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REFLECTION AND TRANSMISSION AT DIELECTRIC-FRACTAL INTERFACE

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Abstract—The transmission and reflection of electromagnetic waves at dielectric-fractal interface is studied, the fractal exhibits quasi fractional space properties. The closed form expressions for transmission and reflection coefficients are formulated for such an interface. The classical results are obtained when integer dimensions, instead of fractional dimension are inserted in the said expressions. This work can be used to study behavior of electromagnetic waves in slabs and waveguides filled with fractal media.

1. INTRODUCTION

Shapes and forms are found in nature that cannot be described by Euclidean geometry, for instance the way trees branch, the roughness of ocean floor, the geometry of clouds and the dust in planetary disks etc.. These are cases of high spatial complexity. This kind of complexity is evident at microscopic level as well as in biological tissues for instance; branching of bronchial tubes in lungs, cells, proteins and plasma. It is difficult to describe such structures because we cannot attribute a characteristic length to them. Mandelbrot [1] first introduced the term, "Fractal" to differentiate pure geometries from other types which do not fit into a simple classification. Fractal is used to describe the degree of irregularity or fragmentation of a sample or structure which is identical at all scales. The term, "self similar" is also used in the same context. Fractals are characterized by a fractional dimension, D. Hence complex structures can be modeled at microscopic and macroscopic levels [2, 3]. The use of fractional dimension analysis is

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well established in areas of physics [4–7]. In the last few years the fundamentals for fractional dimension analysis have been studied in depth and various new breakthroughs have resulted [9, 10] beginning from Stillinger's [8], "Axiomatic basis for spaces with noninteger dimension".

It is important to apply the generalization of electromagnetic theory in fractional space in order to extract full benefits of fractal models, which are becoming popular due to small number of parameters that define a medium of greater complexity and a rich structure. Theoretical and experimental investigations of the problem of interaction of electromagnetic waves with different inhomogeneous media possessing fractal properties have been carried out [11–14] and useful results regarding scattering of electromagnetic waves and fractional calculus have been presented. It was proposed that smoothing of microscopic characteristics over the physically infinitesimal volume transforms the initial fractal distribution into fractional continuous model that uses fractional integrals [15]. Solutions to Poisson's and Laplace equation for scalar potential in fractional space have been discussed in [16, 17]. Recently Zubair, et al. [18–23] have worked on electromagnetic wave propagation in fractional space, and have given solutions to plane, cylindrical and spherical waves in D-dimensional fractional space. Antenna radiation in fractional space was also investigated and presented by Mughal, et al. [24]. Similarly Faraday's and Ampere's laws were derived for fractional space and Maxwell's electromagnetic stress tensor was reformulated, by Martin, et al. [25]. The electromagnetic radiation from fractal structures has also been an area of interest in the last few years [26–29]. The study of transmission and reflection at an interface of fractional space medium, will facilitate us to analyze fractional waveguides and wave behavior in fractional space slabs.

The quasi fractional space is used in this analysis, because such a boundary was realized by Attiya [30], in which the magnetic field properties are equivalent to complementary fractional space wave equation.

In this paper the expressions for transmission and reflection coefficients are derived for parallel and perpendicular polarizations at an integer-fractal interface. The general solution for plane waves in fractional space is described, which is then used to formulate the expressions for electric and magnetic fields. The resulting expressions reduce to classical results when the dimension D is an integer. In Section 2, the geometry of the problem is given, field equations for parallel polarization are given, transmission and reflection coefficients are derived. This is also done for perpendicular polarization. In

Section 3, it is shown that classical results can be recovered from fractional space, when integer dimensions are inserted. And finally results are discussed.

2. REFLECTION AND TRANSMISSION AT DIELECTRIC-FRACTAL INTERFACE

In this section, behavior of TEM fields in lossless media, at a planar Dielectric-Fractal interface is investigated at oblique incidence. The boundary is assumed to be infinite. Furthermore, the permeability of the two media is the same, i.e., $\mu_1 = \mu_2 = \mu_0$. And fractionality exists in z direction only. Figure 1 shows the geometry of incident, reflected and transmitted waves at the dielectric-fractal interface.

2.1. Parallel Polarization

For this polarization, the electric field is parallel to the plane of incidence. The field equations for incident and reflected waves can be written as follows:

$$\mathbf{E}_i = \left(\hat{a_x}\cos\theta_i - \hat{a_z}\sin\theta_i\right) E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{1}$$

$$\mathbf{H}_i = \hat{a_y}\frac{E_0}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{2}$$

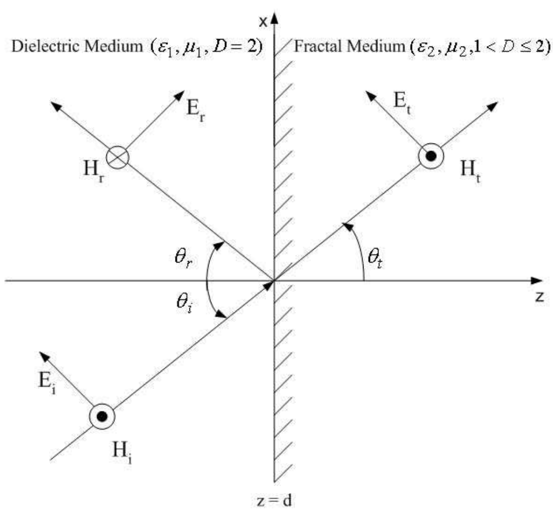

Figure 1. Geometry of the incident, reflected and transmitted waves at the interface.

where $E_i = E_0$ and $H_i = E_0/\eta_1$. The wave impedance $\eta_1 = \sqrt{\mu_1/\epsilon_1}$, and the wave number $\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$.

$$\mathbf{E}_r = (\hat{a_x}\cos\theta_r + \hat{a_z}\sin\theta_r)\,\Gamma E_0 e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{3}$$

$$\mathbf{H}_r = -\hat{a_y}\frac{\Gamma E_0}{\eta_1}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{4}$$

where $E_r = \Gamma E_0$ and $H_r = \Gamma E_0/\eta_1$. The wave impedance $\eta_1 = \sqrt{\mu_1/\epsilon_1}$, and the wave number $\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$.

The transmitted fields can be described in a similar manner, where $\hat{a_x}$, $\hat{a_y}$ and $\hat{a_z}$ are the vectors, the exponential function is used to describe wave propagation in x direction and Hankel function of second kind of order n is used to represent wave propagation in z direction as is done by Zubair, et al. [20]. Since fractionality is assumed to exist in z axis only, Hankel function of second kind of order n is being used. Thereby the wave equation for transmitted electric field can be written as follows:

$$\mathbf{E}_t = (\hat{a_x}\cos\theta_t - \hat{a_z}\sin\theta_t)TE_0 e^{-j\beta_2 x\sin\theta_t}(\beta_2 z\cos\theta_t)^n\Big[H_n^{(2)}(\beta_2 z\cos\theta_t)\Big] \tag{5}$$

where $E_t = TE_0$. The wave impedance $\eta_2 = \sqrt{\mu_2/\epsilon_2}$, and the wave number $\beta_2 = \omega\sqrt{\mu_2\epsilon_2}$. $H_n^{(2)}(\beta_2 z\cos\theta_t)$ is Hankel function of second kind of order n. It is used to represent positive traveling waves. Also $n = |3 - D|/2$, and D is the dimension.

The wave equation for magnetic field used to describe the transmitted magnetic field is:

$$\mathbf{H}_t = \hat{a_y}\frac{TE_0}{\eta_2}e^{-j\beta_2 x\sin\theta_t}(\beta_2 z\cos\theta_t)^{n_h}[H_{n_h}^{(2)}(\beta_2 z\cos\theta_t)] \tag{6}$$

where $H_t = TE_0/\eta_2$. The wave impedance $\eta_2 = \sqrt{\mu_2/\epsilon_2}$, and the wave number $\beta_2 = \omega\sqrt{\mu_2\epsilon_2}$. $H_n^{(2)}(\beta_2 z\cos\theta_t)$ is Hankel function of second kind of order n. It is used to represent positive traveling waves. Also $n_h = |D - 1|/2$, and D is the dimension.

The transmission and reflection coefficient are the unknowns. These can be expressed in terms of incident angle θ_i and other parameters of the given media by applying boundary conditions on tangential components of electric and magnetic fields which are continuous at boundary $z = d$. Since the tangential component of electric and magnetic fields must be continuous at the boundary $z = d$, the following must hold true for both electric and magnetic fields:

$$\mathbf{E}_{ix}(z = d) + \mathbf{E}_{rx}(z = d) = \mathbf{E}_{tx}(z = d) \tag{7}$$

$$\mathbf{H}_{ix}(z = d) + \mathbf{H}_{rx}(z = d) = \mathbf{H}_{tx}(z = d) \tag{8}$$

Inserting field expressions in (7) and (8) results in:

$$\begin{aligned}
&\cos\theta_i e^{-j\beta_1(x\sin\theta_i + d\cos\theta_i)} + \Gamma\cos\theta_r e^{-j\beta_1(x\sin\theta_r - d\cos\theta_r)} \\
=\;& T\cos\theta_t e^{-j\beta_2 x\sin\theta_t}(\beta_2 d\cos\theta_t)^n [H_n^{(2)}(\beta_2 d\cos\theta_t)] \qquad (9)\\
&\frac{1}{\eta_1}\left[e^{-j\beta_1(x\sin\theta_i + d\cos\theta_i)} - \Gamma e^{-j\beta_1(x\sin\theta_r - d\cos\theta_r)}\right] \\
=\;& \frac{T}{\eta_2} e^{-j\beta_2 x\sin\theta_t}(\beta_2 d\cos\theta_t)^{n_h}\left[H_{n_h}^{(2)}(\beta_2 d\cos\theta_t)\right] \qquad (10)
\end{aligned}$$

It must be noted that (9) and (10) are functions of x and z. For continuity condition to hold at $z = d$, for all x, the variation of x must be same on both sides of interface. Hence,

$$\beta_1\sin\theta_i = \beta_1\sin\theta_r = \beta_2\sin\theta_t \qquad (11)$$

As a consequence the following two relations are obtained:

$$\theta_i = \theta_r \qquad (12)$$

$$\beta_1\sin\theta_i = \beta_2\sin\theta_t \qquad (13)$$

Inserting (12) and (13) into electric and magnetic field equations, and then solving them simultaneously, expressions for transmission coefficient, T and reflection coefficient, Γ are obtained:

$$T_{\parallel} = \frac{2\eta_2\cos\theta_i e^{-j\beta_1 d\cos\theta_i}}{\eta_1\cos\theta_i A + \eta_2\cos\theta_t B} \qquad (14)$$

$$\Gamma_{\parallel} = \frac{e^{-j\beta_1 d\cos\theta_i}(\eta_2\cos\theta_t B - \eta_1\cos\theta_i A)}{e^{j\beta_1 d\cos\theta_i}(\eta_2\cos\theta_t B + \eta_1\cos\theta_i A)} \qquad (15)$$

where,

$$A = (\beta_2 d\cos\theta_t)^{n_h}\left[H_{n_h}^{(2)}(\beta_2 d\cos\theta_t)\right] \qquad (16)$$

$$B = (\beta_2 d\cos\theta_t)^n H_n^{(2)}(\beta_2 d\cos\theta_t) \qquad (17)$$

For parallel polarization case, Brewster's angle can also be formulated. Setting (15) equal to zero, the following expression is obtained:

$$\cos\theta_i = \sqrt{\frac{\mu_2\epsilon_1}{\mu_1\epsilon_2}}\frac{B\cos\theta_t}{A} \qquad (18)$$

Using (13), (18) can be written as:

$$\sin\theta_i = \sqrt{\frac{\epsilon_2 A/\epsilon_1 - \mu_2 B/\mu_1}{\epsilon_2 A/\epsilon_1 - \epsilon_1 B/\epsilon_2}} \qquad (19)$$

Since sine function cannot exceed unity, (19) exists only if:

$$\frac{\epsilon_2 A}{\epsilon_1} - \frac{\mu_2 B}{\mu_1} \leq \frac{\epsilon_2 A}{\epsilon_1} - \frac{\epsilon_1 B}{\epsilon_2} \tag{20}$$

since $\mu_1 = \mu_2$, (19) reduces to:

$$\sin\theta_i = \sqrt{\frac{\epsilon_2 A/\epsilon_1 - B}{\epsilon_2 A/\epsilon_1 - \epsilon_1 B/\epsilon_2}} \tag{21}$$

Hence,

$$\theta_i = \sin^{-1}\sqrt{\frac{\epsilon_2 A/\epsilon_1 - B}{\epsilon_2 A/\epsilon_1 - \epsilon_1 B/\epsilon_2}} \tag{22}$$

2.2. Perpendicular Polarization

For this polarization, field expressions for Electric and Magnetic field intensities are as follows:

$$\mathbf{E}_i = \hat{a_y} E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{23}$$

$$\mathbf{H}_i = (-\hat{a_x}\cos\theta_i + \hat{a_z}\sin\theta_i)\,\frac{E_0}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{24}$$

where $E_i = E_0$ and $H_i = E_0/\eta_1$. The wave impedance $\eta_1 = \sqrt{\mu_1/\epsilon_1}$, and the wave number $\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$.

$$\mathbf{E}_r = \hat{a_y}\Gamma E_0 e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \tag{25}$$

$$\mathbf{H}_r = (\hat{a_x}\cos\theta_r + \hat{a_z}\sin\theta_r)\,\frac{\Gamma E_0}{\eta_1}\, e^{-j\beta_1(x\sin\theta_r + z\cos\theta_r)} \tag{26}$$

where $E_r = \Gamma E_0$ and $H_r = \Gamma E_0/\eta_1$. The wave impedance $\eta_1 = \sqrt{\mu_1/\epsilon_1}$, and the wave number $\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$.

The transmitted fields can be described in a similar manner, where $\hat{a_x}$, $\hat{a_z}$ and $\hat{a_y}$ are the vectors, the exponential function is used to describe wave propagation in x direction and Hankel function of second kind of order n is used to represent wave propagation in z direction. Since fractionality is assumed to exist in z axis only, Hankel function of second kind of order n is being used. Accordingly the wave equation for transmitted electric and magnetic fields is:

$$\mathbf{E}_t = \hat{a_y} T E_0 e^{-j\beta_2 x\sin\theta_t} B \tag{27}$$

$$\mathbf{H}_t = (-\hat{a_x}\cos\theta_t + \hat{a_z}\sin\theta_t)\,\frac{TE_0}{\eta_2} e^{-j\beta_2 x\sin\theta_t} A \tag{28}$$

where $E_t = TE_0$, $H_t = TE_0/\eta_2$. The wave impedance $\eta_2 = \sqrt{\mu_2/\epsilon_2}$, and the wave number $\beta_2 = \omega\sqrt{\mu_2\epsilon_2}$. $H_n^{(2)}(\beta_2 z\cos\theta_t)$ is Hankel function of second kind of order n. It is used to represent positive traveling waves. Also $n = |3-D|/2$ and $n_h = |D-1|/2$, and D is the dimension.

By following a procedure similar to that for parallel polarization, expressions for transmission and reflection coefficients for perpendicular polarization are found to be, as follows:

$$T_{\perp} = \frac{2\eta_2 \cos\theta_i e^{-j\beta_1 d\cos\theta_i}}{\eta_1 \cos\theta_t B + \eta_2 \cos\theta_i A} \tag{29}$$

$$\Gamma_{\perp} = \frac{e^{-j\beta_1 d\cos\theta_i}(\eta_2 \cos\theta_i A - \eta_1 \cos\theta_t B)}{e^{j\beta_1 d\cos\theta_i}(\eta_2 \cos\theta_i A + \eta_1 \cos\theta_t B} \tag{30}$$

where,

$$A = (\beta_2 d\cos\theta_t)^{n_h}\left[H_{n_h}^{(2)}(\beta_2 d\cos\theta_t)\right] \tag{31}$$

$$B = (\beta_2 d\cos\theta_t)^{n} H_{n}^{(2)}(\beta_2 d\cos\theta_t) \tag{32}$$

3. RESULTS AND DISCUSSION

The results obtained in the previous section are for fractional space. Insertion of integer dimensions in expressions for transmission and reflection coefficients of both parallel and perpendicular polarizations gives back the classical results. Hence for $D = 2$, n becomes equal to 1/2 and n_h becomes equal to 1/2 as well. Hankel function of second

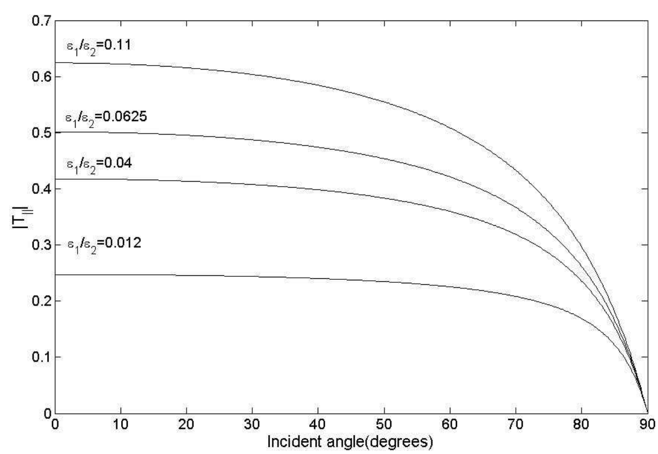

Figure 2. Transmission coefficient for parallel polarization for $D = 2$ versus varying incident angles.

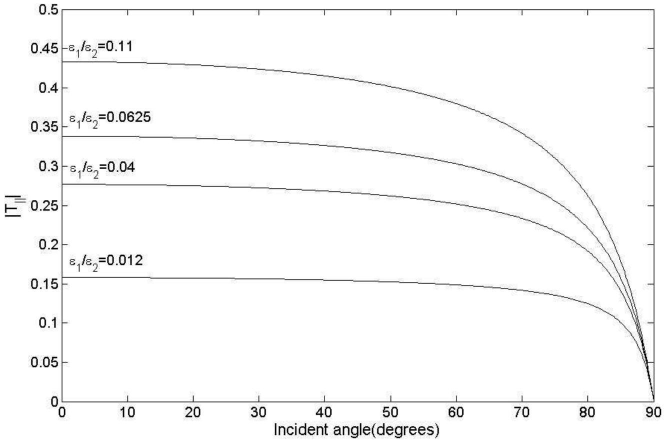

Figure 3. Transmission coefficient for parallel polarization for $D = 1.5$ versus varying incident angles.

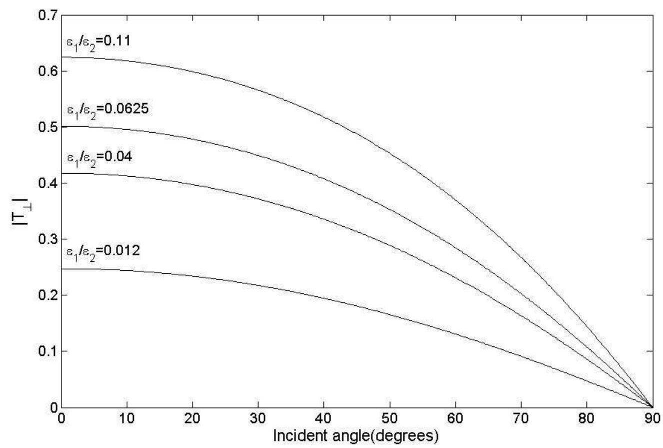

Figure 4. Transmission coefficient for perpendicular polarization for $D = 2$ versus varying incident angles.

kind is now of order 1/2, which can be expressed in exponential form as follows [31]:

$$H_n^{(2)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{-jz} \tag{33}$$

Using this definition of Hankel function, in (14), (15) and (22), the following is obtained:

$$T_{\parallel} = \frac{\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \tag{34}$$

$$\Gamma_{\parallel} = \frac{(\eta_2 \cos\theta_t - \eta_1 \cos\theta_i)}{(\eta_2 \cos\theta_t + \eta_1 \cos\theta_i)} \tag{35}$$

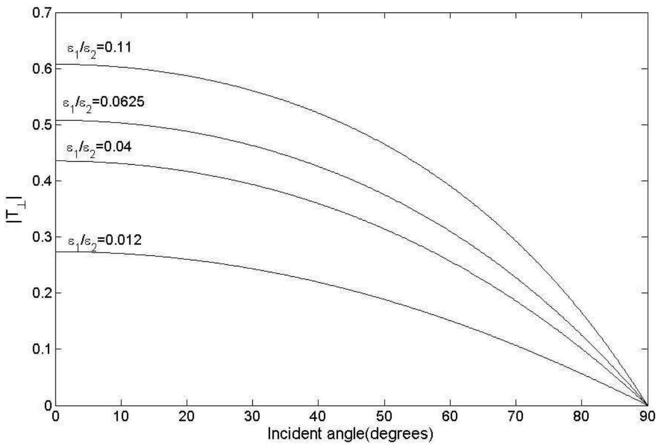

Figure 5. Transmission coefficient for perpendicular polarization for $D = 1.5$ versus varying incident angles.

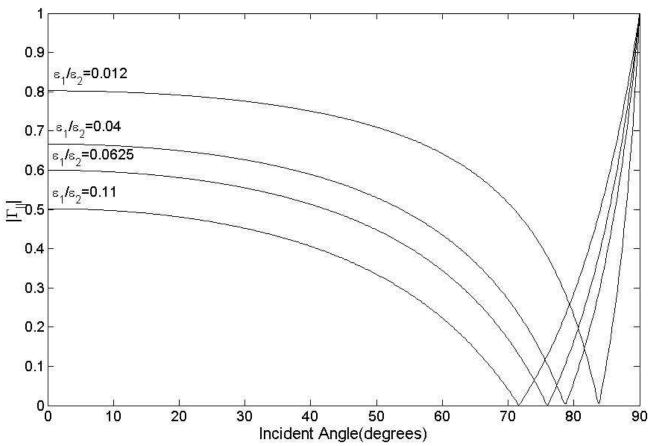

Figure 6. Reflection coefficient for parallel polarization for $D = 2$ versus varying incident angles.

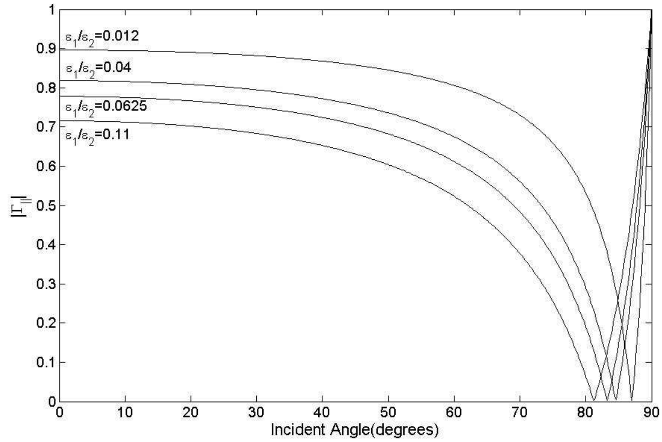

Figure 7. Reflection coefficient for parallel polarization for $D = 1.5$ versus varying incident angles.

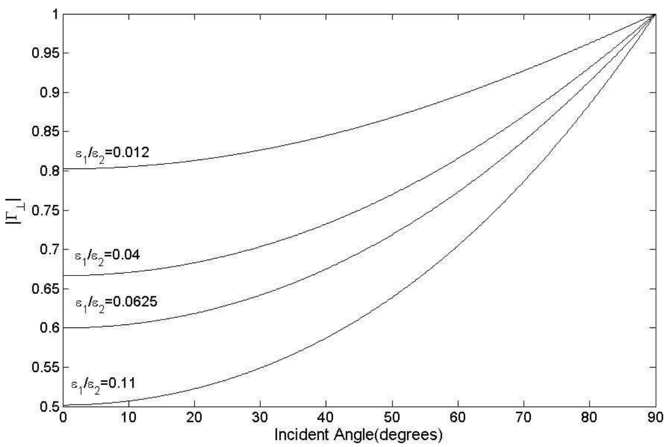

Figure 8. Reflection coefficient for perpendicular polarization for $D = 2$ versus varying incident angles.

and

$$\theta_i = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_2 + \epsilon_1}} \tag{36}$$

which are the same expressions as those obtained by Balanis [31] for transmission, reflection coefficients and Brewster's angle for parallel

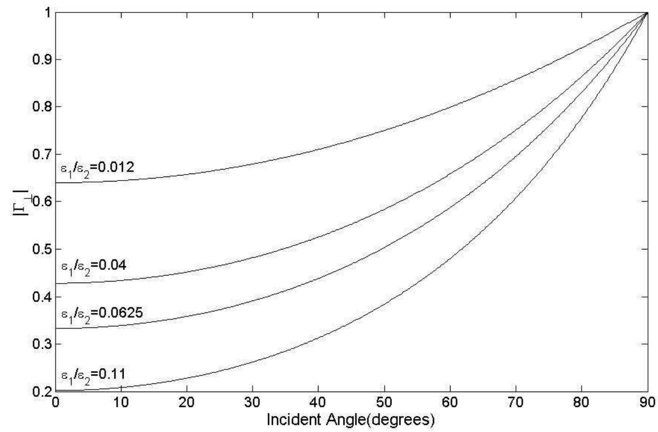

Figure 9. Reflection coefficient for parallel polarization for $D = 1.5$ versus varying incident angles.

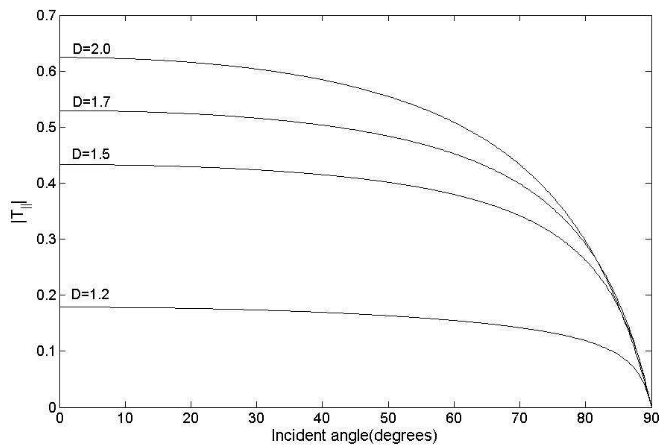

Figure 10. Transmission coefficient for parallel polarization for varying dimension, D for $\epsilon_1/\epsilon_2 = 0.11$.

polarization. Similarly for perpendicular polarization, on inserting dimension, $D = 2$, classical results are recovered which are same as those mentioned in Balanis [31].

Transmission coefficient was plotted against varying incident angles for materials with different permittivities. Plots of transmission coefficient with $D = 2$ and $D = 1.5$ were obtained for comparison in integer and fractional space. This was investigated for both parallel and perpendicular polarization, as shown in Figure 2 through Figure 5. The same was done for reflection coefficient as presented in Figure 6 through Figure 9. For a fixed ratio of permittivities of the two media, transmission coefficient was plotted against varying angles of incidence for different fractional dimensions. It was observed that as

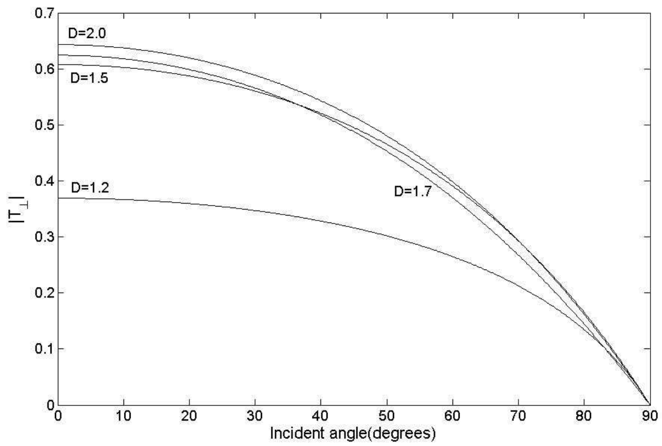

Figure 11. Transmission coefficient for perpendicular polarization for varying dimension, D for $\epsilon_1/\epsilon_2 = 0.11$.

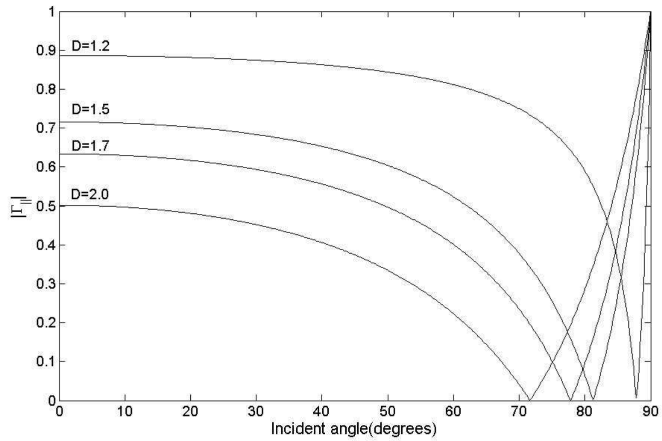

Figure 12. Reflection coefficient for parallel polarization for varying dimension, D for $\epsilon_1/\epsilon_2 = 0.11$.

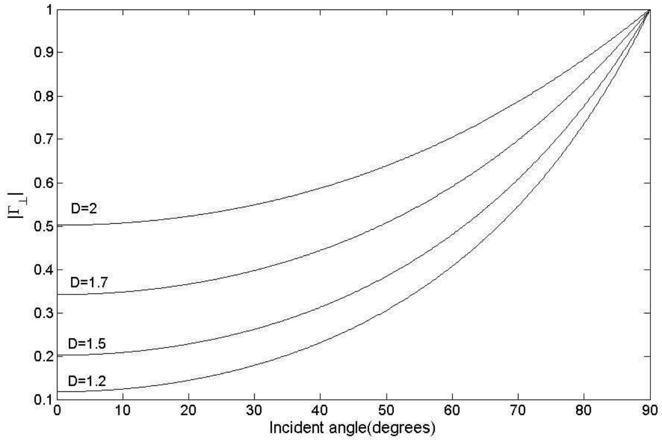

Figure 13. Reflection coefficient for perpendicular polarization for varying dimension, D for $\epsilon_1/\epsilon_2 = 0.11$.

$1/D$ decreases, transmission coefficient increases. This is true for both parallel and perpendicular polarization, as it is evident in Figure 10 and Figure 11. Similarly Figure 12 and Figure 13 show the reflection coefficient versus incident angle for different fractal dimensions.

It can be observed that Brewster's angle for parallel polarization changes considerably with changing dimensions. In Figure 6, the Brewster's angle is concurrent with the familiar Brewster's law and the results obtained by Balanis [31]. However in Figure 7, with D=1.5 the angle increases with decreasing ratios of permittivities, and Figure 12 shows that as the dimension decreases, brewster's angle increases.

4. CONCLUSION

Reflection and transmission coefficients were formulated in this paper for dielectric-quasi fractal interface. It was assumed, fractionality exists only along z-axis of the fractional space and that the permeability of the fractional medium is approximately same as that of integer space. It was found that as the dimension D increases the transmission coefficient for both parallel and perpendicular polarizations also increases. This work will provide a foundation for investigating the behavior of electromagnetic fields and waves inside slabs, waveguides, and other multiple interfaces filled with fractal media.

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