c,m}(\rho_a) - E^-_{\phi c,m}(\rho_a) = \sum_{n=0}^{\infty}\Big[C^E_{mn}F^-_{mn}(\rho_a) + D^E_{mn}G^+_{mn}(\rho_a)\Big]. \tag{13d}$$

and

$$H^+_{\rho c,m}(\rho_a) - H^-_{\rho c,m}(\rho_a) = \sum_{n=0}^{\infty}\Big[-A^H_{mn}F^+_{mn}(\rho_a) + B^H_{mn}G^-_{mn}(\rho_a)\Big], \tag{14a}$$

$$H^+_{\rho s,m}(\rho_a) - H^-_{\rho s,m}(\rho_a) = \sum_{n=0}^{\infty}\Big[C^H_{mn}F^+_{mn}(\rho_a) + D^H_{mn}G^-_{mn}(\rho_a)\Big], \tag{14b}$$

$$H^{+}_{\phi s,m}(\rho_a) - H^{-}_{\phi s,m}(\rho_a) = \sum_{n=0}^{\infty}\Big[A^H_{mn}F^{-}_{mn}(\rho_a) - B^H_{mn}G^{+}_{mn}(\rho_a)\Big], \quad (14c)$$

$$H^{+}_{\phi c,m}(\rho_a) - H^{-}_{\phi c,m}(\rho_a) = \sum_{n=0}^{\infty}\Big[C^H_{mn}F^{-}_{mn}(\rho_a) + D^H_{mn}G^{+}_{mn}(\rho_a)\Big]. \quad (14d)$$

where

$$F^{\pm}_{mn}(\rho_a) = \int_0^{\infty}\Big[J_{|m-1|}(\eta\rho_a)J_{|m-1|+2n+1}(\eta) \pm J_{m+1}(\eta\rho_a)J_{m+2n+2}(\eta)\Big]\eta^{-1}d\eta \quad (15a)$$

$$G^{\pm}_{mn}(\rho_a) = \int_0^{\infty}\Big[J_{|m-1|}(\eta\rho_a)J_{|m-1|+2n+2}(\eta) \pm J_{m+1}(\eta\rho_a)J_{m+2n+3}(\eta)\Big]\eta^{-2}d\eta \quad (15b)$$

These integrals are of the form of the discontinuous Weber-Schafheitlin's integral. The edge conditions for the present problem are E_t, $H_t \sim O(1)$ [36–38] and it may readily be verified that $F^{\pm}_{mn}(\rho_a) = G^{\pm}_{mn}(\rho_a) = 0$ for $\rho_a \geq 1$, and $F^{\pm}_{mn}(\rho_a) \sim O(1)$ and $G^{\pm}_{mn}(\rho_a) \sim O(1)$ near the edge $\rho_a \simeq 1$. To derive the spectrum functions $\widetilde{f}(\xi)$ and $\widetilde{g}(\xi)$ of the vector potentials we first determine the spectrum functions of the electromagnetic field, since they are related to each other. We substitute (13) and (14) into (10) and perform the integration, then the spectrum functions of the surface electromagnetic field are determined. The result is

$$\begin{aligned}
\widetilde{E}_{\rho c,m}(\xi) &= \sum_{n=0}^{\infty}\Big[A^E_{mn}\Xi^{-}_{mn}(\xi) - B^E_{mn}\Gamma^{+}_{mn}(\xi)\Big],\\
\widetilde{E}_{\phi s,m}(\xi) &= \sum_{n=0}^{\infty}\Big[-A^E_{mn}\Xi^{+}_{mn}(\xi) + B^E_{mn}\Gamma^{-}_{mn}(\xi)\Big],\\
\widetilde{E}_{\rho s,m}(\xi) &= \sum_{n=0}^{\infty}\Big[C^E_{mn}\Xi^{-}_{mn}(\xi) + D^E_{mn}\Gamma^{+}_{mn}(\xi)\Big],\\
\widetilde{E}_{\phi c,m}(\xi) &= \sum_{n=0}^{\infty}\Big[C^E_{mn}\Xi^{+}_{mn}(\xi) + D^E_{mn}\Gamma^{-}_{mn}(\xi)\Big]. \quad (16a)
\end{aligned}$$

$$\begin{aligned}
\widetilde{H}_{\rho c,m}(\xi) &= \sum_{n=0}^{\infty}\Big[A^H_{mn}\Xi^{-}_{mn}(\xi) - B^H_{mn}\Gamma^{+}_{mn}(\xi)\Big],\\
\widetilde{H}_{\phi s,m}(\xi) &= \sum_{n=0}^{\infty}\Big[-A^H_{mn}\Xi^{+}_{mn}(\xi) + B^H_{mn}\Gamma^{-}_{mn}(\xi)\Big],
\end{aligned}$$

$$\widetilde{H}_{\rho s,m}(\xi) = \sum_{n=0}^{\infty}\Big[C^H_{mn}\Xi^-_{mn}(\xi) + D^H_{mn}\Gamma^+_{mn}(\xi)\Big],$$
$$\widetilde{H}_{\phi c,m}(\xi) = \sum_{n=0}^{\infty}\Big[C^H_{mn}\Xi^+_{mn}(\xi) + D^H_{mn}\Gamma^-_{mn}(\xi)\Big]. \tag{17a}$$

In the above equations the functions $\Xi^{\pm}_{mn}(\xi)$ and $\Gamma^{\pm}_{mn}(\xi)$ are defined by

$$\Xi^{\pm}_{mn}(\xi) = \Big[J_{m+2n}(\xi) \pm J_{m+2n+2}(\xi)\Big]\xi^{-1},$$
$$\Gamma^{\pm}_{mn}(\xi) = \Big[J_{m+2n+1}(\xi) \pm J_{m+2n+3}(\xi)\Big]\xi^{-2}. \tag{17b}$$

It is readily found that the spectrum functions $\widetilde{f}_{cm}(\xi) \sim \widetilde{g}_{sm}(\xi)$ can be expressed in terms of spectrum functions of electromagnetic field, that is,

$$\widetilde{f}^{\pm}_{cm}(\xi) = \frac{1}{2}\Bigg[\frac{1}{j\sqrt{\xi^2-\kappa^2}}\sum_{n=0}^{\infty}\Big[A^E_{mn}\Xi^-_{mn}(\xi) - B^E_{mn}\Gamma^+_{mn}(\xi)\Big] \mp\frac{Z_0}{\kappa}\sum_{n=0}^{\infty}\Big[C^H_{mn}\Xi^+_{mn}(\xi) + D^H_{mn}\Gamma^-_{mn}(\xi)\Big]\Bigg]\xi \tag{18a}$$

$$\widetilde{f}^{\pm}_{sm}(\xi) = \frac{1}{2}\Bigg[\frac{1}{j\sqrt{\xi^2-\kappa^2}}\sum_{n=0}^{\infty}\Big[C^E_{mn}\Xi^-_{mn}(\xi) + D^E_{mn}\Gamma^+_{mn}(\xi)\Big] \mp\frac{Z_0}{\kappa}\sum_{n=0}^{\infty}\Big[-A^H_{mn}\Xi^+_{mn}(\xi) + B^H_{mn}\Gamma^-_{mn}(\xi)\Big]\Bigg]\xi \tag{18b}$$

$$\widetilde{g}^{\pm}_{cm}(\xi) = \frac{1}{2}\Bigg[\frac{\kappa Z_0}{j\sqrt{\xi^2-\kappa^2}}\sum_{n=0}^{\infty}\Big[A^H_{mn}\Xi^-_{mn}(\xi) - B^H_{mn}\Gamma^+_{mn}(\xi)\Big] \pm\sum_{n=0}^{\infty}\Big[C^E_{mn}\Xi^+_{mn}(\xi) + D^E_{mn}\Gamma^-_{mn}(\xi)\Big]\Bigg]\xi \tag{18c}$$

$$\widetilde{g}^{\pm}_{sm}(\xi) = \frac{1}{2}\Bigg[\frac{\kappa Z_0}{j\sqrt{\xi^2-\kappa^2}}\sum_{n=0}^{\infty}\Big[C^H_{mn}\Xi^-_{mn}(\xi) + D^H_{mn}\Gamma^+_{mn}(\xi)\Big] \pm\sum_{n=0}^{\infty}\Big[-A^E_{mn}\Xi^+_{mn}(\xi) + B^H_{mn}\Gamma^-_{mn}(\xi)\Big]\Bigg]\xi \tag{18d}$$

3.2. Derivation of the Expansion Coefficients

The equations for the expansion coefficients can be obtained by applying the second boundary condition for $\rho_a \leq 1$ which is given by (12).

$$\begin{aligned}
&\int_0^\infty J'_m(\rho_a\xi)\begin{bmatrix}\tilde{f}^+_{cm}(\xi)\\ \tilde{f}^+_{sm}(\xi)\end{bmatrix}\left(j\sqrt{\xi^2-\kappa^2}-\kappa\zeta_+\right)\\
&-\frac{m}{\xi\rho_a}J_m(\rho_a\xi)\begin{bmatrix}g^+_{sm}(\xi)\\ -\tilde{g}^+_{cm}(\xi)\end{bmatrix}\left(1-\frac{j\zeta_+}{\kappa}\sqrt{\xi^2-\kappa^2}\right)d\xi\\
=\ &j2\begin{bmatrix}(\cos\theta_0-\zeta_+)E_2 j^m J'_m(\kappa\rho_a\sin\theta_0)\\ (1-\zeta_+\cos\theta_0)E_1 j^m\frac{m}{\kappa\rho_a\sin\theta_0}J_m(\kappa\rho_a\sin\theta_0)\end{bmatrix}
\end{aligned}\tag{19a}$$

$$\begin{aligned}
&-\int_0^\infty J'_m(\rho_a\xi)\begin{bmatrix}\tilde{f}^-_{cm}(\xi)\\ \tilde{f}^-_{sm}(\xi)\end{bmatrix}\left(j\sqrt{\xi^2-\kappa^2}-\kappa\zeta_-\right)\\
&-\frac{m}{\xi\rho_a}J_m(\rho_a\xi)\begin{bmatrix}g^-_{sm}(\xi)\\ -\tilde{g}^-_{cm}(\xi)\end{bmatrix}\left(1-\frac{j\zeta_-}{\kappa}\sqrt{\xi^2-\kappa^2}\right)d\xi\\
=\ &j2\begin{bmatrix}(\cos\theta_0+\zeta_-)E_2 j^m J'_m(\kappa\rho_a\sin\theta_0)\\ (1+\zeta_-\cos\theta_0)E_1 j^m\frac{m}{\kappa\rho_a\sin\theta_0}J_m(\kappa\rho_a\sin\theta_0)\end{bmatrix}
\end{aligned}\tag{19b}$$

$$\begin{aligned}
&\int_0^\infty \frac{m}{\xi\rho_a}J_m(\rho_a\xi)\begin{bmatrix}\tilde{f}^+_{sm}(\xi)\\ -\tilde{f}^+_{cm}(\xi)\end{bmatrix}\left(j\sqrt{\xi^2-\kappa^2}-\kappa\zeta_+\right)\\
&+J'_m(\rho_a\xi)\begin{bmatrix}g^+_{cm}(\xi)\\ \tilde{g}^+_{sm}(\xi)\end{bmatrix}\left(1-\frac{j\zeta_+}{\kappa}\sqrt{\xi^2-\kappa^2}\right)d\xi\\
=\ &j2\begin{bmatrix}(1-\zeta_+\cos\theta_0)E_1 j^m J'_m(\kappa\rho_a\sin\theta_0)\\ -(\cos\theta_0-\zeta_+)E_2 j^m\frac{m}{\kappa\rho_a\sin\theta_0}J_m(\kappa\rho_a\sin\theta_0)\end{bmatrix}
\end{aligned}\tag{19c}$$

$$\begin{aligned}
&-\int_0^\infty \frac{m}{\xi\rho_a}J_m(\rho_a\xi)\begin{bmatrix}\tilde{f}^-_{sm}(\xi)\\ -\tilde{f}^-_{cm}(\xi)\end{bmatrix}\left(j\sqrt{\xi^2-\kappa^2}-\kappa\zeta_-\right)\\
&+J'_m(\rho_a\xi)\begin{bmatrix}g^-_{cm}(\xi)\\ \tilde{g}^-_{sm}(\xi)\end{bmatrix}\left(1-\frac{j\zeta_-}{\kappa}\sqrt{\xi^2-\kappa^2}\right)d\xi\\
=\ &j2\begin{bmatrix}(1+\zeta_-\cos\theta_0)E_1 j^m J'_m(\kappa\rho_a\sin\theta_0)\\ -(\cos\theta_0+\zeta_-)E_2 j^m\frac{m}{\kappa\rho_a\sin\theta_0}J_m(\kappa\rho_a\sin\theta_0)\end{bmatrix}
\end{aligned}\tag{19d}$$

where $\zeta\pm$ are the normalized impedances given by $\zeta\pm = \frac{Z_s^\pm}{z_0}$ with an intrinsic impedance Z_0 in free space. Now we substitute the spectrum functions and project the resulting equations into the functional space with elements P_n^m. If we substitute (18) into (19), and apply the projection of P_n^m, we get the relations for expansion coefficients (A^E_{mn}, B^E_{mn}, C^H_{mn}, D^H_{mn}) and (A^H_{mn}, B^H_{mn}, C^E_{mn}, D^E_{mn}). We take surface impedances of upper and lower surfaces equal, i.e., $\zeta_+ = \zeta_- = \zeta$, to simplify the equations.

$$x^{-m/2}J_m(\xi\sqrt{x}) = \sum_{n=0}^{\infty} 2(m+2n+1)\frac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)}\frac{J_{m+2n+1}(\xi)}{\xi}P_n^m$$

$$P_n^m = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+1)}x^{-m/2}\int_0^\infty J_m(\xi\sqrt{x})J_{m+2n+1}(\xi)d\xi$$

Equations for $(A_{mn}^H, B_{mn}^H, C_{mn}^E, D_{mn}^E)$ are given below.

$$\psi_1(\xi) = \left(1 - \frac{\kappa\zeta}{j\sqrt{\xi^2-\kappa^2}}\right), \qquad \psi_2(\xi) = \frac{Z_0}{\kappa}\left(j\sqrt{\xi^2-\kappa^2} - \kappa\zeta\right),$$

$$\psi_3(\xi) = Z_0\left(\frac{\kappa}{j\sqrt{\xi^2-\kappa^2}} - \zeta\right), \quad \psi_4(\xi) = \left(1 - \frac{j\zeta}{\kappa}\sqrt{\xi^2-\kappa^2}\right).$$

$$\begin{aligned}
&\int_0^\infty \psi_1(\xi)\Big[A_{mn}^E\Xi_{mn}^-(\xi) - B_{mn}^E\Gamma_{mn}^+(\xi)\Big]\Big[\alpha_p^m J_{m+2p}(\xi) - \left(\alpha_p^m+2\right) J_{m+2p+2}(\xi)\Big] \\
&-\psi_4(\xi)\Big[-A_{mn}^E\Xi_{mn}^+(\xi) + B_{mn}^E\Gamma_{mn}^-(\xi)\Big]\Big[2m\left(\alpha_p^m+1\right) J_{m+2p+1}(\xi)\xi^{-1}\Big]d\xi \\
&= -4E_2\zeta j^{\,m+1}\Big[\alpha_p^m J_{m+2p}\left(\kappa\sin\theta_0\right) \\
&- \left(\alpha_p^m+2\right) J_{m+2p+2}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-1}
\end{aligned} \tag{20a}$$

$$\begin{aligned}
&\int_0^\infty -\psi_2(\xi)\Big[C_{mn}^H\Xi_{mn}^+(\xi) + D_{mn}^H\Gamma_{mn}^-(\xi)\Big]\Big[\alpha_p^m J_{m+2p}(\xi) - \left(\alpha_p^m+2\right) J_{m+2p+2}(\xi)\Big] \\
&-\psi_3(\xi)\Big[C_{mn}^H\Xi_{mn}^-(\xi) + D_{mn}^H\Gamma_{mn}^+(\xi)\Big]\Big[2m\left(\alpha_p^m+1\right) J_{m+2p+1}(\xi)\xi^{-1}\Big]d\xi \\
&= 4E_2\cos\theta_0 j^{\,m+1}\Big[\alpha_p^m J_{m+2p}(\kappa\sin\theta_0) \\
&- \left(\alpha_p^m+2\right) J_{m+2p+2}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-1}
\end{aligned} \tag{20b}$$

$$\begin{aligned}
&-\int_0^\infty \psi_1(\xi)\Big[A_{mn}^E\Xi_{mn}^-(\xi) - B_{mn}^E\Gamma_{mn}^+(\xi)\Big]\Big[2m\left(\alpha_p^m+1\right) J_{m+2p+1}(\xi)\xi^{-1}\Big] \\
&+\psi_4(\xi)\Big[-A_{mn}^E\Xi_{mn}^+(\xi) + B_{mn}^E\Gamma_{mn}^-(\xi)\Big] \\
&\Big[\left(\alpha_p^m\right) J_{m+2p}(\xi) - \left(\alpha_p^m+2\right) J_{m+2p+2}(\xi)\Big]d\xi \\
&= 4E_2\zeta j^{m+1}\Big[2m\left(\alpha_p^m+1\right) J_{m+2p+1}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-2}
\end{aligned} \tag{20c}$$

$$\int_0^\infty \psi_2(\xi)\left[C_{mn}^H \Xi_{mn}^+(\xi) + D_{mn}^H \Gamma_{mn}^-(\xi)\right]\left[2m\left(\alpha_p^m+1\right)J_{m+2p+1}(\xi)\xi^{-1}\right]$$
$$+\psi_3(\xi)\left[C_{mn}^H \Xi_{mn}^-(\xi) + D_{mn}^H \Gamma_{mn}^+(\xi)\right]$$
$$\left[\left(\alpha_p^m\right)J_{m+2p}(\xi)-\left(\alpha_p^m+2\right)J_{m+2p+2}(\xi)\right]d\xi$$
$$= -4E_2\cos\theta_0 j^{m+1}\left[2m\left(\alpha_p^m+1\right)J_{m+2p+1}(\kappa\sin\theta_0)\right](\kappa\sin\theta_0)^{-2} \quad (20\text{d})$$

These are equations for $(A_{mn}^H, B_{mn}^H, C_{mn}^E, D_{mn}^E)$ are given below.

$$\int_0^\infty -\psi_2(\xi)\left[-A_{mn}^H \Xi_{mn}^+(\xi) + B_{mn}^H \Gamma_{mn}^-(\xi)\right]$$
$$\left[\alpha_p^m J_{m+2p}(\xi)-(\alpha_p^m+2)J_{m+2p+2}(\xi)\right]$$
$$+\left\{\psi_3(\xi)\left[A_{mn}^H \Xi_{mn}^-(\xi) - B_{mn}^H \Gamma_{mn}^+(\xi)\right]\right\}\left[2m\left(\alpha_p^m+1\right)J_{m+2p+1}(\xi)\right]\xi^{-1}d\xi$$
$$= 4E_1 j^{m+1}\left[2m\left(\alpha_p^m+1\right)J_{m+2p+1}(\kappa\sin\theta_0)\right](\kappa\sin\theta_0)^{-2} \quad (21\text{a})$$

$$\int_0^\infty \left\{\psi_1(\xi)\left[C_{mn}^E \Xi_{mn}^-(\xi) + D_{mn}^E \Gamma_{mn}^+(\xi)\right]\right\}$$
$$\left[\alpha_p^m J_{m+2p}(\xi)-\left(\alpha_p^m+2\right)J_{m+2p+2}(\xi)\right]$$
$$+\psi_4(\xi)\left[C_{mn}^E \Xi_{mn}^+(\xi) + D_{mn}^E \Gamma_{mn}^-(\xi)\right]\left[2m\left(\alpha_p^m+1\right)J_{m+2p+1}(\xi)\right]\xi^{-1}d\xi$$
$$= -4E_1\zeta\cos\theta_0 j^{m+1}\left[2m\left(\alpha_p^m+1\right)J_{m+2p+1}(\kappa\sin\theta_0)\right](\kappa\sin\theta_0)^{-2} \quad (21\text{b})$$

$$\int_0^\infty -\psi_2(\xi)\left[-A_{mn}^H \Xi_{mn}^+(\xi) + B_{mn}^H \Gamma_{mn}^-(\xi)\right]\left[2m\left(\alpha_p^m+1\right)J_{m+2p+1}(\xi)\xi^{-1}\right]$$
$$+\psi_3(\xi)\left[A_{mn}^H \Xi_{mn}^-(\xi) - B_{mn}^H \Gamma_{mn}^+(\xi)\right]\left[\alpha_p^m J_{m+2p}(\xi)-\left(\alpha_p^m+2\right)J_{m+2p+2}(\xi)\right]d\xi$$
$$=4E_1 j^{m+1}\left[\alpha_p^m J_{m+2p}(\kappa\sin\theta_0)-\left(\alpha_p^m+2\right)J_{m+2p+2}(\kappa\sin\theta_0)\right](\kappa\sin\theta_0)^{-1} \quad (21\text{c})$$

$$\int_0^\infty \psi_1(\xi)\left[C_{mn}^E \Xi_{mn}^-(\xi) + D_{mn}^E \Gamma_{mn}^+(\xi)\right]\left[2m\left(\alpha_p^m+1\right)J_{m+2p+1}(\xi)\xi^{-1}\right]$$
$$+\psi_4(\xi)\left[C_{mn}^E \Xi_{mn}^+(\xi) + D_{mn}^E \Gamma_{mn}^-(\xi)\right]\left[\alpha_p^m J_{m+2p}(\xi)-\left(\alpha_p^m+2\right)J_{m+2p+2}(\xi)\right]d\xi$$
$$= -4E_1\zeta\cos\theta_0 j^{m+1}\left[\alpha_p^m J_{m+2p}(\kappa\sin\theta_0)\right.$$
$$\left.-\left(\alpha_p^m+2\right)J_{m+2p+2}(\kappa\sin\theta_0)\right](\kappa\sin\theta_0)^{-1} \quad (21\text{d})$$

where $\alpha_p^m = m + 2p$. Through simple manipulations, the Equations (20) and (21) are reduced to matrix equations.

The matrix equations for expansion coefficients $(A_{mn}^E, B_{mn}^E, C_{mn}^H, D_{mn}^H)$ are

$$\sum_{n=0}^{\infty}\left[A_{mn}^E Z_{mp,n}^{(1,1)} - B_{mn}^E Z_{mp,n}^{(1,2)}\right] = H_{m,p}^{(1)},$$
$$\sum_{n=0}^{\infty}\left[-A_{mn}^E Z_{mp,n}^{(2,1)} + B_{mn}^E Z_{mp,n}^{(2,2)}\right] = H_{m,p}^{(2)}, \tag{22a}$$

$$\sum_{n=0}^{\infty}\left[C_{mn}^H Z_{mp,n}^{\prime(1,1)} + D_{mn}^H Z_{mp,n}^{\prime(1,2)}\right] = H_{m,p}^{\prime(1)},$$
$$\sum_{n=0}^{\infty}\left[C_{mn}^H Z_{mp,n}^{\prime(2,1)} + D_{mn}^H Z_{mp,n}^{\prime(2,2)}\right] = H_{m,p}^{\prime(2)} \tag{22b}$$

$$\sum_{n=0}^{\infty} A_{0n}^E Z_{0p,n}^{(1,1)} = H_{0,p}^{(1)}, \quad \sum_{n=0}^{\infty} D_{0n}^H Z_{0p,n}^{\prime(1,2)} = H_{0,p}^{\prime(1)}$$
$$m = 1, 2, 3, \ldots; \quad p = 0, 1, 2, 3, \ldots; \tag{22c}$$

The matrix equations for expansion coefficients $(A_{mn}^H, B_{mn}^H, C_{mn}^E, D_{mn}^E)$ are

$$\sum_{n=0}^{\infty}\left[A_{mn}^H Z_{mp,n}^{\prime(1,1)} - B_{mn}^H Z_{mp,n}^{\prime(1,2)}\right] = K_{m,p}^{\prime(1)},$$
$$\sum_{n=0}^{\infty}\left[A_{mn}^H Z_{mp,n}^{('2,1)} - B_{mn}^H Z_{mp,n}^{\prime(2,2)}\right] = K_{m,p}^{\prime(2)}, \tag{23a}$$

$$\sum_{n=0}^{\infty}\left[C_{mn}^E Z_{mp,n}^{(1,1)} + D_{mn}^E Z_{mp,n}^{(1,2)}\right] = K_{m,p}^{(1)},$$
$$\sum_{n=0}^{\infty}\left[C_{mn}^E Z_{mp,n}^{(2,1)} + D_{mn}^E Z_{mp,n}^{(2,2)}\right] = K_{m,p}^{(2)} \tag{23b}$$

$$\sum_{n=0}^{\infty} A_{0n}^H Z_{0p,n}^{\prime(1,1)} = K_{0,p}^{\prime(2)}, \quad \sum_{n=0}^{\infty} D_{0n}^E Z_{0p,n}^{(1,2)} = K_{0,p}^{(2)}$$
$$m = 1, 2, 3, \ldots; \qquad p = 0, 1, 2, 3, \ldots \tag{23c}$$

These matrix equations can be solved using standard numerical techniques. The elements $Z_{mp,n}^{(1,1)} \sim Z_{mp,n}^{(2,2)}$ and $Z_{mp,n}^{\prime(1,1)} \sim Z_{mp,n}^{\prime(2,2)}$ contains integrals of the form given below.

$$K_\lambda(\alpha,\beta)=\int_0^\infty \frac{\sqrt{\xi^2-\kappa^2}}{\xi^\lambda}J_\alpha(\xi)J_\beta(\xi)d\xi,$$

$$G_\lambda(\alpha,\beta)=\int_0^\infty \frac{1}{\xi^\lambda\sqrt{\xi^2-\kappa^2}}J_\alpha(\xi)J_\beta(\xi)d\xi,$$

$$L_\lambda(\alpha,\beta)=\int_0^\infty \frac{J_\alpha(\xi)J_\beta(\xi)}{\xi^\lambda}\,d\xi. \tag{24}$$

These integrals converge when $\alpha+\beta>\lambda-1$ and $\lambda>1$ for $K(\alpha,\beta)$ and $\alpha+\beta>\lambda-1$ and $\lambda>-1$ for $G(\alpha,\beta)$ and are discussed in detail in appendix B of [21]. The $L(\alpha,\beta,\lambda)$ is a special case of the Weber-Schafheitlin's integral.

3.3. Far Field Expression

Here we derive the far field expressions of A_z^d and F_z^d given in (7) directly by applying the stationary phase method of integration. Application of the standard process of the method yields the result given by

$$I_{nt}=\exp\left(j\frac{m+1}{2}\pi\right)\frac{\exp(-j\kappa R)}{\kappa R}\widetilde{P}(\kappa\sin\theta)\frac{\cos\theta}{\sin\theta} \tag{25}$$

If we apply this formula to the vector potential given in (7) we have

$$\begin{aligned}A_z^d(\mathbf{r})=&\;\mu_0a^2Y_0\frac{\exp(-jkR)}{R}\frac{1}{\sin\theta}\\ &\left\{\sum_{n=0}^{\infty}j\left[A_{0n}^E\frac{J_{2n+2}(\kappa\sin\theta)}{(\kappa\sin\theta)}-Z_0\cos\theta D_{0n}^H\frac{J_{2n+3}(\kappa\sin\theta)}{(\kappa\sin\theta)^2}\right]\right.\\ &-\frac{1}{2}\sum_{m=1}^{\infty}j^{m+1}\sum_{n=0}^{\infty}\left\{\left[A_{mn}^E\Xi_{mn}^-(\kappa\sin\theta)-B_{mn}^E\Gamma_{mn}^+(\kappa\sin\theta)\right]\right.\\ &+Z_0\cos\theta\left[C_{mn}^H\Xi_{mn}^+(\kappa\sin\theta)+D_{mn}\Gamma_{mn}^-(\kappa\sin\theta)\right]\cos m\phi\\ &+Z_0\cos\theta\left[-A_{mn}^H\Xi_{mn}^+(\kappa\sin\theta)+B_{mn}^H\Gamma_{mn}^-(\kappa\sin\theta)\right]\\ &\left.\left.+\left[C_{mn}^E\Xi_{mn}^-(\kappa\sin\theta)+D_{mn}^E\Gamma_{mn}^+(\kappa\sin\theta)\right]\sin m\phi\right\}\right\}\end{aligned} \tag{26a}$$

$$F_z^d(\mathbf{r}) = \epsilon_0 a^2 \frac{\exp(-jkR)}{R} \frac{1}{\sin\theta}$$
$$\left\{ \sum_{n=0}^{\infty} j \left[Z_0 A_{0n}^H \frac{J_{2n+2}(\kappa\sin\theta)}{(\kappa\sin\theta)} - \cos\theta D_{0n}^E \frac{J_{2n+3}(\kappa\sin\theta)}{(\kappa\sin\theta)^2} \right] \right.$$
$$-\frac{1}{2} \sum_{m=1}^{\infty} j^{m+1} \sum_{n=0}^{\infty} \left\{ Z_0 \left[A_{mn}^H \Xi_{mn}^-(\kappa\sin\theta) - B_{mn}^H \Gamma_{mn}^+(\kappa\sin\theta) \right] \right.$$
$$- \cos\theta \left[C_{mn}^E \Xi_{mn}^+(\kappa\sin\theta) + D_{mn}^E \Gamma_{mn}^-(\kappa\sin\theta) \right] \cos m\phi$$
$$+ \cos\theta \left[A_{mn}^E \Xi_{mn}^+(\kappa\sin\theta) - B_{mn}^E \Gamma_{mn}^-(\kappa\sin\theta) \right]$$
$$\left. \left. + Z_0 \left[C_{mn}^H \Xi_{mn}^-(\kappa\sin\theta) + D_{mn}^H \Gamma_{mn}^+(\kappa\sin\theta) \right] \sin m\phi \right\} \right\} \tag{26b}$$

In the far region we have the relations

$$\begin{aligned} E_\theta &= -j\omega A_\theta = j\omega\sin\theta A_z, \\ H_\theta &= -j\omega F_\theta = j\omega\sin\theta F_z = -Y_0 E_\phi, \\ A_\phi &= Z_0 \sin\theta F_z. \end{aligned} \tag{27}$$

3.4. Physical Optics Approximate Solutions

We consider here physical optics solutions for comparison with the KP solutions.

3.4.1. E-polarization

The incident and reflected electromagnetic plane wave at the plane $z = 0$ may be represented as follows.

$$\mathbf{E}^i = \hat{y} E_1 \exp[jk(x\sin\theta_0 + z\cos\theta_0)] \tag{28a}$$
$$\mathbf{H}^i = (\hat{x}\cos\theta_0 - \hat{z}\sin\theta_0)\, E_1 Y_0 \exp[jk(x\sin\theta_0 + z\cos\theta_0) \tag{28b}$$
$$\mathbf{E}^r = A\hat{y} E_1 \exp[jk(x\sin\theta_0 - z\cos\theta_0)] \tag{28c}$$
$$\mathbf{H}^r = A\,(-\hat{x}\cos\theta_0 - \hat{z}\sin\theta_0)\, E_1 Y_0 \exp[jk(x\sin\theta_0 - z\cos\theta_0) \tag{28d}$$

Applying surface impedance boundary condition(SIBC) at $z = 0$ plane, we get the reflection coefficient $A = \frac{-1+\cos\theta_0\zeta^+}{1+\cos\theta_0\zeta^+}$. The total field on the disk is

$$\begin{aligned} \mathbf{E}^{tot} &= \hat{y} \frac{2E_1\zeta^+\cos\theta_0}{\zeta^+\cos\theta_0 + 1} \exp[jk(x\sin\theta_0)], \\ \mathbf{H}^{tot} &= \hat{x} \frac{2E_1 Y_0\cos\theta_0}{\zeta^+\cos\theta_0 + 1} \exp[jk(x\sin\theta_0)]. \end{aligned} \tag{29}$$

The corresponding currents will be

$$\mathbf{M} = -\hat{n} \times \mathbf{E}^{tot}, \qquad \mathbf{J} = \hat{n} \times \mathbf{H}^{tot}. \tag{30}$$

In our case, unit normal vector $\hat{n}$ is $\hat{z}$. The far field expressions of the vector potentials are given by

$$\begin{aligned} A_y &= \frac{\mu E_1 Y_0 \cos\theta_0}{2\pi(\zeta^+ \cos\theta_0 + 1)R} \exp(-jkR) \int_S \exp\left[jk\left(x' \sin\theta_0\right)\right] \exp \\ &\quad \left[jk \sin\theta \left(x' \cos\phi + y' \sin\phi\right)\right] dx'dy' \\ A_y &= \frac{\kappa\mu E_1 Y_0 \cos\theta_0}{(\zeta^+ \cos\theta_0 + 1)} G_0(R) \frac{J_1(\kappa\Theta)}{\Theta} \qquad (31a) \\ F_x &= \frac{\epsilon_0 E_1 \zeta^+ \cos\theta_0}{2\pi(\zeta^+ \cos\theta_0 + 1)R} \exp(-jkR) \int_S \exp[jk(x' \sin\theta_0)] \exp \\ &\quad \left[jk \sin\theta(x' \cos\phi + y' \sin\phi)\right] dx'dy' \\ F_x &= \frac{\kappa\epsilon_0 E_1 \zeta^+ \cos\theta_0}{(\zeta^+ \cos\theta_0 + 1)} G_0(R) \frac{J_1(\kappa\Theta)}{\Theta} \qquad (31b) \end{aligned}$$

where $\Theta = \sqrt{(\sin\theta_0 + \sin\theta\cos\phi)^2 + \sin\theta\sin\phi}$ and $G_0(R) = \frac{\exp(-jkR)}{R}$.

3.4.2. H-polarization

In this case, We got the far field expressions in a similar fashion as for E-polarization. We are just writing the final results

$$\begin{aligned} A_x &= \frac{\kappa\mu E_2 Y_0 \cos\theta_0}{(\zeta^+ + \cos\theta_0)} G_0(R) \frac{J_1(\kappa\Theta)}{\Theta}, \\ F_y &= -\frac{\kappa\epsilon_0 E_2 \zeta^+}{(\zeta^+ + \cos\theta_0)} G_0(R) \frac{J_1(\kappa\Theta)}{\Theta}. \end{aligned} \tag{32}$$

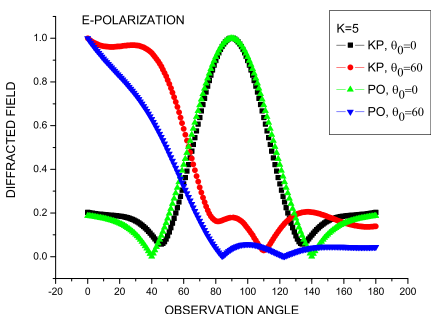

Figure 2. Comparison of KP and PO methods.

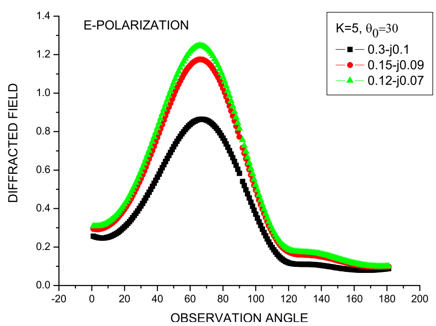

Figure 3. Effect of surface impedance of circular disk.

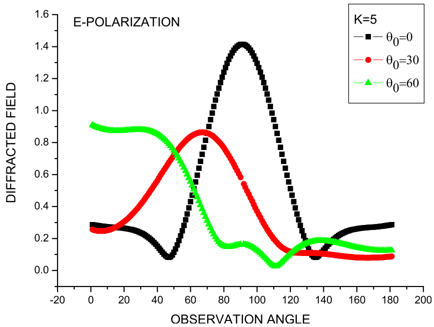

Figure 4. Effect of angle of incidence.

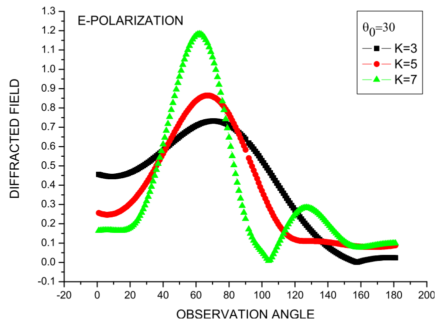

Figure 5. Effect of disk size.

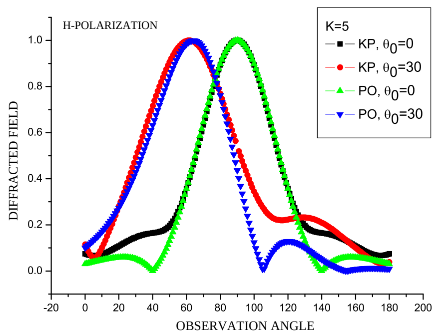

Figure 6. Comparison of KP and PO methods.

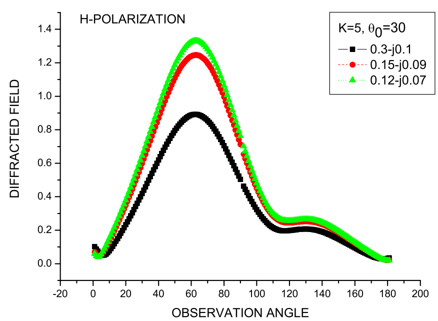

Figure 7. Effect of surface impedance of circular disk.

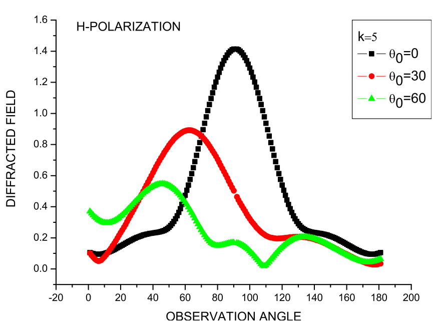

Figure 8. Effect of angle of incidence.

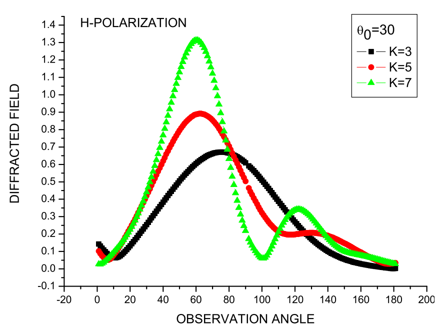

Figure 9. Effect of disk size.

The far electric field is derived as

$$E_\theta \sim -j\omega[A_\theta + Z_0 F_\phi], \qquad E_\phi \sim -j\omega[A_\phi - Z_0 F_\theta]. \tag{33}$$

where $A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi$ and $F_\phi = -F_x \sin\phi + F_y \cos\phi$.

4. RESULTS AND DISCUSSION

To study the scattering properties of impedance disk, expansion coefficients $Am \sim Dm$ are computed. We have taken $m = 2 * \kappa$ in our numerical computations. The theoretical expressions for the far field are given by (27) for the impedance disk. The patterns for E-polarization and H-polarization are shown by $E_2 = 0$ and $E_1 = 0$ respectively. The plane of incidence is xz-plane ($\phi_0 = 0, \pi$). Fig. 2 to Fig. 9 show the far field patterns of circular disk in the ϕ-cut plane $\phi = 0, \pi$. The normalized radii are $\kappa = ka = 3$, $ka = 5$ and, $ka = 7$ respectively. In all these figures, the normal incidence is for $\theta_0 = 0$. In all results, the value of surface impedance ($\zeta = 0.3 - j0.1$) is used except where the results are shown for different values of surface impedances which are mentioned in figures explicitly. In these figures, the field patterns obtained using the physical optics (PO) method are also included for comparison. The PO results are obtained using (28) $\sim$ (33). It is observed from the comparison that the PO and KP results agree well for normal incidence ($\theta_0 = 0$) but the degree of discrepancy increases as the angle of incidence becomes large. It is due to the fact that the PO approximation inaccuracy increases for shadow region contribution. The values of the normalized surface impedance ζ ($0.3 - j0.1$, $0.15 - j0.09$, $0.12 - j0.07$) are taken from [39] which correspond to 5%, 10%, and 20% respectively gravimetric moisture content in San Antonio Gray Clay Loam with a density of $1.4\,\mathrm{g/cm^3}$. We also observe that the scattered field increases as the surface impedance of the disk decreases and it approaches to perfect electric conductor(PEC) disk scattering [21] case as the surface impedance leads to zero, as expected. Because PEC boundary condition is a special case of surface impedance boundary condition. We observe through Figs. 4 and 8 that the peak of the field patterns shifts as the incidence angle changes.

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