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ROBUST CALCULATIONS OF MAXIMUM RATIO COMBINING DIVERSITY GAINS BASED ON STOCHASTIC MEASUREMENTS

X. Chen*

Chalmers University of Technology, Gothenburg 412 96, Sweden

Abstract—Previous work on maximum ratio combining (MRC) diversity has derived a closed-form cumulative distribution function (CDF), referred to as Lee's formula, for spatially correlated Rayleigh fading channels. It is usually believed that (due to its singularity) Lee's formula will result in large numerical error when two eigenvalues of a diversity antenna's covariance matrix are close to each other. This letter shows that the limit of Lee's formula converges to the true CDF as eigenvalues converge to each other, which implies that Lee's formula is robust in determining diversity gains of arbitrary antennas based on stochastic measurements.

1. INTRODUCTION

Diversity antenna techniques have been used to improve the communication reliability in multipath fading environments for decades [1, 2]. Consequently the diversity gain has become a parameter for the characterization of multi-element antennas (MEAs) [1–7]. This letter focuses on the maximum ratio combining (MRC) diversity gain [5–7] in rich scattering Rayleigh fading environments. The cumulative distribution function (CDF) of the MRC output of a correlated MEA in a rich scattering Rayleigh fading environment was derived by Lee [5]

$$F(\gamma) = 1 - \sum_{i=1}^{M} \frac{\lambda_i^{M-1} \exp(-\gamma/\lambda_i)}{\prod\limits_{k \neq i}^{M} (\lambda_i - \lambda_k)} \tag{1}$$

where γ is the instantaneous signal-to-noise ratio (SNR), M is the number of antenna elements, and λ_i ($i = 1 \ldots M$) as the ith eigenvalue

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* Corresponding author: Xiaoming Chen (xiaoming.chen@chalmers.se).

of the covariance matrix

$$\mathbf{R} = E[\mathbf{h}\mathbf{h}^H] \tag{2}$$

where $\mathbf{h}$ is the complex column-vector fading channel. We thereby refer to (1) as Lee's (CDF) formula.

Lee's formula is valid for MEAs with distinct eigenvalues of covariance matrices, which has been validated by measurements [6]. However, Lee's formula has apparent singularity when any two eigenvalues of the MEA's covariance matrix are equal. Therefore, it is usually believed that Lee's formula will result in large numerical error when two eigenvalues are close to each other. In this letter, however, we will show that the limit of Lee's formula converges to the true CDF as eigenvalues converge to each other when the covariance matrix is estimated by sample mean,

$$\hat{\mathbf{R}} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{h}_n\mathbf{h}_n^H \tag{3}$$

where $\mathbf{h}_n$ is the nth realization of random channel vector $\mathbf{h}$, and N is the number of realizations. Therefore, it can be used for MRC diversity evaluations of arbitrary antennas based on stochastic measurements in Rayleigh fading environments. We hereafter refer to (3) as sample covariance matrix and its eigenvalue, $\hat{\lambda}_i$ ($i = 1 \ldots M$), as sample eigenvalues. In a typical diversity antenna measurement, N is always finite so that $\hat{\lambda}_i$ deviate from λ_i with large probability.

2. DIVERSITY GAIN

The effective diversity gain (EDG) is defined as the output SNR improvement of a diversity antenna compared with that of a single antenna with unity efficiency at 1% outage probability level [1]. For Rayleigh fading, the output SNR of a single antenna has an exponential distribution, $1 - \exp(-\gamma)$ [1]. The CDF of the MRC output SNR in Rayleigh fading is known for two cases:

- eigenvalues are all different from each other as (1);
- eigenvalues are all equal, $\lambda_i = \lambda$ ($i = 1 \ldots M$), as

$$F(\gamma) = 1 - \exp\left(-\frac{\gamma}{\lambda}\right)\sum_{i=1}^{M}\frac{(\gamma/\lambda)^{i-1}}{(i-1)!}. \tag{4}$$

The CDF expressions with arbitrary equal eigenvalues are unknown in general, and have to be approximated by empirical

CDFs from measured samples. We first consider a two-element diversity antenna with unity efficiencies and no correlation. The covariance matrix with perfect estimation $\mathbf{R}$ is an identity matrix with equal eigenvalues of unity. In this case, there would have been singularity if we use Lee's formula (1). Nevertheless, in practice MEAs' covariance matrices and eigenvalues in multipath fading environments are unknown and have to be estimated from measurement samples. Thus $\hat{\lambda}_1 \neq \hat{\lambda}_2$ with large probability. The question is if there will be large numerical error using Lee's formula? To answer that, we generate independent and identically distributed (i.i.d.) complex Gaussian channel, represented by $\mathbf{h}_w$, with Frobenius norm satisfying $E[\|\mathbf{h}_w\|_F^2] = M$, where $M = 2$ in this case. The channel seen by the diversity antenna can then be expressed as $\mathbf{h} = \mathbf{R}^{1/2}\mathbf{h}_w$ where $\mathbf{R}^{1/2}$ is the Cholesky decomposition of $\mathbf{R}$, which is identity matrix $\mathbf{I}$ in this case. The sample covariance matrix $\hat{\mathbf{R}}$ is estimated using (3), which deviates from $\mathbf{I}$ (due to finite sample number N) with large probability.

Fig. 1 shows the EDG (as a function of the number of channel realizations) obtained using Lee's formula with sample eigenvalues $\hat{\lambda}_i$ of $\hat{\mathbf{R}}$ against that obtained using empirical CDF from generated channel realizations. Surprisingly, the EDG converges to the true value, i.e., 11.7 dB, much faster than that of the empirical CDF. It is surprising to see that there is no noticeable error due to close-to-singularity problem in spite of the fact that the estimation error of sample eigenvalues reduces (and thereby become closer to each other) with increasing

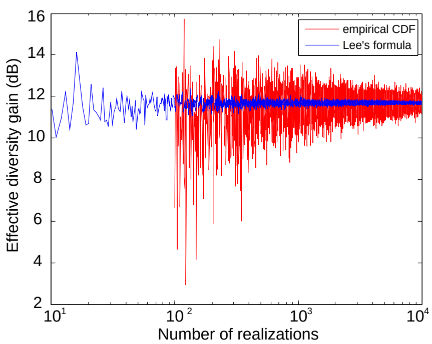

Figure 1. Numerically simulated EDG as a function of number of realizations for two-element antenna with unity efficiency and no correlation.

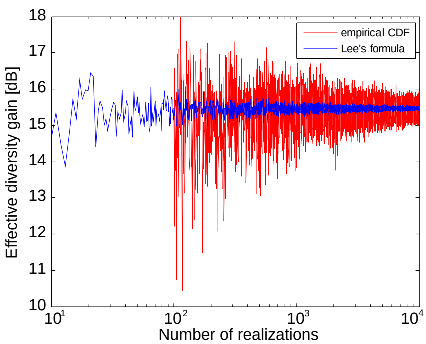

Figure 2. Numerically simulated effective diversity gain as a function of number of realizations for three-element antenna with unity efficiency and a uniform correlation of 0.5.

sample number.

We then consider a three-element antenna with a covariance matrix

$$\mathbf{R} = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix}, \tag{5}$$

so that two of the eigenvalues are equal ($\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 0.5$). Repeat the same simulation procedure as above, the EDGs are calculated and shown in Fig. 2 as a function of the number of channel realizations. Similar results are observed, i.e., EDG obtained using Lee's formula with sample eigenvalues not only converge to the true value but also converge much faster than that obtained from empirical CDF. The faster convergence of Lee's formula is because that the sample eigenvalues converge faster the empirical CDF at 1% level.

3. CONVERGENCE IN DISTRIBUTION

Although the fact of that the limit of Lee's formula converge to the true CDF as eigenvalues converge to each other was not found or shown in previous literature, its proof is straightforward. First we consider the case when $M = 2$, and $\lambda_i \to \lambda$ ($i = 1, 2$). In this case, Lee's formula reduces to

$$F_{Lee}(\gamma) = 1 - \frac{\lambda_1 \exp(-\gamma/\lambda_1) - \lambda_2 \exp(-\gamma/\lambda_2)}{\lambda_1 - \lambda_2}; \tag{6}$$

the true CDF, i.e., the distribution of the MRC output of a two-element antenna with unity efficiency and no correlation, is

$$F(\gamma) = 1 - \exp\left(-\frac{\gamma}{\lambda}\right)\left(1 + \frac{\gamma}{\lambda}\right). \tag{7}$$

Namely, we need to show that $F_{Lee}(\gamma) \to F(\gamma)$ as $\lambda_i \to \lambda$.

Proof: Without loss of generality, let $\lambda_2 = \lambda$ and $\lambda_1 = \lambda + \varepsilon$ for any $\varepsilon > 0$ ($\lambda_i \to \lambda$ is equivalent to $\varepsilon \to 0$), and substitute these into (6),

$$F_{Lee}(\gamma) = 1 - \frac{\lambda}{\varepsilon}\left[\exp\left(-\frac{\gamma}{\lambda+\varepsilon}\right) - \exp\left(-\frac{\gamma}{\lambda}\right)\right] - \exp\left(-\frac{\gamma}{\lambda+\varepsilon}\right) \tag{8}$$

Using Taylor expansion to the first order,

$$\begin{aligned} \lim_{\varepsilon\to 0} F_{Lee}(\gamma) &= \lim_{\varepsilon\to 0}\left\{1 - \frac{\lambda}{\varepsilon}\left[\frac{\gamma}{\lambda^2}\exp\left(-\frac{\gamma}{\lambda}\right)\varepsilon + o(\varepsilon)\right]\right. \\ &\qquad \left. -\left[\exp\left(-\frac{\gamma}{\lambda}\right) + \frac{\gamma}{\lambda^2}\exp\left(-\frac{\gamma}{\lambda}\right)\varepsilon + o(\varepsilon)\right]\right\} \\ &= 1 - \frac{\gamma}{\lambda}\exp\left(-\frac{\gamma}{\lambda}\right) - \exp\left(-\frac{\gamma}{\lambda}\right) = F(\gamma) \end{aligned} \tag{9}$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$.

It is, however, difficult to prove that the limit of Lee's formula converges to the true CDF with arbitrary antenna element number (and therefore arbitrary number of equal eigenvalues) from CDF formula directly. We have to resort to characteristic function for more general proof. The characteristic function of the MRC output with an arbitrary MEA is [8]

$$\phi(z) = E[\exp(jz\gamma)] = \frac{1}{\det(\mathbf{I} + z\mathbf{R})} = \prod_{i=1}^{M} \frac{1}{1 + z\lambda_i}. \tag{10}$$

It is self-evident that $\phi(z)$ converges as $\lambda_i \to \lambda$ $(i = 1 \dots M)$. Since $\phi(z)$ is the Fourier transform of the probability density function (PDF) of γ, to show that the limit of Lee's formula converges to the true CDF is equivalent to show that the convergence of $\phi(z)$.

Proof: Let F_i $(i = 1 \dots M)$ be the CDF of γ when i eigenvalues of the covariance matrix of the MEAs are equal. Namely, F_1 denotes Lee's formula, F_{Lee}, (1) and F_M represents the classical MRC output CDF with i.i.d. antenna branches (4). Due to bijection between CDF and characteristic function, there exists one $\phi_i(z)$ for each F_i uniquely. The continuity theorem [9] states that if $\phi_1(z)$ converges to $\phi_i(z)$, then F_1 converges to F_i.

This is a general proof, yet it is rather abstract, for this reason, we would like to keep the proof for $M = 2$ case for better convergence illustration.

4. CONCLUSION

In this letter, we showed that EDGs obtained using Lee's formula with sample eigenvalues not only converge to the true values for MEAs with arbitrary number of equal eigenvalues, but also converge much faster than the EDGs obtained from empirical CDF. A proof of limit-convergence was given. This work verified that Lee's formula can be used in diversity measurement with arbitrary MEAs without any singularity problem.

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