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STABILITY AND DISPERSION ANALYSIS FOR THREE-DIMENSIONAL (3-D) LEAPFROG ADI-FDTD METHOD

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Abstract—Stability and dispersion analysis for the three-dimensional (3-D) leapfrog alternate direction implicit finite difference time domain (ADI-FDTD) method is presented in this paper. The leapfrog ADI-FDTD method is reformulated in the form similar to conventional explicit FDTD method by introducing two auxiliary variables. The auxiliary variables serve as perturbations of the main fields variables. The stability of the leapfrog ADI-FDTD method is analyzed using the Fourier method and the eigenvalues of the Fourier amplification matrix are obtained analytically to prove the unconditional stability of the leapfrog ADI-FDTD method. The dispersion relation of the leapfrog ADI-FDTD method is also presented.

1. INTRODUCTION

Unconditionally stable implicit finite difference time domain (FDTD) methods such as alternate direction implicit (ADI) FDTD [1–5], split-step FDTD [6, 7] and locally one-dimensional (LOD) [8, 9] FDTD methods are attractive over the conventional explicit FDTD [10] method as they are not constrained by the Courant-Friedrich-Levy (CFL) stability criterion [11]. The spatial mesh size can be reduced independently of the time step size. Thus, implicit FDTD methods are advantageous over explicit FDTD methods as they can model fine structures without increasing the simulation time.

To improve the efficiency of implicit FDTD methods in modeling fine structures, conformal ADI-FDTD methods have been proposed [12]. It overcomes the problem of staircase meshing by using distorted cells at the boundaries and has better stability than other conformal FDTD methods.

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Alternatively, hybrid ADI-FDTD subgridding method can also be employed. In this method, FDTD method is used to model the coarse grid while ADI-FDTD method is used to model the fine grid [13]. ADI-FDTD method is unconditionally stable and can employ the larger time step size of the FDTD method. The time step size for the two methods is the same and there is no need for time interpolation when compared to FDTD subgridding methods [14]. However, ADI-FDTD is a split-step method and FDTD is a leapfrog method. In the hybrid ADI-FDTD subgridding method, the FDTD method is updated twice for every update in the ADI-FDTD method.

Recently, the leapfrog ADI-FDTD method was introduced to better synchronize the FDTD and ADI-FDTD methods [15]. The leapfrog ADI-FDTD method employs the same Yee cells and leapfrog scheme as the conventional explicit FDTD method. Numerical examples highlighting the stability of the method have been presented in [15–17]. Since it is an implicit method, it requires treatment of the boundary condition [18]. In [16], perfectly matched layer (PML) was implemented into the leapfrog ADI-FDTD method to model open structures. In [17], convolutional perfectly matched layer (CPML) was implemented into the leapfrog ADI-FDTD method to simulate a bow-tie antenna. These examples demonstrated the stability of the leapfrog ADI-FDTD method only in numerical way with some boundary conditions. Till date, there is no rigorous analysis for the stability of the leapfrog ADI-FDTD method. Furthermore, the dispersion relation of the leapfrog ADI-FDTD method is also not clear.

In this paper, we present the rigorous analysis of the stability and dispersion for the three-dimensional (3-D) leapfrog ADI-FDTD method. The organization of the paper is as follows. Section 2 reviews the leapfrog ADI-FDTD method and presents all the pertaining implicit update equations of the method. In Section 3, we will reformulate the leapfrog ADI-FDTD method in the form similar to the conventional explicit FDTD method by introducing two auxiliary variables. The auxiliary variables serve as perturbations of the main fields variables. In Section 4, we analyze the stability of the leapfrog ADI-FDTD method using the Fourier method. The eigenvalues of the Fourier amplification matrix are obtained analytically and it is proven that the leapfrog ADI-FDTD method is unconditionally stable. In Section 5, the dispersion relation of the leapfrog ADI-FDTD method will be presented.

2. REVIEW OF 3-D LEAPFROG ADI-FDTD METHOD

In this section, we review the 3-D leapfrog ADI-FDTD method. The leapfrog ADI-FDTD method uses the Yee cells and it has been derived from the conventional ADI-FDTD method [15]. The conventional ADI-FDTD method is a split-step approach with two procedures. By taking the previous time step of the second procedure in the conventional ADI-FDTD method and substituting the equations (cf. [15], Equations (10)–(11)) into the tridiagonal implicit equations of the first procedure, one can obtain (1) and (2) in the following.

Tridiagonal implicit equations of $E^{n+\frac{1}{2}}$ for the leapfrog ADI-FDTD method:

$$\left(I-\frac{\Delta t^2}{4}\epsilon_x^{-1}\partial y\mu_z^{-1}\partial y\right)E_x^{n+\frac{1}{2}} = \left(I-\frac{\Delta t^2}{4}\epsilon_x^{-1}\partial y\mu_z^{-1}\partial y\right)E_x^n + \Delta t\left(\epsilon_x^{-1}\partial yH_z^n - \epsilon_x^{-1}\partial zH_y^n\right) \tag{1a}$$

$$\left(I-\frac{\Delta t^2}{4}\epsilon_y^{-1}\partial z\mu_x^{-1}\partial z\right)E_y^{n+\frac{1}{2}} = \left(I-\frac{\Delta t^2}{4}\epsilon_y^{-1}\partial z\mu_x^{-1}\partial z\right)E_y^n + \Delta t\left(\epsilon_y^{-1}\partial zH_x^n - \epsilon_y^{-1}\partial xH_z^n\right) \tag{1b}$$

$$\left(I-\frac{\Delta t^2}{4}\epsilon_z^{-1}\partial x\mu_y^{-1}\partial x\right)E_z^{n+\frac{1}{2}} = \left(I-\frac{\Delta t^2}{4}\epsilon_z^{-1}\partial x\mu_y^{-1}\partial x\right)E_z^n + \Delta t\left(\epsilon_z^{-1}\partial xH_y^n - \epsilon_z^{-1}\partial yH_x^n\right) \tag{1c}$$

where ϵ_α is the permittivity of the medium and α is the x, y, z directions respectively.

Tridiagonal implicit equations of H^{n+1} for the leapfrog ADI-FDTD method:

$$\left(I-\frac{\Delta t^2}{4}\mu_x^{-1}\partial y\epsilon_z^{-1}\partial y\right)H_x^{n+1} = \left(I-\frac{\Delta t^2}{4}\mu_x^{-1}\partial y\epsilon_z^{-1}\partial y\right)H_x^n + \Delta t\left(\mu_x^{-1}\partial zE_y^{n+\frac{1}{2}} - \mu_x^{-1}\partial yE_z^{n+\frac{1}{2}}\right) \tag{2a}$$

$$\left(I-\frac{\Delta t^2}{4}\mu_y^{-1}\partial z\epsilon_x^{-1}\partial z\right)H_y^{n+1} = \left(I-\frac{\Delta t^2}{4}\mu_y^{-1}\partial z\epsilon_x^{-1}\partial z\right)H_y^n + \Delta t\left(\mu_y^{-1}\partial xE_z^{n+\frac{1}{2}} - \mu_y^{-1}\partial zE_x^{n+\frac{1}{2}}\right) \tag{2b}$$

$$\left(I-\frac{\Delta t^2}{4}\mu_z^{-1}\partial x\epsilon_y^{-1}\partial x\right)H_z^{n+1} = \left(I-\frac{\Delta t^2}{4}\mu_z^{-1}\partial x\epsilon_y^{-1}\partial x\right)H_z^n + \Delta t\left(\mu_z^{-1}\partial yE_x^{n+\frac{1}{2}} - \mu_z^{-1}\partial xE_y^{n+\frac{1}{2}}\right) \tag{2c}$$

where μ_α is the permeability of the medium. Note that the leapfrog ADI-FDTD method solves (1)–(2) implicitly and no explicit update is needed as in the conventional ADI-FDTD method. In addition, the leapfrog ADI-FDTD method updates E and H fields implicitly in one direction only. In particular, referring to (1a) and (2a), we can observed that E_x and H_x are updated only in the y direction. Although the leapfrog ADI-FDTD method is derived from the conventional ADI-FDTD method, the leapfrog ADI-FDTD method does not retain the second procedure which alternates the direction of update. The tridiagonal implicit equations (1) and (2) can be solved by using the Thomas algorithm which is a special type of Gaussian elimination method.

3. REFORMULATION OF LEAPFROG ADI-FDTD METHOD

In this section, we reformulate the leapfrog ADI-FDTD method in the form similar to the conventional explicit FDTD method by introducing two auxiliary variables e and h. Let us consider the 3-D wave propagation in a medium with permittivity ϵ and permeability μ. $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic component vectors

$$\mathbf{E} = [E_x \quad E_y \quad E_z]^T, \quad \mathbf{H} = [H_x \quad H_y \quad H_z]^T \tag{3}$$

The splitting matrix operators of Maxwell's equations may be selected as (cf. [3], Equations (33)–(34))

$$\mathbf{A}_{12} = \begin{bmatrix} 0 & 0 & \epsilon^{-1}\partial y \\ \epsilon^{-1}\partial z & 0 & 0 \\ 0 & \epsilon^{-1}\partial x & 0 \end{bmatrix} \tag{4a}$$

$$\mathbf{A}_{21} = \begin{bmatrix} 0 & \mu^{-1}\partial z & 0 \\ 0 & 0 & \mu^{-1}\partial x \\ \mu^{-1}\partial y & 0 & 0 \end{bmatrix} \tag{4b}$$

$$\mathbf{B}_{12} = \begin{bmatrix} 0 & -\epsilon^{-1}\partial z & 0 \\ 0 & 0 & -\epsilon^{-1}\partial x \\ -\epsilon^{-1}\partial y & 0 & 0 \end{bmatrix} \tag{4c}$$

$$\mathbf{B}_{21} = \begin{bmatrix} 0 & 0 & -\mu^{-1}\partial y \\ -\mu^{-1}\partial z & 0 & 0 \\ 0 & -\mu^{-1}\partial x & 0 \end{bmatrix} \tag{4d}$$

where ∂_x, ∂_y, ∂_z are the spatial difference operators for the first derivatives along x, y, z directions respectively.

Using the matrix operators above, the tridiagonal implicit E update equations in (1) can be expressed as

$$\left(\mathbf{I}-\frac{\Delta t^2}{4}\mathbf{A}_{12}\ \mathbf{A}_{21}\right)\ \mathbf{E}^{n+\frac{1}{2}}\ =\ \left(\mathbf{I}-\frac{\Delta t^2}{4}\mathbf{A}_{12}\mathbf{A}_{21}\right)\mathbf{E}^{n-\frac{1}{2}} + \Delta t\left(\mathbf{A}_{12}+\mathbf{B}_{12}\right)\ \mathbf{H}^{n} \tag{5}$$

where $\mathbf{I}$ is a 3×3 identity matrix.

We can rewrite (5) as

$$\mathbf{E}^{n+\frac{1}{2}}=\mathbf{E}^{n-\frac{1}{2}}+\frac{\Delta t^2}{4}\mathbf{A}_{12}\mathbf{A}_{21}\left(\mathbf{E}^{n+\frac{1}{2}}-\mathbf{E}^{n-\frac{1}{2}}\right)+\Delta t\mathbf{A}_{12}\mathbf{H}^{n}+\Delta t\mathbf{B}_{12}\mathbf{H}^{n} \tag{6a}$$

$$=\mathbf{E}^{n-\frac{1}{2}}+\Delta t\mathbf{A}_{12}\left[\frac{\Delta t}{4}\mathbf{A}_{21}\left(\mathbf{E}^{n+\frac{1}{2}}-\mathbf{E}^{n-\frac{1}{2}}\right)+\mathbf{H}^{n}\right]+\Delta t\mathbf{B}_{12}\mathbf{H}^{n} \tag{6b}$$

$$=\mathbf{E}^{n-\frac{1}{2}}+\Delta t\left(\mathbf{A}_{12}\mathbf{h}^{n}+\mathbf{B}_{12}\mathbf{H}^{n}\right) \tag{6c}$$

where we introduce the auxiliary variable

$$\mathbf{h}^{n}=\mathbf{H}^{n}+\frac{\Delta t}{4}\mathbf{A}_{21}\left(\mathbf{E}^{n+\frac{1}{2}}-\mathbf{E}^{n-\frac{1}{2}}\right) \tag{7}$$

Equation (6c) becomes the conventional explicit FDTD E update equation when $\mathbf{h}^n=\mathbf{H}^n$. It can be observed that the auxiliary variable $\mathbf{h}$ in (7) is a perturbation of $\mathbf{H}$. Next, we cast (6c) and (7) in matrix form compactly as

$$\begin{bmatrix}\mathbf{I} & -\Delta t\mathbf{A}_{12}\\ -\frac{\Delta t}{4}\mathbf{A}_{21} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{E}^{n+\frac{1}{2}}\\ \mathbf{h}^{n}\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \Delta t\mathbf{B}_{12}\\ -\frac{\Delta t}{4}\mathbf{A}_{21} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{E}^{n-\frac{1}{2}}\\ \mathbf{H}^{n}\end{bmatrix} \tag{8}$$

Similarly, the tridiagonal implicit H update equations in (2) can be expressed as

$$\left(\mathbf{I}-\frac{\Delta t^2}{4}\mathbf{B}_{21}\mathbf{B}_{12}\right)\mathbf{H}^{n+1}=\left(\mathbf{I}-\frac{\Delta t^2}{4}\mathbf{B}_{21}\mathbf{B}_{12}\right)\mathbf{H}^{n}+\Delta t\left(\mathbf{A}_{21}+\mathbf{B}_{21}\right)\mathbf{E}^{n+\frac{1}{2}} \tag{9}$$

Following the procedure above, we can rewrite (9) as

$$\mathbf{H}^{n+1}=\mathbf{H}^{n}+\frac{\Delta t^2}{4}\mathbf{B}_{21}\mathbf{B}_{12}\left(\mathbf{H}^{n+1}-\mathbf{H}^{n}\right)+\Delta t\mathbf{A}_{21}\mathbf{E}^{n+\frac{1}{2}}+\Delta t\mathbf{B}_{21}\mathbf{E}^{n+\frac{1}{2}} \tag{10a}$$

$$=\mathbf{H}^{n}+\Delta t\mathbf{B}_{21}\left[\frac{\Delta t}{4}\mathbf{B}_{12}\left(\mathbf{H}^{n+1}-\mathbf{H}^{n}\right)+\mathbf{E}^{n+\frac{1}{2}}\right]+\Delta t\mathbf{A}_{21}\mathbf{E}^{n+\frac{1}{2}} \tag{10b}$$

$$=\mathbf{H}^{n}+\Delta t\left(\mathbf{A}_{21}\ \mathbf{E}^{n+\frac{1}{2}}+\mathbf{B}_{21}\mathbf{e}^{n+\frac{1}{2}}\right) \tag{10c}$$

where the auxiliary variable

$$\mathbf{e}^{n+\frac{1}{2}}=\mathbf{E}^{n+\frac{1}{2}}+\frac{\Delta t}{4}\mathbf{B}_{12}(\mathbf{H}^{n+1}-\mathbf{H}^{n}) \tag{11}$$

Equation (10c) becomes the conventional explicit FDTD H update equation when $\mathbf{e}^{n+\frac{1}{2}} = \mathbf{E}^{n+\frac{1}{2}}$. The auxiliary variable $\mathbf{e}$ in (11) is a perturbation of $\mathbf{E}$. Next, we cast (10c) and (11) in matrix form compactly as

$$\begin{bmatrix} \mathbf{I} & -\frac{\Delta t}{4}\mathbf{B}_{12} \\ -\Delta t\mathbf{B}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e}^{n+\frac{1}{2}} \\ \mathbf{H}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\frac{\Delta t}{4}\mathbf{B}_{12} \\ \Delta t\mathbf{A}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{E}^{n+\frac{1}{2}} \\ \mathbf{H}^{n} \end{bmatrix} \tag{12}$$

Equations (8) and (12) can be combined as

$$\begin{bmatrix} \mathbf{I} & -\Delta t\mathbf{A}_{12} & \mathbf{0} & \mathbf{0} \\ -\frac{\Delta t}{4}\mathbf{A}_{21} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{I} & -\frac{\Delta t}{4}\mathbf{B}_{12} \\ -\Delta t\mathbf{A}_{21} & \mathbf{0} & -\Delta t\mathbf{B}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{E}^{n+\frac{1}{2}} \\ \mathbf{h}^{n} \\ \mathbf{e}^{n+\frac{1}{2}} \\ \mathbf{H}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \Delta t\mathbf{B}_{12} \\ -\frac{\Delta t}{4}\mathbf{A}_{21} & \mathbf{I} \\ \mathbf{0} & -\frac{\Delta t}{4}\mathbf{B}_{12} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{E}^{n-\frac{1}{2}} \\ \mathbf{H}^{n} \end{bmatrix} \tag{13a}$$

Solving (13a) for $\mathbf{E}^{n+\frac{1}{2}}$ and $\mathbf{H}^{n+1}$ reduces symbolically to

$$\begin{bmatrix} \mathbf{E}^{n+\frac{1}{2}} \\ \mathbf{H}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\Delta t\mathbf{A}_{12} & \mathbf{0} & \mathbf{0} \\ -\frac{\Delta t}{4}\mathbf{A}_{21} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{I} & -\frac{\Delta t}{4}\mathbf{B}_{12} \\ -\Delta t\mathbf{A}_{21} & \mathbf{0} & -\Delta t\mathbf{B}_{21} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & \Delta t\mathbf{B}_{12} \\ -\frac{\Delta t}{4}\mathbf{A}_{21} & \mathbf{I} \\ \mathbf{0} & -\frac{\Delta t}{4}\mathbf{B}_{12} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{E}^{n-\frac{1}{2}} \\ \mathbf{H}^{n} \end{bmatrix}. \tag{13b}$$

The E and H fields are staggered in time and updated in one full time step iteratively. Finally (13b) can be expressed in matrix operator form compactly as

$$\begin{bmatrix} \mathbf{E}^{n+\frac{1}{2}} \\ \mathbf{H}^{n+1} \end{bmatrix} = \mathbf{M} \begin{bmatrix} \mathbf{E}^{n-\frac{1}{2}} \\ \mathbf{H}^{n} \end{bmatrix} \tag{14}$$

where $\mathbf{M}$ is a 6×6 matrix.

4. STABILITY ANALYSIS

The stability of the leapfrog ADI-FDTD method will be analyzed using the Fourier (Von Neumann) method in this section [19, 20]. The matrix $\mathbf{M}$ in (14) is transformed to the Fourier domain to obtained the Fourier amplification matrix $\mathbf{G}_F$. The eigenvalues of $\mathbf{G}_F$ will determine the stability of the leapfrog ADI-FDTD method. Note that subscript "F" denotes Fourier domain.

By using MATLAB, (14) can be written in the Fourier domain as

$$\mathbf{U}_F^{n+1} = \mathbf{G}_F \mathbf{U}_F^n \tag{15}$$

where

$$\mathbf{U}_F = [E_{x0} \quad E_{y0} \quad E_{z0} \quad H_{x0} \quad H_{y0} \quad H_{z0}]^T \tag{16}$$

$$\mathbf{G}_F = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{-j2bP_z}{\Omega_y} & \frac{j2bP_y}{\Omega_y} \\ 0 & 1 & 0 & \frac{j2bP_z}{\Omega_z} & 0 & \frac{-j2bP_x}{\Omega_z} \\ 0 & 0 & 1 & \frac{-j2bP_y}{\Omega_x} & \frac{-j2bP_x}{\Omega_x} & 0 \\ 0 & \frac{j2dP_z}{\Omega_y} & \frac{-j2dP_y}{\Omega_y} & \frac{g_{4,4}}{\Omega_x\Omega_y\Omega_z} & \frac{4bdP_xP_y}{\Omega_x\Omega_y} & \frac{4bdP_xP_z}{\Omega_y\Omega_z} \\ \frac{-j2dP_z}{\Omega_z} & 0 & \frac{-j2dP_x}{\Omega_z} & \frac{4bdP_xP_y}{\Omega_x\Omega_z} & \frac{g_{5,5}}{\Omega_x\Omega_y\Omega_z} & \frac{4bdP_yP_z}{\Omega_y\Omega_z} \\ \frac{j2dP_y}{\Omega_x} & \frac{-j2dP_x}{\Omega_x} & 0 & \frac{4bdP_xP_z}{\Omega_x\Omega_z} & \frac{4bdP_yP_z}{\Omega_x\Omega_y} & \frac{g_{6,6}}{\Omega_x\Omega_y\Omega_z} \end{bmatrix} \tag{17}$$

$$\begin{aligned} g_{4,4} &= b^3d^3P_x^2P_y^2P_z^2 + b^2d^2\left(P_x^2P_y^2 - 3P_x^2P_z^2 - 3P_y^2P_z^2\right) \\ &\quad + bd\left(P_x^2 - 3P_y^2 - 3P_z^2\right) + 1 \end{aligned} \tag{18}$$

$$\begin{aligned} g_{5,5} &= b^3d^3P_x^2P_y^2P_z^2 + b^2d^2\left(P_y^2P_z^2 - 3P_x^2P_y^2 - 3P_x^2P_z^2\right) \\ &\quad + bd\left(P_y^2 - 3P_x^2 - 3P_z^2\right) + 1 \end{aligned} \tag{19}$$

$$\begin{aligned} g_{6,6} &= b^3d^3P_x^2P_y^2P_z^2 + b^2d^2\left(P_x^2P_z^2 - 3P_x^2P_y^2 - 3P_y^2P_z^2\right) \\ &\quad + bd\left(P_z^2 - 3P_x^2 - 3P_y^2\right) + 1 \end{aligned} \tag{20}$$

$$P_\alpha = \frac{-2\sin\left(\frac{k_\alpha \Delta_\alpha}{2}\right)}{\Delta_\alpha} \tag{21}$$

$$\Omega_\alpha = 1 + bdP_\alpha^2, \quad \alpha = x, y, z \tag{22}$$

$$b = \frac{\Delta t}{2\epsilon}, \quad d = \frac{\Delta t}{2\mu} \tag{23}$$

Here, P_α corresponds to the spatial difference operator ∂_α in the Fourier domain and k is the wavenumber. $E_{\alpha 0}$ and $H_{\alpha 0}$ are the initial conditions of E_α and H_α fields respectively in the Fourier domain.

The eigenvalues of $\mathbf{G}_F$ are obtained using MATLAB as

$$\lambda_1 = \lambda_2 = 1 \tag{24}$$

$$\lambda_3 = \lambda_4 = \frac{Q + \sqrt{-S^2}}{R} \tag{25}$$

$$\lambda_5 = \lambda_6 = \frac{Q - \sqrt{-S^2}}{R} \tag{26}$$

where

$$R = \Omega_x\Omega_y\Omega_z = \left(1 + bdP_x^2\right)\left(1 + bdP_y^2\right)\left(1 + bdP_z^2\right) \tag{27}$$

$$\begin{aligned} Q = {} & b^3d^3P_x^2P_y^2P_z^2 - b^2d^2\left(P_x^2P_y^2 + P_x^2P_z^2 + P_y^2P_z^2\right) \\ & -bd\left(P_x^2 + P_y^2 + P_z^2\right) + 1 \end{aligned} \tag{28}$$

$$\begin{aligned} S^2 = {} & R^2 - Q^2 = 4bd\left(b^3d^3P_x^2P_y^2P_z^2 + 1\right) \\ & \cdot\left[P_x^2 + P_y^2 + P_z^2 + bd\left(P_x^2P_y^2 + P_x^2P_z^2 + P_y^2P_z^2\right)\right] \end{aligned} \tag{29}$$

Since R and Q are real numbers and (29) is true, all the eigenvalues of $\mathbf{G}_F$ have unity magnitude. The leapfrog ADI-FDTD method is thus unconditionally stable and non-dissipative. The eigenvalues of $\mathbf{G}_F$ for the leapfrog ADI-FDTD method are also the same as the eigenvalues of $\mathbf{G}_F$ for the conventional ADI-FDTD method. Note that Q in [20] has a typo error and it should read (28).

5. DISPERSION ANALYSIS

The dispersion of the leapfrog ADI-FDTD method can be analyzed by assuming the field to be a monochromatic wave with angular frequency ω [20, 21]. The dispersion relation of the leapfrog ADI-FDTD method can be deduced from the eigenvalues of $\mathbf{G}_F$. Since the eigenvalues of the leapfrog ADI-FDTD method are the same as the conventional ADI-FDTD method, the dispersion relation of the leapfrog ADI-FDTD method is also the same as that of the conventional ADI-FDTD method and is given as

$$\sin^2(\omega\Delta t) = \frac{4bd\left(b^3d^3P_x^2P_y^2P_z^2+1\right)\left[P_x^2+P_y^2+P_z^2+bd\left(P_x^2P_y^2+P_x^2P_z^2+P_y^2P_z^2\right)\right]}{\left(1+bdP_x^2\right)^2\left(1+bdP_y^2\right)^2\left(1+bdP_z^2\right)^2} \tag{30}$$

For illustration, the angular frequency ω is chosen to be 3 GHz with uniform cell size $\Delta = \Delta_x = \Delta_y = \Delta_z = \frac{\lambda}{50}$. The Courant limit time step size is $\Delta t_{CFL} = \frac{\Delta}{c\sqrt{3}}$ where c is the speed of light in freespace. We also denote CFLN as $\frac{\Delta t}{\Delta t_{CFL}}$.

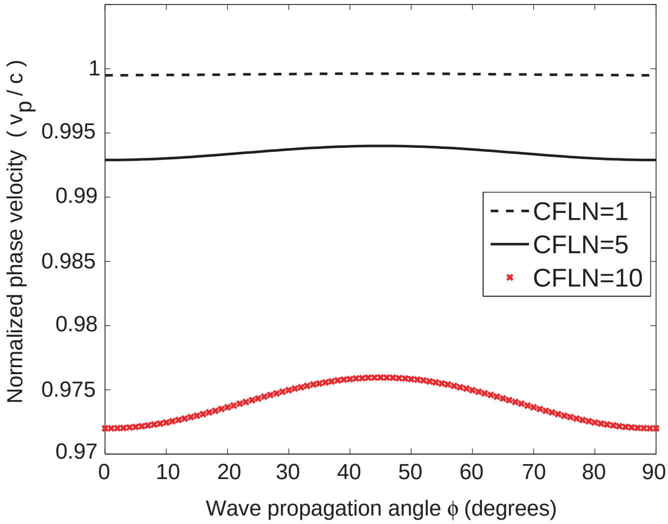

Figure 1. Normalized phase velocity (v_p/c) vs. wave propagation angle ϕ for CFLN = 1, 5 and 10 at $\theta = 45°$.

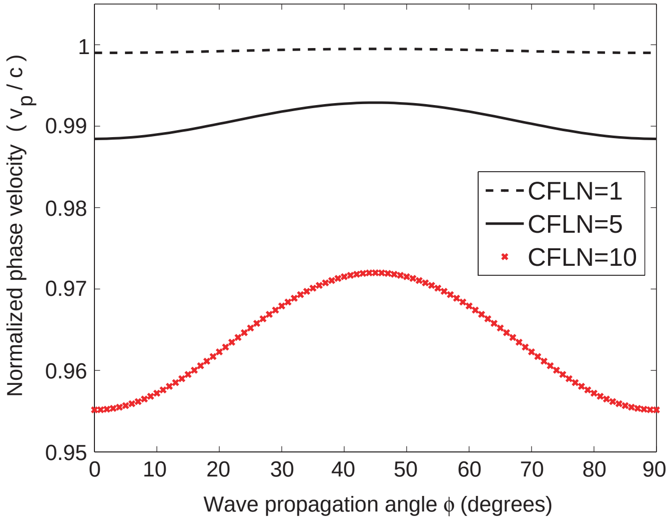

Figure 2. Normalized phase velocity (v_p/c) vs. wave propagation angle ϕ for CFLN = 1, 5 and 10 at $\theta = 90°$.

The numerical phase velocity is given as

$$v_p = \frac{\omega}{k} \tag{31}$$

where the wavenumber k can be obtained by solving (30) with the parameters given above. The normalized phase velocity (v_p/c) vs. wave propagation angle ϕ for CFLN = 1, 5 and 10 at $\theta = 45°$ and $90°$ is plotted in Figures 1 and 2 respectively. It is apparent that the numerical phase velocity error increases with CFLN and the numerical phase velocity is slightly anisotropic. Referring to Figure 1, the normalized phase velocity approaches 1 for CFLN = 1 while the normalized phase velocity is approximately 0.975 for CFLN = 10. Referring to Figure 2 for CFLN = 10, we can clearly observed that the normalized phase velocity varies with wave propagation angle ϕ and it is maximum at wave propagation angle $\phi = 45°$.

6. CONCLUSION

This paper has presented the stability and dispersion analysis of the leapfrog ADI-FDTD method. The leapfrog ADI-FDTD method was reviewed and the implicit update equations presented. The leapfrog ADI-FDTD method was reformulated in the form similar to conventional explicit FDTD method by introducing two auxiliary variables. The auxiliary variables serve as perturbations of the main fields variables. We analyzed the stability of the leapfrog ADI-FDTD method using the Fourier method. The eigenvalues of the Fourier amplification matrix are obtained analytically and it is proven that the leapfrog ADI-FDTD method is unconditionally stable. The dispersion relation of the leapfrog ADI-FDTD method was also presented. The leapfrog ADI-FDTD method has the same eigenvalues and dispersion relation as the conventional ADI-FDTD method.

The leapfrog ADI-FDTD method is unconditionally stable and employs the same leapfrog scheme as the conventional explicit FDTD method. It should be very useful for modeling of fine structures using hybrid subgridding methods.

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