BEAM PROPAGATION FACTOR OF PARTIALLY CO-HERENT LAGUERRE-GAUSSIAN BEAMS IN NON-KOLMOGOROV TURBULENCE

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Abstract—In order to study beam-propagation factor (M^2-factor) of partially coherent Laguerre-Gaussian (PCLG) beams in non-Kolmogorov turbulence, a generalized exponent and a generalized amplitude factor are introduced. Based on the extended Huygens-Fresnel principle and second-order moments of the Wigner distribution function (WDF), the analytical formula of M^2-factor for PCLG beams in non-Kolmogorov turbulence is derived. The corresponding numerical results are also calculated. Results show that for PCLG beams propagating in non-Kolmogorov turbulence, the bigger the beam order or outer scale is, or the smaller the correlation length, $\tilde{C}_n^2$, or inner scale is, the smaller the value of the normalized M^2-factor is. Furthermore, the normalized M^2-factor of PCLG beams increases with the increasing of α until it reaches the maximum point, then it gradually decreases the increasing of α.

1. INTRODUCTION

For a long time, Kolmogorov's power spectrum of refractive index fluctuations has been widely accepted and shown good agreement with experimental results [1]. However, the Kolmogorov spectrum is only effective in inertial subrange. In last several decades, it has been experimentally indicated that turbulence in the upper troposphere and stratosphere deviates from predictions of the Kolmogorov model [2, 3], and in the case of laser propagation along the vertical direction, the turbulence is no longer homogeneous in three dimensions, since the vertical component is suppressed [4]. Taking into account the slope

variation of the atmospheric power spectrum, Toselli et al. introduced a non-Kolmogorov power spectrum [5, 6] by using a generalized exponent and a generalized amplitude factor instead of a constant standard exponent value 11/3 and a constant value 0.033 associated with the conventional Kolmogorov power spectrum. The parameter "α" is the power-law for the spectrum of the index of refraction fluctuations, and the generalized power spectrum reduces to the conventional Kolmogorov spectrum when the exponent value $\alpha = 11/3$ [5, 6]. Based on this model, average spreading of a Gaussian beam array, spreading and direction of Gaussian–Schell model beam and second-order statistics of stochastic electromagnetic beams in non-Kolmogorov turbulence have been studied [3, 7, 8]. However, to our knowledge, the propagation of other types of beams in non-Kolmogorov turbulence have rarely been taken into account, even though the propagation properties of various types of laser beams in Kolmogorov turbulence have been widely studied [9–12].

Parameter properties of laser beams (such as the kurtosis parameter, M^2-factor, polarization characteristics and scintillation, etc.) have been discussed in many publications [13–16]. Recently, the M^2-factor is a very useful beam parameter for characterizing laser beams and can be regarded as a beam quality factor in many practical applications [17]. The M^2-factor of partially coherent beams in free space has been studied by Gori, et al. [18, 19]. Amarande [13], Zhou and Zheng [14], Baida and Luo [20] have studied beam propagation factor of flattened Gaussian beams, higher-order cosh-Gaussian beam and hard-edge diffracted cosh-Gaussian beams. Dan and Zhang have studied M^2-factor of partially coherent flat-topped beams [21]. Yuan et al. have calculated coherent and partially coherent hollow beam propagation in Kolmogorov turbulent atmosphere [22]. Especially, Chu has investigated the Hermite-Gaussian beam quality and its beam shape under some conditions in non-Kolmogorov turbulence [23].

In this paper, the main aim is to study the propagation properties of M^2-factor for PCLG beams in non-Kolmogorov turbulence by using the extended Huygens-Fresnel principle and second-order moments of the WDF. The analytical expression in non-Kolmogorov is derived. Some numerical examples have been discussed.

2. THEORETICAL MODEL

2.1. WDF of PCLG Beams in a Turbulent Atmosphere

The electric field distribution of LG beams at the source plane $(z = 0)$ is given by [24]

$$E_0(\boldsymbol{\rho}', 0) = \frac{(-1)^n}{2^{2n+m} \cdot n!} \sum_{t=0}^{n} \sum_{s=0}^{m} i^s \binom{n}{t} \binom{m}{s} H_{2t+m-s}\left(\frac{\sqrt{2}x'}{w_0}\right)$$

$$H_{2n-2t+s}\left(\frac{\sqrt{2}y'}{w_0}\right) \exp\left(-\frac{\boldsymbol{\rho}'^2}{w_0^2}\right) \tag{1}$$

where $\boldsymbol{\rho} \equiv (x, y)$ is the two-dimensional position vector in the source plane. $H_{2t+m-s}(\cdot)$ and $H_{2n-2t+s}(\cdot)$ are Hermite polynomials. w_0 is the waist width of the fundamental Gaussian beam. For the case of the beam orders $m = n = 0$, Eq. (1) is reduced to a fundamental Gaussian beam.

Reference [25] introduces a Gaussian term of the spectral degree of coherence. The fully coherent beam can be extended to the partially coherent one, and this type of partially coherent beam can be produced by the fully coherent beam passing through a random phase plate or a liquid crystal [21, 26]. In rectangular coordinate system, a partially coherent beam at the plane $z = 0$ is characterized by the cross-spectral density, which is expressed as [21, 27].

$$\Gamma(\boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2, 0) = \langle E(\boldsymbol{\rho}'_1, 0) E^*(\boldsymbol{\rho}'_2, 0) \rangle_m$$

$$= \frac{1}{2^{4n+2m} \cdot (n!)^2} \sum_{t_1=0}^{n} \sum_{s_1=0}^{m} \sum_{t_2=0}^{n} \sum_{s_2=0}^{m} i^{s_1} (-i)^{s_2} \binom{n}{t_1} \binom{m}{s_1} \binom{n}{t_2} \binom{m}{s_2}$$

$$\times H_{2t_1+m-s_1}\left(\frac{\sqrt{2}x'_1}{w_0}\right) H_{2t_2+m-s_2}\left(\frac{\sqrt{2}x'_2}{w_0}\right) H_{2n-2t_1+s_1}\left(\frac{\sqrt{2}y'_1}{w_0}\right)$$

$$\times H_{2n-2t_2+s_2}\left(\frac{\sqrt{2}y'_2}{w_0}\right) \exp\left[-\frac{\boldsymbol{\rho}'^2_1 + \boldsymbol{\rho}'^2_2}{w_0^2} - \frac{|\boldsymbol{\rho}'_1 - \boldsymbol{\rho}'_2|^2}{2\sigma_0^2}\right] \tag{2}$$

where $\langle \cdot \rangle_m$ denotes average over the field ensemble, and $\boldsymbol{\rho}'_1$ and $\boldsymbol{\rho}'_2$ are two different point vectors in the source plane. σ_0 is the correlation length of the source, and $H(\cdot)$ is Hermite polynomial. If $m = n = 0$, Eq. (2) is reduced to the cross-spectral density function of Gaussian-Schell model beams and $\sigma_0 \to \infty$, and PLG beams reduce to a coherent LG beams.

By using the paraxial form of the extended Huygens-Fresnel principle [3, 7], the cross-spectral density of PCLG beams through the

turbulence can be expressed as [21, 22].

$$W(\boldsymbol{\rho}, \boldsymbol{\rho}_d, z) = \left(\frac{k}{2\pi z}\right)^2 \iiiint W(\boldsymbol{\rho}', \boldsymbol{\rho}'_d, 0)$$
$$\times \exp\left\{\frac{ik}{z}[(\boldsymbol{\rho} - \boldsymbol{\rho}') \cdot (\boldsymbol{\rho}_d - \boldsymbol{\rho}'_d)] - H(\boldsymbol{\rho}_d, \boldsymbol{\rho}'_d, z)\right\} d^2\boldsymbol{\rho}' d^2\boldsymbol{\rho}'_d \quad (3)$$

where $k = 2\pi/\lambda$ is the wave number, λ the wavelength, and term $\exp[-H(\boldsymbol{\rho}_d, \boldsymbol{\rho}'_d, z)]$ the effect of the turbulence. H can be written as [21, 22].

$$H(\boldsymbol{\rho}_d, \boldsymbol{\rho}'_d, z) = 4\pi^2 k^2 z \int_0^1 d\xi \int_0^\infty \left[1 - J_0\left(\kappa \left|\boldsymbol{\rho}'_d \xi + (1-\xi)\boldsymbol{\rho}_d\right|\right)\right] \Phi_n(\kappa)\kappa d\kappa \quad (4)$$

where J_0 is the Bessel function of zero order, κ the magnitude of the spatial wave number, and Φ_n the spatial power spectrum of the refractive index fluctuations of the turbulent atmosphere. To include both inner- and outer-scale effects, we use non-Kolmogorov spectrum expressed in the following form to model the atmospheric turbulence [7, 28].

$$\Phi_n(\kappa) = A(\alpha)\tilde{C}_n^2 \frac{\exp\left[-\kappa^2/\kappa_m^2\right]}{\left(\kappa^2 + \kappa_0^2\right)^{\alpha/2}} \qquad 0 \le \kappa < \infty \quad -3 < \alpha < 4 \quad (5)$$

where $\kappa_0 = \frac{2\pi}{L_0}$ and $\kappa_m = \frac{c(\alpha)}{l_0}$ with L_0 and l_0 are the turbulence outer- and inner-scale parameters, and α is the power law. In the present paper, it is supposed that α holds unchanged along the propagation path. And $c(\alpha) = \left[\Gamma(5 - \frac{\alpha}{2})A(\alpha)\frac{2}{3}\pi\right]^{1/(\alpha-5)}$, the term $\tilde{C}_n^2$ in Eq. (5) is generalized refractive-index structure parameter with units $m^{3-\alpha}$, and Γ denotes the gamma function. $A(\alpha) = \frac{1}{4\pi^2}\Gamma(\alpha - 1)\cos\left(\frac{\alpha\pi}{2}\right)$. The spectrum expressed in Eq. (5) is reduced to conventional Kolmogorov spectrum when $\alpha = 11/3$, $A(\alpha) = 0.033$, $L_0 = \infty$, $l_0 = 0$ and $\tilde{C}_n^2 = C_n^2$.

To evaluate Eq. (3), it is convenient to introduce new variables of integration

$$\boldsymbol{\rho}' = (\boldsymbol{\rho}'_1 + \boldsymbol{\rho}'_2)/2, \qquad \boldsymbol{\rho}'_d = \boldsymbol{\rho}'_1 - \boldsymbol{\rho}'_2$$
$$\boldsymbol{\rho} = (\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)/2, \qquad \boldsymbol{\rho}_d = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$$

where $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ are two arbitrary point vectors in the receiver plane, perpendicular to the direction of propagation of the beam, and the cross-spectral density at the source plane can be expressed as

$$W(\boldsymbol{\rho}', \boldsymbol{\rho}'_d, 0) = \Gamma(\boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2, 0) = \Gamma\left(\boldsymbol{\rho}' + \frac{\boldsymbol{\rho}'_d}{2}, \boldsymbol{\rho}' - \frac{\boldsymbol{\rho}'_d}{2}, 0\right) \quad (6)$$

It is well known that the WDF can characterize partially coherent beams in space and in spatial frequency domain simultaneously and

can be expressed in terms of the cross-spectral density $W(\boldsymbol{\rho}, \boldsymbol{\rho}_d, z)$ as [29, 30].

$$h(\boldsymbol{\rho}, \boldsymbol{\theta}, z) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty} W(\boldsymbol{\rho}, \boldsymbol{\rho}_d, z) \exp(-ik\boldsymbol{\theta} \cdot \boldsymbol{\rho}_d) d^2 \boldsymbol{\rho}_d \quad (7)$$

where vector $\boldsymbol{\theta} = (\theta_x, \theta_y)$ denotes an angle of propagation, and $k\theta_x$ and $k\theta_y$ are the wave vector components along the x-axis and y-axis, respectively.

On the basis of inverse Fourier transform of the Dirac delta function and its property of even function [21], we obtain

$$\delta(\boldsymbol{\rho}'' - \boldsymbol{\rho}') = \frac{1}{(2\pi)^2} \int \exp\left[\pm i\boldsymbol{\kappa}_d \cdot (\boldsymbol{\rho}'' - \boldsymbol{\rho}')\right] d^2 \boldsymbol{\kappa}_d \quad (8)$$

Then, the cross-spectral density of the beams in the source ($z = 0$) can be rewritten as

$$W(\boldsymbol{\rho}', \boldsymbol{\rho}'_d, 0) = \frac{1}{(2\pi)^2} \iint W(\boldsymbol{\rho}'', \boldsymbol{\rho}'_d, 0) \exp[i\boldsymbol{\kappa}_d \cdot (\boldsymbol{\rho}'' - \boldsymbol{\rho}')] d^2 \boldsymbol{\kappa}_d d^2 \boldsymbol{\rho}'' \quad (9)$$

Substituting Eq. (9) into Eq. (3) and using Eq. (4), we obtain

$$W(\boldsymbol{\rho}, \boldsymbol{\rho}_d, z) = \frac{1}{(2\pi)^2} \iint W\left(\boldsymbol{\rho}'', \boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d, 0\right)$$
$$\times \exp\left[-i\boldsymbol{\rho} \cdot \boldsymbol{\kappa}_d + i\boldsymbol{\kappa}_d \cdot \boldsymbol{\rho}'' - H\left(\boldsymbol{\rho}_d, \boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d, z\right)\right] d^2 \boldsymbol{\kappa}_d d^2 \boldsymbol{\rho}'' \quad (10)$$

where

$$W\left(\boldsymbol{\rho}'', \boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d, 0\right)$$

$$= \frac{1}{2^{4n+2m} \cdot (n!)^2} \sum_{t_1=0}^{n} \sum_{s_1=0}^{m} \sum_{t_2=0}^{n} \sum_{s_2=0}^{m} i^{s_1}(-i)^{s_2} \binom{n}{t_1}\binom{m}{s_1}\binom{n}{t_2}\binom{m}{s_2}$$

$$\times H_{2t_1+m-s_1}\left[\frac{\sqrt{2}}{w_0}x'' + \frac{\sqrt{2}}{2w_0}\left(\boldsymbol{\rho}_{dx} + \frac{z}{k}\boldsymbol{\kappa}_{dx}\right)\right]$$

$$H_{2t_2+m-s_2}\left[\frac{\sqrt{2}}{w_0}x'' - \frac{\sqrt{2}}{2w_0}\left(\boldsymbol{\rho}_{dx} + \frac{z}{k}\boldsymbol{\kappa}_{dx}\right)\right]$$

$$\times H_{2n-2t_1+s_1}\left[\frac{\sqrt{2}}{w_0}y'' + \frac{\sqrt{2}}{2w_0}\left(\boldsymbol{\rho}_{dy} + \frac{z}{k}\boldsymbol{\kappa}_{dy}\right)\right]$$

$$H_{2n-2t_2+s_2}\left[\frac{\sqrt{2}}{w_0}y'' - \frac{\sqrt{2}}{2w_0}\left(\boldsymbol{\rho}_{dy} + \frac{z}{k}\boldsymbol{\kappa}_{dy}\right)\right]$$

$$\exp\left[-\frac{2}{w_0^2}\boldsymbol{\rho}''^2 - \frac{1}{\varepsilon^2}\boldsymbol{\rho}_d^2 - 2\frac{1}{\varepsilon^2}\frac{z}{k}\boldsymbol{\rho}_d \cdot \boldsymbol{\kappa}_d - \frac{z^2}{\varepsilon^2 k^2}\boldsymbol{\kappa}_d^2)\right] \quad (11)$$

with $\frac{1}{\varepsilon^2} = \frac{1}{2w_0^2} + \frac{1}{2\sigma_0^2}$.

Substituting Eq. (10) into Eq. (7), using Eq. (11), and performing the integration with respect to $\boldsymbol{\rho}''$, we obtain

$$h(\boldsymbol{\rho},\boldsymbol{\theta},z) = \frac{k^2 w_0^2}{32\pi^3} \frac{1}{2^{2n+m} \cdot (n!)^2} \sum_{t_1=0}^{n}\sum_{s_1=0}^{m}\sum_{t_2=0}^{n}\sum_{s_2=0}^{m}$$

$$i^{s_1}(-i)^{s_2} \binom{n}{t_1}\binom{m}{s_1}\binom{n}{t_2}\binom{m}{s_2}(2t_1+m-s_1)!(2n-2t_1+s_1)!$$

$$\times \iint \exp\Big[-i\boldsymbol{\rho}\cdot\boldsymbol{\kappa}_d - ik\boldsymbol{\theta}\cdot\boldsymbol{\rho}_d - a\boldsymbol{\kappa}_d^2 - b\boldsymbol{\rho}_d^2 + c\boldsymbol{\rho}_d \cdot \boldsymbol{\kappa}_d$$

$$-H\Big(\boldsymbol{\rho}_d, \boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d, z\Big)\Big] \times \Big[\frac{1}{2}\frac{w_0}{\sqrt{2}}i\kappa_{dx} - \frac{\sqrt{2}}{2w_0}\Big(\boldsymbol{\rho}_{dx} + \frac{z}{k}\boldsymbol{\kappa}_{dx}\Big)\Big]^{2t_2-s_2-2t_1+s_1}$$

$$\Big[\frac{1}{2}\frac{w_0}{\sqrt{2}}i\kappa_{dy} - \frac{\sqrt{2}}{2w_0}\Big(\boldsymbol{\rho}_{dy} + \frac{z}{k}\boldsymbol{\kappa}_{dy}\Big)\Big]^{-2t_2+s_2+2t_1-s_1} \times L_{2t_1+m-s_1}^{2t_2-s_2-2t_1+s_1}$$

$$\Big[\frac{1}{w_0^2}\boldsymbol{\rho}_{dx}^2 + 2\frac{z}{k}\frac{1}{w_0^2}\boldsymbol{\rho}_{dx}\boldsymbol{\kappa}_{dx} + \Big(\frac{z^2}{w_0^2k^2} + \frac{w_0^2}{4}\Big)\boldsymbol{\kappa}_{dx}^2\Big] \times L_{2n-2t_1+s_1}^{-2t_2+s_2+2t_1-s_1}$$

$$\Big[\frac{1}{w_0^2}\boldsymbol{\rho}_{dy}^2 + 2\frac{z}{k}\frac{1}{w_0^2}\boldsymbol{\rho}_{dy}\boldsymbol{\kappa}_{dy} + \Big(\frac{z^2}{w_0^2k^2} + \frac{w_0^2}{4}\Big)\boldsymbol{\kappa}_{dy}^2\Big] d^2\boldsymbol{\kappa}_d d^2\boldsymbol{\rho}_d \tag{12}$$

where $a = \frac{1}{8}w_0^2 + \frac{z^2}{\varepsilon^2 k^2}$, $b = \frac{1}{\varepsilon^2} = \frac{1}{2w_0^2} + \frac{1}{2\sigma_0^2}$, $c = -2\frac{1}{\varepsilon^2}\frac{z}{k}$. To derive Eq. (12), we have used the relations $\boldsymbol{\rho}_d^2 = \rho_{dx}^2 + \rho_{dy}^2$, $\boldsymbol{\kappa}_d^2 = \kappa_{dx}^2 + \kappa_{dy}^2$ and the following formula

$$\int \exp(-x^2) H_m(x+y) H_n(x+z)dx = 2^n\sqrt{\pi}m! z^{n-m} L_m^{n-m}(-2yz) \quad (m \leq n) \tag{13}$$

2.2. The Angular Width and M^2-factor of PCLG Beams in Non-kolmogorov Turbulence

Based on the second-order moments of WDF, the M^2-factor of beams can be defined as [21, 22, 30].

$$M^2(z) = k\left(\langle\boldsymbol{\rho}^2\rangle\langle\boldsymbol{\theta}^2\rangle - \langle\boldsymbol{\rho}\cdot\theta\rangle^2\right)^{1/2}$$

$$= k\big[\big(\langle x^2\rangle + \langle y^2\rangle\big)\big(\langle\theta_x^2\rangle + \langle\theta_y^2\rangle\big) - \big(\langle x\theta_x\rangle + \langle y\theta_y\rangle\big)\big]^{1/2} \tag{14}$$

According to definition, moments of the order $n_1 + n_2 + m_1 + m_2$ of WDF, it can be expressed as [24, 31].

$$\langle x^{n_1}y^{n_2}\theta_x^{m_1}\theta_y^{m_2}\rangle = \frac{1}{P}\iiiint x^{n_1}y^{n_2}\theta_x^{m_1}\theta_y^{m_2}h(\boldsymbol{\rho},\boldsymbol{\theta},z)d^2\boldsymbol{\rho}d^2\boldsymbol{\theta} \tag{15}$$

where

$$P = \iint h(\boldsymbol{\rho}, \boldsymbol{\theta}, z) d\boldsymbol{\rho} d\boldsymbol{\theta} \tag{16}$$

Substitute Eq. (12) into Eq. (15) and use Eq. (16) and formulas in [21]

$$\delta(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-isx) dx \tag{17}$$

$$\delta^n(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-ix)^n \exp(-isx) dx \quad (n = 1, 2) \tag{18}$$

$$\int_{-\infty}^{\infty} f(x)\delta^n(x) dx = (-1)^n f^n(0) \quad (n = 1, 2) \tag{19}$$

After some manipulations, we obtain

$$\langle \boldsymbol{\rho}^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle = \frac{\pi w_0^2}{P} \frac{1}{2^{2n+m} \cdot (n!)^2} \sum_{t_1=0}^{n} \sum_{s_1=0}^{m} \sum_{t_2=0}^{n} \sum_{s_2=0}^{m}$$

$$i^{s_1}(-i)^{s_2} \binom{n}{t_1}\binom{m}{s_1}\binom{n}{t_2}\binom{m}{s_2} \times (2t_1 + m - s_1)!$$

$$(2n - 2t_1 + s_1)! \left[2\left(a + \frac{1}{3}\pi^2 z^3 T \right) + \left(\frac{z^2}{w_0^2 k^2} + \frac{w_0^2}{4} \right)(2n + m) \right] \tag{20}$$

$$\langle \theta^2 \rangle = \langle \theta_x^2 \rangle + \langle \theta_y^2 \rangle = \frac{\pi w_0^2}{P} \frac{1}{2^{2n+m} \cdot (n!)^2} \sum_{t_1=0}^{n} \sum_{s_1=0}^{m} \sum_{t_2=0}^{n} \sum_{s_2=0}^{m}$$

$$i^{s_1}(-i)^{s_2} \binom{n}{t_1}\binom{m}{s_1}\binom{n}{t_2}\binom{m}{s_2}(2t_1 + m - s_1)!$$

$$(2n - 2t_1 + s_1)! \left[2\left(\frac{b}{k^2} + \pi^2 z T \right) + \frac{1}{k^2 w_0^2}(2n + m) \right] \tag{21}$$

$$\langle \boldsymbol{\rho} \cdot \boldsymbol{\theta} \rangle = \langle x\theta_x \rangle + \langle y\theta_y \rangle = \frac{\pi w_0^2}{P} \frac{1}{2^{2n+m} \cdot (n!)^2}$$

$$\sum_{t_1=0}^{n} \sum_{s_1=0}^{m} \sum_{t_2=0}^{n} \sum_{s_2=0}^{m} i^{s_1}(-i)^{s_2} \binom{n}{t_1}\binom{m}{s_1}\binom{n}{t_2}\binom{m}{s_2}(2t_1 + m - s_1)!$$

$$(2n - 2t_1 + s_1)! \left[\frac{z}{k^2} \frac{1}{w_0^2}(m + 2n) - \left(\frac{c}{k} - \pi^2 z^2 T \right) \right] \tag{22}$$

$$P = \frac{\pi w_0^2}{2} \frac{1}{2^{2n+m} \cdot (n!)^2} \sum_{t_1=0}^{n} \sum_{s_1=0}^{m} \sum_{t_2=0}^{n} \sum_{s_2=0}^{m} i^{s_1}(-i)^{s_2} \binom{n}{t_1}\binom{m}{s_1}$$

$$\times \binom{n}{t_2}\binom{m}{s_2}(2t_1+m-s_1)!(2n-2t_1+s_1)! \tag{23}$$

where

$$T = \int_0^\infty \Phi_n(\kappa)\kappa^3 d\kappa \tag{24}$$

Substituting Eq. (6) into Eq. (24), the expression of T can be written as

$$T = \frac{A(\alpha)}{2(\alpha-2)}\tilde{C}_n^2 \left[k_m^{2-\alpha}\beta \exp\left(\kappa_0^2\kappa_m^{-2}\right)\Gamma\left(2-\alpha/2, \kappa_0^2\kappa_m^{-2}\right) - 2\kappa_0^{4-\alpha}\right] \tag{25}$$

where $\beta = 2\kappa_0^2 - 2\kappa_m^2 + \alpha\kappa_m^2$.

Substituting Eq. (20), Eq. (21) and Eq. (22) into Eq. (14), the expression of M^2-factor for PCLG beams in the received plane can be expressed as

$$M^2(z) = k\left(\langle\boldsymbol{\rho}^2\rangle\langle\theta^2\rangle - \langle\boldsymbol{\rho}\cdot\theta\rangle^2\right)^{1/2} = i^{s_1}(-i)^{s_2}\frac{\pi k w_0^2}{2^{2n+m}(n!)^2 P}$$

$$\left\{\left\{\sum_{t_1=0}^{n}\sum_{s_1=0}^{m}\sum_{t_2=0}^{n}\sum_{s_2=0}^{m}(2t_1+m-s_1)!(2n-2t_1+s_1)!\binom{n}{t_1}\binom{m}{s_1}\binom{n}{t_2}\binom{m}{s_2}\right.\right.$$

$$\times\left[2a+\frac{\pi^2 z^3 A(\alpha)\tilde{C}_n^2}{3(\alpha-2)}\left[\kappa_m^{2-\alpha}\beta\exp\left(\kappa_0^2\kappa_m^{-2}\right)\Gamma\left(2-\alpha/2, \kappa_0^2\kappa_m^{-2}\right)-2\kappa_0^{4-\alpha}\right]\right.$$

$$\left.+\left(\frac{z^2}{w_0^2 k^2}+\frac{w_0^2}{4}\right)(m+2n)\right]\right\}\left\{\sum_{t_1=0}^{n}\sum_{s_1=0}^{m}\sum_{t_2=0}^{n}\sum_{s_2=0}^{m}(2t_1+m-s_1)!\right.$$

$$(2n-2t_1+s_1)!\binom{n}{t_1}\binom{m}{s_1}\binom{n}{t_2}\binom{m}{s_2}$$

$$\times\left[\frac{2b}{k^2}+\frac{\pi^2 z A(\alpha)\tilde{C}_n^2}{(\alpha-2)}\left[\kappa_m^{2-\alpha}\beta\exp\left(\kappa_0^2\kappa_m^{-2}\right)\Gamma\left(2-\alpha/2, \kappa_0^2\kappa_m^{-2}\right)-2\kappa_0^{4-\alpha}\right]\right.$$

$$\left.+\frac{1}{k^2 w_0^2}(m+2n)\right]\right\}-\left\{\sum_{t_1=0}^{n}\sum_{s_1=0}^{m}\sum_{t_2=0}^{n}\sum_{s_2=0}^{m}(2t_1+m-s_1)!(2n-2t_1+s_1)!\right.$$

$$\binom{n}{t_1}\binom{m}{s_1}\binom{n}{t_2}\binom{m}{s_2}\left[\frac{z}{k^2\frac{1}{w_0^2}(2n+m)}-\frac{c}{k}+\frac{\pi^2 z^2 A(\alpha)\tilde{C}_n^2}{2(\alpha-2)}\right.$$

$$\left.\left.\left.\left[\kappa_m^{2-\alpha}\beta\exp\left(\kappa_0^2\kappa_m^{-2}\right)\Gamma\left(2-\alpha/2, \kappa_0^2\kappa_m^{-2}\right)-2\kappa_0^{4-\alpha}\right]\right]\right\}^2\right\}^{1/2} \tag{26}$$

Equation (26) is the main result of this paper, which presents a powerful tool to study the M^2-factor of PCLG in the receiving plane. One can find that when $\sigma_0 \to \infty$, Eq. (26) reduces to coherent M^2-factor of LG beams, and when $\Phi_n(\kappa) = 0$, Eq. (26) can also turn into M^2-factor of PCLG beams in the free space.

3. NUMERICAL EXAMPLES

Now we study the numerical results of the M^2-factor for PCLG beams on propagation by using the formula derived in above section.

Figure 1 gives the variation of the M^2-factor and the normalized M^2-factor of PCLG beams on propagation in turbulent atmosphere for different α. From Fig. 1, one can see that the M^2-factor and
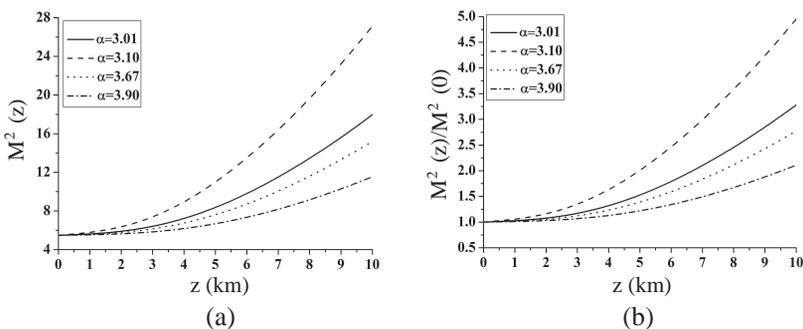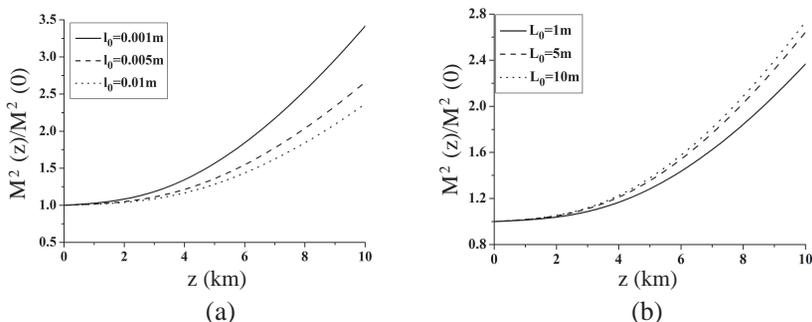

Figure 1. M^2-factor and normalized M^2-factor of PCLG beams versus propagation in non-Kolmogorov turbulence for different α. The calculation parameters are $m = 2$, $n = 1$, $L_0 = 1\,\mathrm{m}$, $l_0 = 0.01\,\mathrm{m}$, $\lambda = 8.50 \times 10^{-7}\,\mathrm{m}$, $\tilde{C}_n^2 = 10^{-15}\,\mathrm{m}^{3-\alpha}$, $\sigma_0 = 0.02\,\mathrm{m}$.

Figure 2. Normalized M^2-factor of PCLG beams versus propagation in non-Kolmogorov turbulence for different L_0. The calculation parameters are $m = 2$, $n = 1$, $\alpha = 3.8$, $\lambda = 8.50 \times 10^{-7}\,\mathrm{m}$, $\tilde{C}_n^2 = 10^{-15}\,\mathrm{m}^{3-\alpha}$, $\sigma_0 = 0.02\,\mathrm{m}$. (a) $L_0 = 1\,\mathrm{m}$. (b) $l_0 = 0.01\,\mathrm{m}$.

normalized M^2-factor of PCLG beams obviously increase with the increasing propagation distance z. In other words, the beam quality decreases as the propagation distance increases. But it firstly grows and then decreases for a fixed propagation distance z with increasing on the value of α.

Figure 2 plots the normalized M^2-factor of PCLG beams propagating in non-Kolmogorov turbulence for different inner scale l_0 and outer scale L_0. It is seen from Fig. 2 that the normalized M^2-factor of PCLG beams decreases with increasing inner scale l_0 when the propagation distance z is fixed(see Fig. 2(a)). And the normalized M^2-factor increases with increasing outer scale L_0 (see Fig. 2(b)). The inner scale l_0, which forms the lower limit of the inertial range, has a smaller value for strong turbulence and a larger value for weak turbulence. The outer scale L_0 forms the upper limit of the inertial range and increases with the increasing strength of turbulence. The decreasing of inner scale l_0 or increasing of outer scale L_0 is equivalent to increasing the strength of the turbulence. In these cases, the laser beam will meet more turbulence cells along its propagation paths, and as a result, M^2-factor of the beam maybe have higher value.

Figure 3 gives the normalized M^2-factor of PCLG beams versus propagation in non-Kolmogorov turbulence for different $\tilde{C}_n^2$. From Fig. 3, one can see that the normalized M^2-factor of PCLG beam increases with increasing of propagation distance. And it is clearly seen that for a given propagation distance the normalized M^2-factor is smaller for PCLG beam with weaker turbulence.
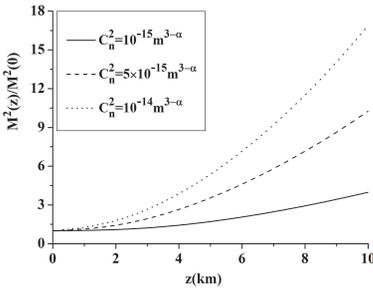

Figure 3. Normalized M^2-factor of PCLG beams versus propagation in non-Kolmogorov turbulence for different $\tilde{C}_n^2$. The calculation parameters are $m = 2$, $n = 1$, $w = 0.02\,\text{m}$, $\sigma_0 = 0.01\,\text{m}$, $L_0 = 1\,\text{m}$, $l_0 = 0.01\,\text{m}$, $\lambda = 8.50 \times 10^{-7}\,\text{m}$, $\alpha = 3.36$.
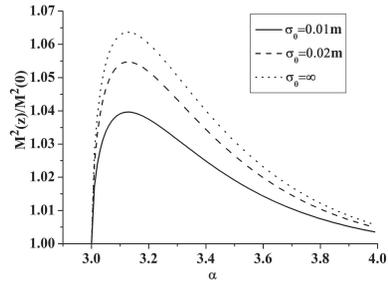
Figure 4. Normalized M^2-factor of PCLG beams versus α in non-Kolmogorov turbulence for different σ_0. The calculation parameters are $m = 2$, $n = 1$, $l_0 = 0.01\,\text{m}$, $\lambda = 8.50 \times 10^{-7}\,\text{m}$, $z = 1\,\text{km}$, $\tilde{C}_n^2 = 10^{-15}\,\text{m}^{3-\alpha}$.
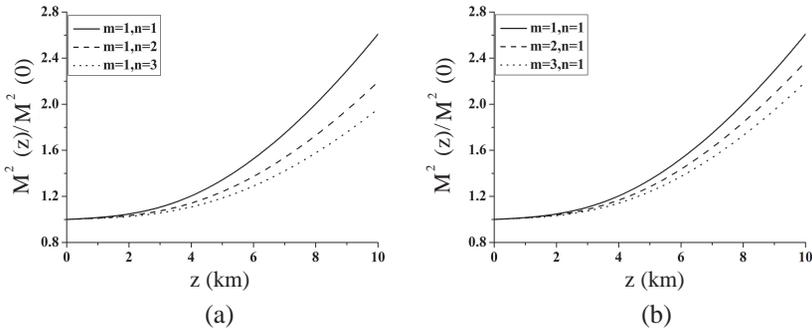
Figure 5. Normalized M^2-factor of PCLG beams versus propagation in non-Kolmogorov turbulence for different m, n. The calculation parameters are $L_0 = 1\,\mathrm{m}$, $l_0 = 0.01\,\mathrm{m}$, $\lambda = 8.50 \times 10^{-7}\,\mathrm{m}$, $\tilde{C}_n^2 = 10^{-15}\,\mathrm{m}^{3-\alpha}$, $\sigma_0 = 0.02\,\mathrm{m}$.

Figure 4 shows the normalized M^2-factor of PLG beams as a function of α on propagation in non-Kolmogorov turbulence for different correlation lengths σ_0. It can be shown in Fig. 4 that the normalized M^2-factor of LG beams increases with the increasing of α until it reaches the maximum point. After the maximum point, the normalized M^2-factor decreases with the increasing of α. The normalized M^2-factor of fully coherent LG beams is worse than that of PCLG beams in atmospheric turbulent. We assume that when the power law approaches the limiting value $\alpha = 3$, the function $A(\alpha)$ approaches zero. Consequently, the refractive-index power spectral density vanishes in this limiting case. The explanation for alpha approaching 4 is that the power spectrum contains fewer eddies of high wave numbers, i.e., the wavefront tilt is the primary aberrations. So the M^2-factor first grows and then decreases with the change of the parameter α.

Figure 5 shows the normalized M^2-factor of PCLG beams on propagation in non-Kolmogorov turbulence for different beam orders m, n. From Fig. 5, one can see that the normalized M^2-factor in atmospheric turbulent becomes worse for PCLG beams with lower beam orders as propagation distance z increases, i.e., the influence of a higher beam order on M^2-factor is less affected by turbulence than that of the lower beam order.

4. CONCLUSIONS

In conclusion, the analytical formulas for the M^2-factor of PCLG beams in non-Kolmogorov turbulence has been derived by using the

extended Huygens-Fresnel principle and second-order moments of the WDF. It is found that the propagation properties of the relative M^2-factor for PCLG beams depend on beam orders, correlation length σ_0, inner scale l_0 and outer scale L_0. The value of the normalized M^2-factor of PCLG beams is smaller for higher beam order, smaller correlation length, smaller $\tilde{C}_n^2$, smaller inner scale and bigger outer scale. And the normalized M^2-factor of PCLG beams increases with the increasing of α until it reaches the maximum point. After the maximum point, the normalized M^2-factor decreases with the increasing of α. These results may be useful in the practical beam propagation.

ACKNOWLEDGMENT

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