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GENERATION OF A WIDE-BAND RESPONSE USING EARLY-TIME AND MIDDLE-FREQUENCY DATA THROUGH THE LAGUERRE FUNCTIONS

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Abstract—In this hybrid combined time domain (TD) and frequency domain (FD) approach, one can generate the early-time response using the method of marching-on-in-time (MOT) and use the method of moment (MOM) to generate the middle-frequency response, as the low frequency data may be unstable. The early-time and the middle-frequency data provide the missing low and high frequency response and the late-time response, respectively. Generation of a wide-band response using partial information of the TD data and FD data has been accomplished by the use of the continuous and discrete Laguerre functions.

1. INTRODUCTION

Typically, the method of moments (MOM), which uses an integral-equation formulation, can be applied to perform the electromagnetic analysis in the frequency domain. However, this approach can become computationally intensive to perform a wide-band analysis. A time-domain approach is preferred to a wide-band analysis. For a time-domain integral equation formulation, the method of marching-on-in-time (MOT) is usually employed. A serious drawback of this algorithm

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is the occurrence of late-time instabilities in the form of high-frequency oscillations.

To overcome high computational demand to generate high-frequency data for MOM and to avoid late-time oscillations for a MOT method, the hybrid TD (time domain) and FD (frequency domain) method has been proposed to interpolate and extrapolate both domain data simultaneously using only early-time and low-frequency data [1–6]. The MOM approach can efficiently generate low-frequency data, while the MOT algorithm can be used to obtain stable early-time data quickly. The basic principle is that the early-time and low-frequency data provide the missing high-frequency response and the missing late-time response, respectively.

From the basic principles, one can also extend the hybrid TD-FD method to generate a wide-band response in both domains using early-time and middle-frequency data, as low frequency data may be unreliable, and one does not need much more computation time to obtain middle-frequency data.

The objective is to generate a wide-band electromagnetic response with high accuracy using the combined early TD and middle FD data. We will use the continuous and discrete Laguerre polynomials to interpolate and extrapolate the wide-band response using the hybrid TD-FD data such as early-time and middle-frequency data, in an electromagnetic analysis.

This paper is organized as follows. In Section 2, we will explain the definitions and properties of the continuous and discrete Laguerre functions, the process of interpolating and extrapolating the wide-band response, and the evaluation of the performance. Section 3 shows one numerical example (horn antenna) to evaluate the performance of the wide-band response using the seven hybrid TD-FD dataset. Finally, some conclusions are presented in Section 4.

2. FORMULATIONS

2.1. Continuous Laguerre Functions

The continuous orthonormal Laguerre functions and its Laplace transform can be defined by [7]

$$\phi_{cn}(t,\sigma) = \sqrt{\sigma}e^{-\frac{\sigma t}{2}}L_n(\sigma t) \tag{1}$$

$$\Phi_{cn}(s,\sigma) = \sqrt{\sigma}\frac{\left(s-\frac{\sigma}{2}\right)^n}{\left(s+\frac{\sigma}{2}\right)^{n+1}} \qquad \sigma > 0; \quad n \geq 0 \tag{2}$$

where σ is the scaling factor with only positive values because its poles should be all on the negative real axis of the S-plane to have stability.

The continuous Laguerre polynomials $L_n(t)$ can be defined by

$$L_n(t) = \frac{e^t}{n!}\frac{d^n(t^n e^{-t})}{dt^n}, \qquad n \geq 0; \qquad t \geq 0 \tag{3}$$

They are causal, i.e., they are nonzero only for $t \geq 0$. A causal electromagnetic response $x(t)$ at a particular location in space for $t \geq 0$ can be expanded by a Laguerre series as

$$x(t) = \sum_{n=0}^{\infty} c_n \phi_{cn}(t, \sigma) \tag{4}$$

The Fourier transform of (1) can be evaluated as

$$\Phi_{cn}(f, l) = \frac{\left(-\frac{1}{2} + j\frac{f}{l}\right)^n}{\sqrt{2\pi l}\left(\frac{1}{2} + j\frac{f}{l}\right)^{n+1}} \tag{5}$$

where $l = \sigma/2\pi$ and $j = \sqrt{-1}$.

2.2. Discrete Laguerre Functions

The discrete Laguerre functions can be defined in the Z-domain as [8, 9]

$$\Phi_{dn}(z, a) = \sqrt{1-a^2}\frac{\left(z^{-1} - a\right)^n}{\left(1 - az^{-1}\right)^{n+1}}, \qquad |a| < 1; \qquad n \geq 0 \tag{6}$$

The constraint on the pole $|a| < 1$ is set to make the functions causal and stable. As the limit Δt approaches zero, (2) and (6) become equal, as they are related by

$$a = e^{-\frac{\sigma \cdot \Delta t}{2}} \quad \text{or} \quad \sigma = -\frac{2}{\Delta t}\ln a \tag{7}$$

One can use a as the scaling factor for both the continuous and discrete Laguerre functions. It is important to note that for the discrete Laguerre functions, the boundary of the scaling factor a is $|a| < 1$. However, the boundary of the scaling factor a is $0 < a < 1$ for the continuous Laguerre functions because it must be a positive number. The TD discrete Laguerre functions can be written as

$$\begin{aligned} \phi_{d0}(k, a) &= a\phi_{d0}(k-1, a) + \sqrt{1-a^2}\delta(k) \\ \phi_{dn+1}(k, a) &= a\phi_{dn+1}(k-1, a) + \phi_{dn}(k-1, a) - a\phi_{dn}(k, a) \end{aligned} \tag{8}$$

where $\delta(k)$ is a Kronecker delta. The FD discrete Laguerre functions can be obtained by putting $z = e^{j2\pi f \cdot \Delta t}$ in (6) resulting in

$$\Phi_{dn}(f, a) = \Delta t \cdot \sqrt{1-a^2}\frac{\left(e^{-j2\pi f \cdot \Delta t} - a\right)^n}{\left(1 - ae^{-j2\pi f \cdot \Delta t}\right)^{n+1}}, \qquad |a| < 1 \tag{9}$$

A causal sampled-sequence of the electromagnetic response $x(k)$ can be expanded by a Laguerre series as

$$x(k) = \sum_{n=0}^{\infty} c_n \phi_{dn}(k, a) \qquad k \geq 0; \quad |a| \ < 1 \tag{10}$$

2.3. Process of Interpolation and Extrapolation

Let M_1 and M_2 be the number of TD and FD samples that are given for the functions $x(t)$ and $X(f)$, respectively. The total number of available samples is M_t in the TD and M_f in the FD. It means one can utilize only early-time data (x_1) and low- or middle-frequency data (X_1). From these relationships, TD and FD samples can be defined as follows:

$$\begin{cases} x_1 = \{x(t_0), x(t_1), \ldots, x(t_{M_1-1})\}^T \\ x_2 = \{x(t_{M_1}), x(t_{M_1+1}), \ldots, x(t_{M_t-1})\}^T \\ X_1 = \{X(f_P), X(f_{P+1}), \ldots, X(f_{P+M_2-1})\}^T \\ X_2 = \left[\begin{array}{l} \{X(f_{M_2}), X(f_{M_2+1}), \ldots, X(f_{M_f-1})\}^T, \text{at } P = 0 \\ \{X(f_0), \ldots, X(f_{P-1}), X(f_{P+M_2}) \ldots X(f_{M_f-1})\}^T, \\ \qquad\qquad \text{at } P = 1, 2, \ldots M_f - M_2 - 1. \end{array} \right. \end{cases} \tag{11}$$

The matrix representation of this hybrid TD-FD data using (4), and (10) would be

$$\begin{bmatrix} \begin{matrix} \phi_0(t_0, a) & \ldots & \phi_{N-1}(t_0, a) \\ \vdots & \ddots & \vdots \\ \phi_0(t_{M-1}, a) & \ldots & \phi_{N-1}(t_{M_1-1}, a) \end{matrix} \\ \text{Re}\begin{pmatrix} \Phi_0(f_0, a) & \ldots & \Phi_{N-1}(f_0, a) \\ \vdots & \ddots & \vdots \\ \Phi_0(f_{M_2-1}, a) & \ldots & \Phi_{N-1}(f_{M_2-1}, a) \end{pmatrix} \\ \text{Im}\begin{pmatrix} \Phi_0(f_0, a) & \ldots & \Phi_{N-1}(f_0, a) \\ \vdots & \ddots & \vdots \\ \Phi_0(f_{M_2-1}, a) & \ldots & \Phi_{N-1}(f_{M_2-1}, a) \end{pmatrix} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} x(t_0) \\ \vdots \\ x(t_{M_1-1}) \\ \text{Re}\begin{pmatrix} X(f_0) \\ \vdots \\ X(f_{M_2-1}) \end{pmatrix} \\ \text{Im}\begin{pmatrix} X(f_0) \\ \vdots \\ X(f_{M_2-1}) \end{pmatrix} \end{bmatrix} \tag{12}$$

where 'Re' and 'Im' in the matrix equation are the real and imaginary parts of the transfer function, respectively. The unknown coefficients c_n can be obtained by solving this matrix equation with the total least-square implementation of the Singular Value Decomposition (SVD) method [10].

To evaluate the performance of the interpolation and extrapolation, we compute the *estimated error* following the normalized mean square errors (MSEs) in the time and frequency domain as

$$E_{est} = \frac{\left\|\hat{x}' - x\right\|_2}{\left\|x\right\|_2} + \frac{\left\|\hat{X}' - X\right\|_2}{\left\|X\right\|_2} \tag{13}$$

where $\| \bullet \|_2$ is the L^2-norm of a vector. $\hat{x}'$ and $\hat{X}'$ are the estimated TD and FD data.

3. NUMERICAL SIMULATION: HORN ANTENNA

A horn antenna is used as one example to validate the above methodology. Typically, we use a Gaussian input pulse as the excitation for solving the TD problem as

$$g(t) = \frac{4}{\sigma\sqrt{\pi}} U_0 e^{-\left(\frac{4(t-t_0)}{\sigma}\right)^2} \tag{14}$$

where U_0 is the amplitude of the input pulse, σ is the width of the Gaussian pulse, and t_0 is the delay to make $g(t) \approx 0$ for $t < 0$. In FD, the Gaussian pulse is given by

$$G(f) = U_0 e^{-\left(\frac{(2\pi f\sigma)^2}{64} + j2\pi f t_0\right)} \tag{15}$$

In our computation, U_0 is chosen to be 1 V. We obtained the S-parameter (S_{11}) for the horn antenna which is fed by a probe, using the HOBBIES (High Order Basis Based Integral Equation Solver) simulation program as shown Fig. 1 [11]. The computed frequency range is from dc to 7.65 GHz (Δf = 30 MHz, 256 data points). We can obtain the FD data of the horn antenna with the Gaussian pulse excitation by multiplication of the horn antenna response from the HOBBIES and the spectrum of the Gaussian plane wave. The Gaussian excitation voltage has the parameters with $\sigma = 0.5208$ ns and $t_0 = 0.9766$ ns. Theoretically, the TD data generated by the MOT method should be the same as the TD data from IDFT of the FD data. Thus, one can obtain the TD data from $t = 0$ to $t = 6.478$ ns ($\Delta t = 0.03255$ ns, 200 data points) using IDFT of the FD data.

Assume that only the first 60 TD data points (0 to 1.921 ns) and the first 70 FD data points (0 to 2.07 GHz) are available to solve

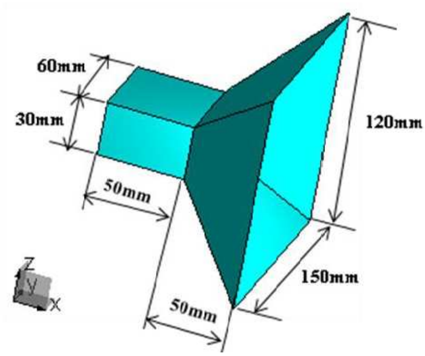


Figure 1. HOBBIES simulation model for the horn antenna.

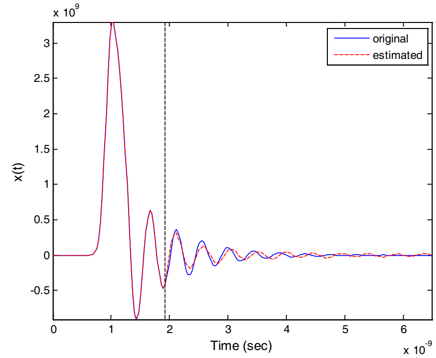


Figure 2. Estimated TD data using the continuous Laguerre function.

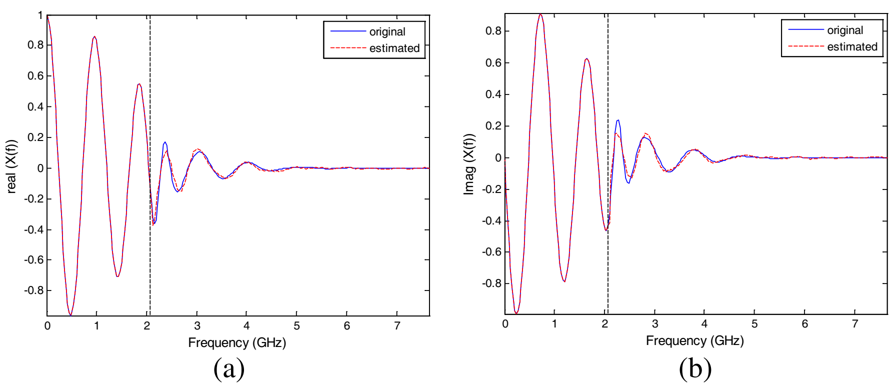


Figure 3. (a) Estimated Real and (b) Imaginary FD data using the continuous Laguerre function.

the matrix Equation (12). Figs. 2 and 3 plot the interpolation and extrapolation of the data using the continuous Laguerre function. We choose the degree of Laguerre expansion $N = 350$, the scaling factor $a = 0.6$. The estimated MSE is 1.08e-1. If one utilizes the middle 70 FD data (0.9 to 2.97 GHz) to apply the Hybrid method with the same scaling factor instead of the first 70 FD data, one can get a better estimated data as shown in Figs. 4 and 5. In this case, the estimated MSE is 6.05e-4.

We now evaluate the performance of the interpolation and extrapolation using the same TD data (the first 60 points) and different FD data (70 points, shifted to the right with 30 incremental points)

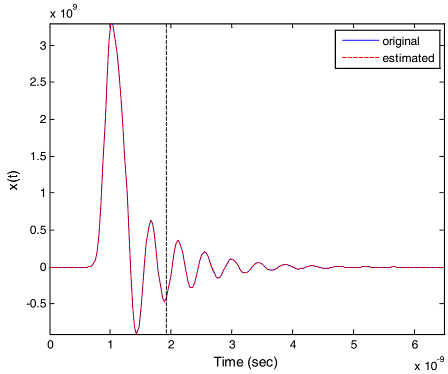
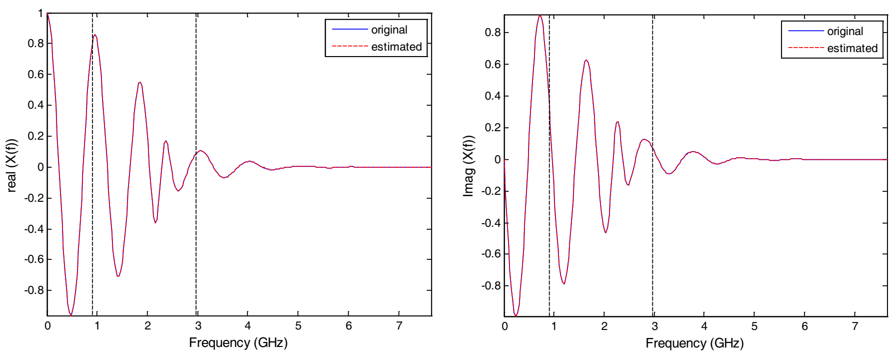


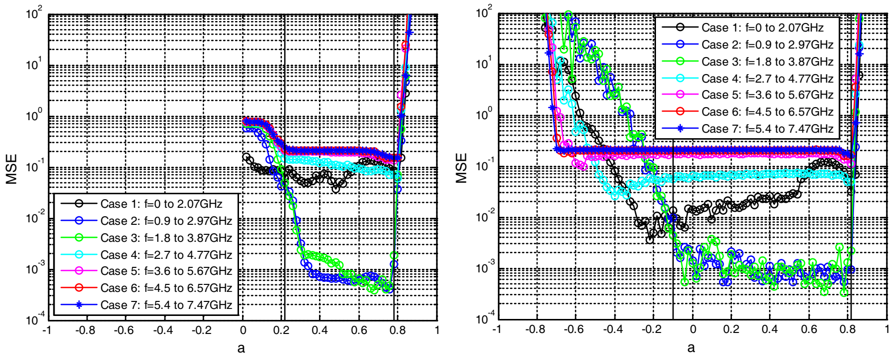
Figure 4. Estimated TD data using the continuous Laguerre function.



(a) (b)

Figure 5. (a) Estimated Real and (b) Imaginary FD data using the continuous Laguerre function.

versus the scaling factor a for both the continuous Laguerre functions and the discrete Laguerre functions. Fig. 6 shows the results of the MSE for seven different hybrid TD-FD dataset. From Fig. 6, one can recognize that the discrete Laguerre function has higher performance for the interpolation and extrapolation than the continuous Laguerre function of case 1. Also, one can get better performance if one utilizes the hybrid early TD data and the middle FD data (case 2 and 3) to apply both the continuous Laguerre ($0.22 \leq a \leq 0.78$) and discrete Laguerre ($-0.1 \leq a \leq 0.82$) techniques. However, if one use high FD data (case 4-7), the hybrid TD-FD method does not work too well, as the early time data and the high frequency data may contain redundant information and thus may not be mutually complementary.



(a) (b)

Figure 6. MSE for the same TD data and different FD data: (a) continuous Laguerre function and (b) discrete Laguerre function for the horn antenna.

4. CONCLUSION

In this paper, we have presented the continuous and discrete Laguerre functions to interpolate and extrapolate the wide-band response in both TD and FD simultaneously using seven different hybrid TD-FD dataset. Even though computing the middle-frequency response needs more time and memory than generating the low-frequency response, one can generate the wide-band response with better performance using both the continuous and discrete Laguerre functions with the hybrid early-time data and middle-frequency data.

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