Progress In Electromagnetics Research C, Vol. 26, 167–179, 2012

THE CLOSE-FORM SOLUTION FOR SYMMETRIC BUTLER MATRICES

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Abstract—The design of a $2^n \times 2^n$ Butler matrix is usually based on an iterative process. In this paper, recurrence relations behind this process are found, and the close-form solutions, i.e., non-recursive functions of n, are reported. These solutions allow the direct derivation of the scattering matrix coefficients of symmetric and large Butler matrices.

1. INTRODUCTION

When used in beam-forming applications, the $2^n \times 2^n$ Butler matrix is a multi-input/multi-output device with 2^n inputs (beam ports) and 2^n outputs (antenna array ports) that allows synthesizing 2^n radiating beams. Introduced in 1961 by Butler and Lowe [1], this passive multi-port device is composed of 3 dB-couplers, phase shifters and cross-overs (see Figure 1). Several articles discuss procedures for designing Butler matrices. In 1964, Moody [2] reported the design of a symmetric Butler matrix based on an iterative process. Shelton and Kelleher [3] proposed in 1961 a reduced scattering matrix which has analogous properties to those of the Butler scattering matrix (with respect to some conditions) while Allen [4] analyzed the orthogonality of the Butler matrix. In 1967, Jaeckle [5] reported an alternative design with different values

Received 14 November 2011, Accepted 19 December 2011, Scheduled 22 December 2011
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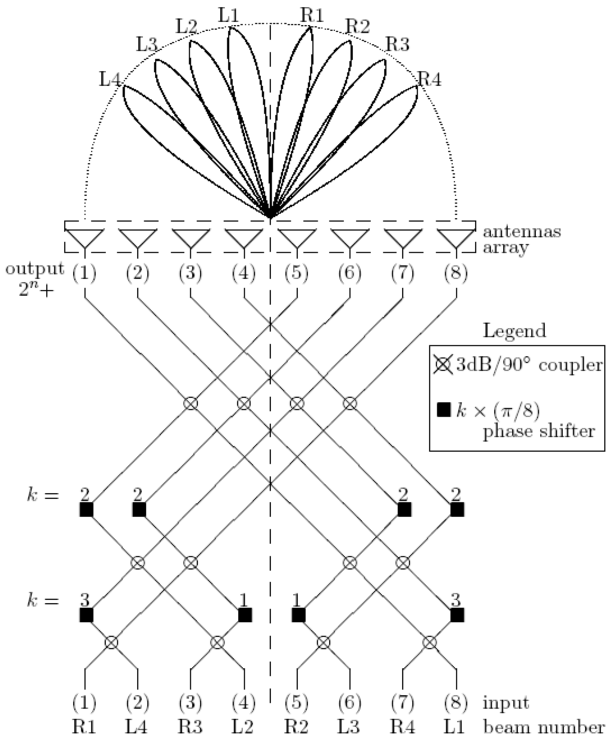

Figure 1. Architecture of a $2^3 \times 2^3$ symmetric Butler matrix.

of phase shifts. More recently, in 1987 Macnamara [6] published a detailed and systematic procedure for designing asymmetric Butler matrices with 3 dB/180° couplers. Besides the theory of Butler matrix, research works have been reported for reducing size [7] or solving problems due to cross-overs [8]. In order to remove the cross-overs, new technologies have been recently applied such as the Substrate Integrated Waveguide (SIW) technology on one layer [9] or on two layers [10]. However, all these works, including the precursor analysis of Butler and Lowe [1], are based on an iterative construction of the Butler matrix. For large matrix dimensions, such derivation may be fastidious and time-consuming. To the authors' knowledge the close-form solution of the recurrence relations behind the iterative process in case of symmetric and lossless Butler matrix has not been reported yet.

In this paper, the recurrence relations used for synthesizing $2^n \times 2^n$ Butler matrices are derived, and their close-form solutions, i.e., non-recursive functions of n, are reported for the first time. These solutions allow the direct calculation of the scattering parameters of Butler matrices.

2. RECURRENCE RELATIONS FOR CALCULATING THE SCATTERING PARAMETERS OF LOSSLESS AND SYMMETRIC BUTLER MATRICES

The scattering matrix (S-matrix) of reciprocal devices having M uncoupled inputs and N uncoupled outputs with all ports matched is given by Eq. (1) [3, 4]:

$$[S] = \begin{bmatrix} 0 & \dots & \dots & 0 & S_{1(M+1)} & \dots & \dots & S_{1(M+N)} \\ \vdots & \ddots & & \vdots & \vdots & & & \vdots \\ 0 & \dots & \dots & 0 & S_{m(M+1)} & \dots & \dots & S_{m(M+N)} \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 0 & \dots & \dots & 0 & S_{M(M+1)} & \dots & \dots & S_{M(M+N)} \\ S_{(M+1)1} & \dots & \dots & S_{(M+1)M} & 0 & \dots & \dots & 0 \\ \vdots & & & \vdots & \vdots & \ddots & & \vdots \\ S_{(M+n)1} & \dots & \dots & S_{(M+n)M} & 0 & \dots & \dots & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ S_{(M+N)1} & \dots & \dots & S_{(M+N)M} & 0 & \dots & \dots & 0 \end{bmatrix} \tag{1}$$

Let $[S_r]$ designates the $N \times M$ non-zero S-matrix appearing at the bottom left quarter of the S-matrix of Eq. (1):

$$[S_r] = \begin{bmatrix} S_{(M+1)1} & \dots & S_{(M+1)M} \\ \vdots & \ddots & \vdots \\ S_{(M+N)1} & \dots & S_{(M+N)M} \end{bmatrix} \tag{2}$$

If $[S_r]$ is unitary, that is, if $[S_r]\,[S_r]^{*T} = [I]$ where $[S_r]^{*T}$ is the conjugate transpose matrix of $[S_r]$ and $[I]$ is the unit matrix with dimension $N \times N$, then the S-matrix given by Eq. (1) is also unitary and can be rewritten as follows [4]:

$$[S] = \begin{bmatrix} [0] & [S_r]^T \\ [S_r] & [0] \end{bmatrix} \tag{3}$$

Moreover, when $M = N = 2^n$, the matrix given in Eq. (1) may represent the S-matrix of a $2^n \times 2^n$ Butler matrix. As an example, following [1] and [2], the S-matrix $[S_r]$ associated with the Butler matrix shown in Figure 1 is given by:

$$[S_r] = \frac{1}{2\sqrt{2}} \times \begin{bmatrix} e^{-j5\frac{\pi}{8}} & e^{-j9\frac{\pi}{8}} & e^{-j6\frac{\pi}{8}} & e^{-j10\frac{\pi}{8}} & e^{-j5\frac{\pi}{8}} & e^{-j9\frac{\pi}{8}} & e^{-j8\frac{\pi}{8}} & e^{-j12\frac{\pi}{8}} \\ e^{-j6\frac{\pi}{8}} & e^{-j2\frac{\pi}{8}} & e^{-j11\frac{\pi}{8}} & e^{-j7\frac{\pi}{8}} & e^{-j8\frac{\pi}{8}} & e^{-j4\frac{\pi}{8}} & e^{-j15\frac{\pi}{8}} & e^{-j11\frac{\pi}{8}} \\ e^{-j7\frac{\pi}{8}} & e^{-j11\frac{\pi}{8}} & e^{-j0\frac{\pi}{8}} & e^{-j4\frac{\pi}{8}} & e^{-j11\frac{\pi}{8}} & e^{-j15\frac{\pi}{8}} & e^{-j6\frac{\pi}{8}} & e^{-j10\frac{\pi}{8}} \\ e^{-j8\frac{\pi}{8}} & e^{-j4\frac{\pi}{8}} & e^{-j5\frac{\pi}{8}} & e^{-j\frac{\pi}{8}} & e^{-j14\frac{\pi}{8}} & e^{-j10\frac{\pi}{8}} & e^{-j13\frac{\pi}{8}} & e^{-j9\frac{\pi}{8}} \\ e^{-j9\frac{\pi}{8}} & e^{-j13\frac{\pi}{8}} & e^{-j10\frac{\pi}{8}} & e^{-j14\frac{\pi}{8}} & e^{-j\frac{\pi}{8}} & e^{-j5\frac{\pi}{8}} & e^{-j4\frac{\pi}{8}} & e^{-j8\frac{\pi}{8}} \\ e^{-j10\frac{\pi}{8}} & e^{-j6\frac{\pi}{8}} & e^{-j15\frac{\pi}{8}} & e^{-j11\frac{\pi}{8}} & e^{-j4\frac{\pi}{8}} & e^{-j0\frac{\pi}{8}} & e^{-j11\frac{\pi}{8}} & e^{-j7\frac{\pi}{8}} \\ e^{-j11\frac{\pi}{8}} & e^{-j15\frac{\pi}{8}} & e^{-j4\frac{\pi}{8}} & e^{-j8\frac{\pi}{8}} & e^{-j7\frac{\pi}{8}} & e^{-j11\frac{\pi}{8}} & e^{-j2\frac{\pi}{8}} & e^{-j6\frac{\pi}{8}} \\ e^{-j12\frac{\pi}{8}} & e^{-j8\frac{\pi}{8}} & e^{-j9\frac{\pi}{8}} & e^{-j5\frac{\pi}{8}} & e^{-j10\frac{\pi}{8}} & e^{-j6\frac{\pi}{8}} & e^{-j9\frac{\pi}{8}} & e^{-j5\frac{\pi}{8}} \end{bmatrix} \tag{4}$$

Butler and Lowe [1], Moody [2] and Macnamara [6] have used an iterative process for deriving the S-parameters of Butler matrices. The recurrence relations associated with this process are now established and in Section 3. These relations are solved, and the close-form expressions for the S-parameters of $[S_r]$ in case of a $2^n \times 2^n$ Butler matrix are reported.

By analyzing the systematic design of a Butler matrix established by Moody [2], it can be noted that the term $S_{(M+1)1} = S_{(2^n+1)1}$ of a lossless $2^n \times 2^n$ Butler matrix can be written as:

$$S_{(2^n+1)1} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left(j\Psi_0^{(n)}\right) \tag{5}$$

$$\text{with } \Psi_0^{(n)} = -\sum_{i=0}^{n-2}\left(\frac{\pi}{2} - 2^i\left|\Delta\phi_1\right|\right) \text{ where } \Delta\phi_1 = -\frac{\pi}{2^n} \tag{6}$$

Moreover, it can be observed that the term $S_{(2^n+1)m}$ for $m = 2, 3, \ldots, 2^n$ depends on the term $S_{(2^n+1)(m-1)}$ as follows:

- If m is even, then:

$$S_{(2^n+1)(m)} = S_{(2^n+1)(m-1)} \cdot \exp\left(-j\pi/2\right) \tag{7}$$

- If m is odd, then:

$$\begin{aligned} S_{(2^n+1)(m)} &= S_{(2^n+1)(m-1)} \cdot \exp\left[j\left(\chi_m \cdot \frac{\pi}{2^{n_m}} + \sum_{i=n_m+1}^{n} \frac{\pi}{2^i}\right)\right] \\ &= S_{(2^n+1)(m-1)} \cdot \exp\left\{j\left[\frac{\pi}{2^{n_m}}\left(\chi_m + 1\right) - \frac{\pi}{2^n}\right]\right\} \end{aligned} \tag{8}$$

The value of n_m in Eq. (8) is reported in Table 1 for any (odd) index m.

Table 1. Value of n_m versus the odd index m of the term $S_{(2^n+1)m}$.

m	3	5	7	9	11	13	15	...	$2^{N_m-1}+1$	...	$2^{N_m}-1$
n_m	2	3		4				...	N_m		

Table 2. Values of χ_m versus the odd index m of the term $S_{(2^n+1)m}$ when $N_m = 5$.

N_m	5							
m	17	19	21	23	25	27	29	31
$x_2^{(m)}$	1	1	1	1	0	0	0	0
$x_3^{(m)}$	1	1	0	0	1	1	0	0
$x_4^{(m)}$	1	0	1	0	1	0	1	0
χ_m	29	13	21	5	25	9	17	1

On the other hand, χ_m in Eq. (8) is given by Eq. (9):

$$\chi_m = 1 + \sum_{i=2}^{N_m-1} x_i^{(m)} 2^i \tag{9}$$

where the $N_m - 2$ values of $x_i^{(m)}$ are determined for all ranks m having the same N_m. For a given N_m, $x_i^{(m)}$ is deduced from the sequence of 2^{N_m-1-i} "1", followed by a sequence of 2^{N_m-1-i} "0". This sequence of "1" and "0" is then repeated 2^{i-2} times. For deriving χ_m from Eq. (9) the following steps can be followed:

- **Step 1:** Find the value of N_m associated with m by using Table 1;
- **Step 2:** Derive the values of the $x_i^{(m)}$ $(i = 2, 3, \ldots, N_{m-1})$ for rank N_m and identify the corresponding values $x_i^{(m)}$;
- **Step 3:** Calculate χ_m using Eq. (9).

Table 2 summarizes the results of these steps when $N_m = 5$. It can be deduced, for example, that $\chi_{17} = 1 \times 2^2 + 1 \times 2^3 + 1 \times 2^4 + 1 = 29$ and $\chi_{21} = 1 \times 2^2 + 0 \times 2^3 + 1 \times 2^4 + 1 = 21$.

The iterative approach described by Moody in [2] for designing a lossless $2^n \times 2^n$ Butler matrix allows deriving the term $S_{(2^n+l)(m)}$ with $l = 2, 3, \ldots, 2^n$ as follows:

$$S_{(2^n+l)(m)} = S_{(2^n+l-1)(m)} \cdot \exp\left[j\Delta\phi_m\right] \tag{10}$$

where the determination of the phase gradient $\Delta\phi_m$ is illustrated with Figure 2 [2, 6].

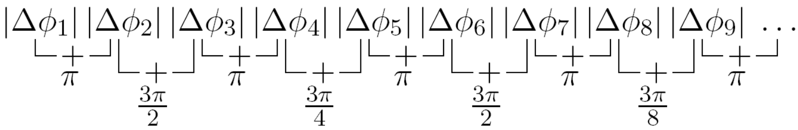

Figure 2. Illustration of the determination of the phase gradient $\Delta\phi m$ [2, 6].

Equations (7)–(8) and (10) are the recurrence relationships that govern the computation of the scattering parameter $S_{(2^n+l)(m)}$.

The value of the phase gradient $\Delta\phi_1$ is such that $|\Delta\phi_1| = 1 \cdot \frac{\pi}{M}$ while the other gradients $\Delta\phi_m$ are determined by recursive expressions combining the Eq. (11) reported in [6] with Eqs. (12)–(14), that is:

$$|\Delta\phi_{2p-1}| + |\Delta\phi_{2p}| = 2^n \times \frac{\pi}{M} = \pi \forall p \in [1; M], \ m \in \mathbb{N} \tag{11}$$

$$\left|\Delta\phi_{M-(p-1)}\right| = |\Delta\phi_p| \, \forall p \in [1; M], p \in \mathbb{N} \tag{12}$$

$$\left|\Delta\phi_{\frac{M}{2^i}}\right| + \left|\Delta\phi_{\frac{M}{2^i}+1}\right| = 3 \times 2^i \frac{\pi}{M} \forall i \in [1, n-1], \ i \in \mathbb{N} \tag{13}$$

$$\left|\Delta\phi_{\frac{M}{2^i}-1-m}\right| + \left|\Delta\phi_{\frac{M}{2^i}-m}\right| = \left|\Delta\phi_{\frac{M}{2^i}+1+m}\right| + \left|\Delta\phi_{\frac{M}{2^i}+2+m}\right| \tag{14}$$

$$\forall i \in [1; n-1], \ i \in \mathbb{N}, \qquad \forall m \in \left[0; \frac{M}{2^i} - 2\right], \ m \in \mathbb{N}$$

Note that the phase gradient $\Delta\phi_m$ takes alternatively negative and positive values.

3. CLOSE-FORM EXPRESSIONS OF THE *S*-PARAMETERS FOR LOSSLESS AND SYMMETRIC BUTLER MATRICES

If we combine Eqs. (7) and (8) and use Eq. (5), the S-parameter $S_{(2^n+1)(m)}$ can be derived as follows:

- If m is even, then:

$$S_{(2^n+1)(m)} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left\{ j \left[\Psi_0^{(n)} - m\frac{\pi}{4} - \pi\left(\frac{m}{2} - 1\right)\left(\frac{1}{2}\right)^n + \sum_{i=3,5,\ldots}^{m-1} \frac{\pi}{2^{n_i}} (1 + \chi_i) \right] \right\} \tag{15}$$

- If m is odd, then:

$$S_{(2^n+1)(m)} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left\{j\left[\Psi_0^{(n)} - (m-1)\frac{\pi}{4} - \pi\frac{m-1}{2}\left(\frac{1}{2}\right)^n + \sum_{i=3,5,\ldots}^{m} \frac{\pi}{2^{n_i}}(1+\chi_i)\right]\right\} \quad (16)$$

Eqs. (15)–(16) are the close-form expressions of the $S_{(2^n+1)(m)}$ of any $2^n \times 2^n$ Butler matrix. They are derived from the analysis previously described in Section 2 of the iterative process proposed in [2]. They can be established by mathematical induction as shown in Appendix A.

From the knowledge of the phase gradient $\Delta\phi_m$, the S-parameter $S_{(2^n+l)(m)}$ given in Eq. (10) can then be deduced:

$$S_{(2^n+l)(m)} = S_{(2^n+1)(m)} \cdot \exp\left[j(l-1)\Delta\phi_m\right] \quad (17)$$

When m is even, $S_{(2^n+l)(m)}$ can be determined by combining Eq. (15) and Eq. (17); when m is odd, Eq. (16) and Eq. (17) are combined. The resulting S-parameter $S_{(2^n+l)(m)}$ is then given by:

- when m is even, then:

$$S_{(2^n+l)(m)} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left\{j\left[\Psi_0^{(n)} + (l-1)\Delta\phi_m - m\frac{\pi}{4} - \pi\left(\frac{m}{2}-1\right)\left(\frac{1}{2}\right)^n + \sum_{i=3,5,\ldots}^{m-1} \frac{\pi}{2^{n_i}}(1+\chi_i)\right]\right\} \quad (18)$$

- when m is odd, then:

$$S_{(2^n+l)(m)} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left\{j\left[\Psi_0^{(n)} + (l-1)\Delta\phi_m - (m-1)\frac{\pi}{4} - \pi\frac{m-1}{2}\left(\frac{1}{2}\right)^n + \sum_{i=3,5,\ldots}^{m} \frac{\pi}{2^{n_i}}(1+\chi_i)\right]\right\} \quad (19)$$

Eqs. (18) and (19) are the close-form expressions of $S_{(2^n+l)(m)}$ (with $l > 1$) of any $2^n \times 2^n$ Butler matrix. These expressions can be established by mathematical induction as shown in Appendix A.

4. CONCLUSION

For the first time to our knowledge, general close-form expressions have been derived to determine the scattering matrix of a lossless and large symmetrical $2^n \times 2^n$ Butler matrix. These close-form expressions have been established by mathematical inductions. They allow a direct computation of the Butler scattering matrix from any value of n.

ACKNOWLEDGMENT

This work is supported by COBHAM and the French Space Agency (CNES).

APPENDIX A. THE CLOSE-FORM EXPRESSIONS OF BUTLER SCATTERING MATRIX DERIVED FROM MATHEMATICAL INDUCTION

The close-form expressions reported in this paper are established by mathematical induction in this appendix.

A.1 Let us show by mathematical induction that the scattering coefficient $S_{(2^n+1)(m)}$ is given by:

- when m is even, then:

$$S_{(2^n+1)(m)} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left\{j\left[\Psi_0^{(n)} - m\frac{\pi}{4} - \pi\left(\frac{m}{2}-1\right)\left(\frac{1}{2}\right)^n + \sum_{i=3,5,\dots}^{m-1} \frac{\pi}{2^{n_i}}\left(1+\chi_i\right)\right]\right\} \tag{A1}$$

- when m is odd, then:

$$S_{(2^n+1)(m)} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left\{j\left[\Psi_0^{(n)} - (m-1)\frac{\pi}{4} - \pi\frac{m-1}{2}\left(\frac{1}{2}\right)^n + \sum_{i=3,5,\dots}^{m} \frac{\pi}{2^{n_i}}\left(1+\chi_i\right)\right]\right\} \tag{A2}$$

Step 1: It is straightforward to show that these expressions are true for $m = 1$, 2 and 3. As a matter of fact:

- Applying $m = 1$ in Eq. (A2), the following expression is obtained:

$$S_{(2^n+1)(1)} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left(j\Psi_0^{(n)}\right) \tag{A3}$$

where $\Psi_0^{(n)}$ is given by:

$$\Psi_0^{(n)} = -\sum_{i=0}^{n-2}\left(\frac{\pi}{2} - 2^i\frac{\pi}{2^n}\right) = -\frac{\pi}{2}(n-2) - \frac{\pi}{2^n} \tag{A4}$$

The same expression is obtained from the Moody's iterative approach [2].

- Applying $m = 2$ in Eq. (A1) it remains the following expression:

$$S_{(2^n+1)(2)} = \left(\frac{1}{\sqrt{2}}\right)^n \exp\left[j\left(\Psi_0^{(n)} - \frac{\pi}{2}\right)\right] \tag{A5}$$

Again, the same expression is obtained from the Moody's iterative approach [2].

- when $m = 3$, Eq. (A2) is used:

$$\begin{aligned} S_{(2^n+1)(3)} &= \left(\frac{1}{\sqrt{2}}\right)^n \exp\left[j\left(\Psi_0^{(n)} - \frac{\pi}{2} - \frac{\pi}{2^n} + \frac{\pi}{2^{n_3}}(1+\chi_3)\right)\right] \\ &= \left(\frac{1}{\sqrt{2}}\right)^n \exp\left[j\left(\Psi_0^{(n)} - \frac{\pi}{2^n}\right)\right] \end{aligned} \tag{A6}$$

On the other hand, analyzing Moody's systematic design [2], Eq. (A7) is obtained:

$$S_{(2^n+1)(3)} = \left(\frac{1}{\sqrt{2}}\right)^n \exp\left[-j\left(\frac{\pi}{2} + \sum_{i=1}^{n-2}\left(\frac{\pi}{2} - 2^i\frac{\pi}{2^n}\right)\right)\right] \tag{A7}$$

Then by dividing Eq. (A6) by Eq. (A7), the result is 1. Therefore, these two expressions are equal.

Consequently for the first values of m, the Eqs. (A1)–(A2) are true.
Step 2: Assuming that Eqs. (A1) and (A2) are true at rank m, let us show that they are true at rank $m+1$:

- If $m+1$ is even, using the recurrence relation Eq. (7) and Eq. (A2) (since m is odd), we find:

$$S_{(2^n+1)(m+1)} = S_{(2^n+1)(m)} \exp(-j\pi/2) \tag{A8}$$

$$S_{(2^n+1)(m+1)} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left\{j\left[\Psi_0^{(n)} - (m+1)\frac{\pi}{4} - \pi\left(\frac{m+1}{2} - 1\right)\left(\frac{1}{2}\right)^n + \sum_{i=3,5,\ldots}^{m} \frac{\pi}{2^{n_i}}(1+\chi_i)\right]\right\} \tag{A9}$$

Eq. (A9) is identical to Eq. (A1) in which m is replaced by $m+1$.

- If $m+1$ is odd, using the recurrence relation Eq. (8) and Eq. (A1) (since m is even), we obtain:

$$S_{(2^n+1)(m+1)} = S_{(2^n+1)(m)} \cdot \exp\left\{j\left[\frac{\pi}{2^{n_{m+1}}}\left(\chi_{m+1}+1\right) - \frac{\pi}{2^n}\right]\right\} \quad \text{(A10)}$$

$$S_{(2^n+1)(m+1)} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left\{j\left[\Psi_0^{(n)} - m\frac{\pi}{4} - \pi\frac{m}{2}\left(\frac{1}{2}\right)^n + \sum_{i=3,5,\ldots}^{m+1} \frac{\pi}{2^{n_i}}\left(\chi_i+1\right)\right]\right\} \quad \text{(A11)}$$

Eq. (A11) is identical to Eq. (A2) in which m is replaced by $m+1$.

Consequently, to deduce the step 2, if Eqs. (A1) and (A2) are assumed to be true at rank m, it is shown that they are also true at rank $m+1$.

Step 3: From step 1 and step 2, we conclude that Eqs. (A1) and (A2) are true for any rank m.

A.2 Let us show by mathematical induction that the scattering coefficient $S_{(2^n+l)(m)}$ is given by:

- when m is even, then:

$$S_{(2^n+l)(m)} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left\{j\left[\Psi_0^{(n)} + (l-1)\,\Delta\phi_m - m\frac{\pi}{4} - \pi\left(\frac{m}{2}-1\right)\left(\frac{1}{2}\right)^n + \sum_{i=3,5,\ldots}^{m-1} \frac{\pi}{2^{n_i}}\left(1+\chi_i\right)\right]\right\} \quad \text{(A12)}$$

- when m is odd, then:

$$S_{(2^n+l)(m)} = \left(\frac{1}{\sqrt{2}}\right)^n \cdot \exp\left\{j\left[\Psi_0^{(n)} + (l-1)\,\Delta\phi_m - (m-1)\frac{\pi}{4} - \pi\frac{m-1}{2}\left(\frac{1}{2}\right)^n + \sum_{i=3,5,\ldots}^{m} \frac{\pi}{2^{n_i}}\left(1+\chi_i\right)\right]\right\} \quad \text{(A13)}$$

Step 1: Eqs. (A12)–(A13) are tested for $2^n \times 2^n$ Butler matrices when $n = 1, 2$ and 3. The resulting matrix $[S_r]$ is then compared with the matrix $[S_r]$ derived from the iterative process reported by Moody [2]:

- For $n = 1$, the Butler matrix is reduced to a 90° coupler. From [2] the $[S_r]$ matrix is then given as follows:

$$[S_r] = \left(\frac{1}{\sqrt{2}}\right)^1 \begin{bmatrix} 1 & e^{-j\pi/2} \\ e^{-j\pi/2} & 1 \end{bmatrix} \quad \text{(A14)}$$

The same scattering matrix is obtained from the close-forms Eqs. (A12)–(A13) by taking $n = 1$ and $m = 1$ in Eq. (A12), $m = 2$ in Eq. (A13) and $l = 1$ and 2 in both equations.

- For $n = 2$, analyzing a 4×4 Butler matrix, the following $[S_r]$ matrix is derived from [2]:

$$[S_r] = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} e^{-j\pi/4} & e^{-j3\pi/4} & e^{-j2\pi/4} & e^{-j4\pi/4} \\ e^{-j2\pi/4} & e^{-j0\pi/4} & e^{-j5\pi/4} & e^{-j3\pi/4} \\ e^{-j3\pi/4} & e^{-j5\pi/4} & e^{-j0\pi/4} & e^{-j2\pi/4} \\ e^{-j4\pi/4} & e^{-j2\pi/4} & e^{-j3\pi/4} & e^{-j\pi/4} \end{bmatrix} \tag{A15}$$

  By substituting n by 2 in Eqs. (A12)–(A13), the same matrix is obtained.

- For $n = 3$, the $[S_r]$ matrix is given in Eq. (4). The same matrix is obtained from Eqs. (A12)–(A13).

Consequently for the first values of m, the Eqs. (A12)–(A13) are true.

Step 2: Assuming that Eqs. (A12)–(A13) are true at rank n, let us show that they are true at rank $n + 1$:

Following [2], it can be observed that scattering parameters corresponding to an odd rank in the first line of a $2^{n+1} \times 2^{n+1}$ Butler matrix allows deriving the overall $[S_r]$ matrix. As a matter of fact:

(1) in the first line of the $[S_r]$ matrix the scattering parameters associated with an even rank can be deduced from odd rank parameters by subtracting a phase of $\pi/2$ induced by the first coupler;

(2) the other lines of the $[S_r]$ matrix can be derived from the first one using Eq. (17).

Consequently, we consider only the scattering parameters having an odd rank in the first line of the $[S_r]$ matrix.

Moody graph [2] allows deriving the $2^{n+1} \times 2^{n+1}$ Butler matrix and the corresponding $[S_r]$ matrix. Eq. (A13) gives the same matrices. As a matter of fact, concerning the phase of the scattering parameters (the comparisons of the magnitudes are straightforward):

$\arg\left(S_{(2^{n+1}+1)(1)}\right)$:

$$\begin{cases} \text{from Moody [2]: } -\sum_{i=0}^{n-1}\left(\frac{\pi}{2} - 2^i\left|\Delta\phi_1\right|\right) \\ \text{from Eq. (A13): } \Psi_0^{(n+1)} = -\frac{\pi}{2}(n-1) - \frac{\pi}{2^{n+1}} \end{cases} \tag{A16}$$

$$\Rightarrow \frac{\pi}{2}(1-n) - \frac{\pi}{2^{n+1}} \tag{A17}$$

$\arg\left(S_{(2^{n+1}+1)(3)}\right)$:

$$\begin{cases} \text{from Moody [2]:} \; -\dfrac{\pi}{2} - \displaystyle\sum_{i=1}^{n-1}\left(\dfrac{\pi}{2} - 2^i\left|\Delta\phi_1\right|\right) \\ \text{from eq. (A13):} \; \Psi_0^{(n+1)} - \dfrac{\pi}{2} - \dfrac{\pi}{2^{n+1}} + \dfrac{\pi}{2^{n_3}}\left(1+\chi_3\right) \end{cases} \tag{A18}$$

$$\Rightarrow \frac{\pi}{2}\left(1-n\right) - \frac{\pi}{2^n} \tag{A19}$$

$\arg\left(S_{(2^{n+1}+1)(5)}\right)$:

$$\begin{cases} \text{from Moody [2]:} \; -\left(\dfrac{\pi}{2} - \left|\Delta\phi_5\right|\right) - \dfrac{\pi}{2} - \displaystyle\sum_{i=2}^{n-1}\left(\dfrac{\pi}{2} - 2^i\left|\Delta\phi_1\right|\right) \\ \text{from eq. (A13):} \; \Psi_0^{(n+1)} - \pi - \dfrac{2\pi}{2^{n+1}} + \displaystyle\sum_{i=3,5}^{5}\dfrac{\pi}{2^{n_i}}\left(1+\chi_i\right) \end{cases} \tag{A20}$$

$$\Rightarrow -\frac{\pi}{2}n - \frac{3\pi}{2^{n+1}} + \frac{3\pi}{4} \tag{A21}$$

Consequently, to conclude the above-mentioned step 2, if Eqs. (A12) and (A13) are assumed to be true at rank n, they are also true at rank $n+1$.

Step 3: From step 1 and step 2, we conclude that Eqs. (A12) and (A13) are true for any rank n.

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