Progress In Electromagnetics Research B, Vol. 36, 89–111, 2012

CARTESIAN MULTIPOLE EXPANSIONS AND TENSORIAL IDENTITIES

E. Radescu, Jr. and G. Vaman*

Institute of Atomic Physics, P. O. Box MG-6, Bucharest, Romania

Abstract—We establish the exact formulas of multipole expansion in Cartesian coordinates for the most general distribution of charges and currents (including toroidal sources).

1. INTRODUCTION

The subject of multipole expansion of the electromagnetic field is treated in many textbooks on classical electrodynamics. Nevertheless, the correct relation between the radiation source and the radiation field was explained only together with the introduction of the class of toroid moments and distributions [1] besides the usual electric and magnetic ones.

In [1], it was shown that the class of toroid multipoles is indispensable for the complete parametrization of an arbitrary distribution of charges and currents, both in classical and quantum electrodynamics. It was shown also the necessity of introducing one more class of local electromagnetic characteristics (in addition to the moments), which is usually omitted — the mean square radii. The review paper [2] contains in addition particular applications of toroid moments in condensed matter physics and many clarifications about the mathematical machinery needed for a correct multipole analysis.

Recently, multipole expansion technique was applied in modern fields such as nanostructures, near-field diffraction, highly directional antennas. The special properties of toroid moments make them very interesting in constructing metamaterials with particular characteristics. Controlling the trajectory and polarization of light in media containing toroidal metamolecules is also an attractive subject (see [3–5] and references therein). Such advanced researches which

Received 7 September 2011, Accepted 2 November 2011, Scheduled 8 November 2011
* Corresponding author: G. Vaman (vaman@ifin.nipne.ro).

mainly exploit the special symmetry properties of the toroid multipole family require a correct knowledge of all the same order multipolar contributions.

The multipole expansions from [1, 2] are written in spherical coordinates, and this formalism have the advantage of displaying directly the multipoles in the form of irreducible spherical tensors. On the other hand, it requires the knowledge of the special function properties. The multipole expansions in Cartesian coordinates are easy to handle for the first multipoles, but for higher order multipoles one has to decompose the tensors into irreducible representations of the rotation group SO(3) [6]. This mechanism of reduction is illustrated in [2], Eq. (2.8) up to the order 3, and it is shown how the toroid dipole appears together with the charge octupole moment, the magnetic quadrupole moment, and the first mean square radius of the electric dipole.

Our aim here is to establish the exact formulas for the multipole expansion of a source in Cartesian coordinates, which, as far as we know, were not written anywhere. This paper is organized as follows: in the next section we write the expansion of an arbitrary distribution of charges and currents in irreducible Cartesian multipoles. We prove these formulas starting from the the parametrization of the charge and current densities in terms of spherical differential operators and distributions from [2]. Section 3 contains another derivation of the Cartesian expansions, based on some combinatorics formulas and on properties of the hypergeometric functions. The end section is devoted to conclusions and the Appendix contains some useful mathematical formulas.

2. THE CARTESIAN MULTIPOLE EXPANSION OF AN ARBITRARY DISTRIBUTION OF CHARGES AND CURRENTS

We prove in this section the following formulas for the exact multipole expansion in Cartesian coordinates of the most general source of charges and currents (which satisfy the continuity equation):

$$\rho(\vec{r},t) = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^l(2l+1)!!}{2^n n! l! (2l+2n+1)!!}\overline{r^{2n}_{i_1\ldots i_l}}(t)\Delta^n\partial_{i_1}\ldots\partial_{i_l}\delta(\vec{r}), \quad (1)$$

$$\begin{aligned}\vec{j}(\vec{r},t) &= c\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^l(2l+1)!!}{l 2^n n! l! (2l+2n+1)!!}\\ &\quad \cdot\left\{\frac{1}{(2l-1)!!}\overline{\rho^{2n}_{i_1\ldots i_l}}(t)(\vec{r}\times\nabla)\Delta^n\partial_{i_1}\ldots\partial_{i_l}\delta(\vec{r})\right.\end{aligned}$$

$$+\left[\dot{\overline{r^0_{i_1\ldots i_l}}}(t)\delta_{n,0}\Delta^{-1}-\frac{1}{(2l-1)!!}\overline{R^{2n}_{i_1\ \ldots i_l}}(t)\right]\nabla\times(\vec{r}\times\nabla)$$
$$\left.\cdot\Delta^n\partial_{i_1}\ldots\partial_{i_l}\delta(\vec{r})-l\dot{\overline{r^{2n}_{i_1\ \ldots i_l}}}(t)\nabla\Delta^{n-1}\partial_{i_1}\ldots\partial_{i_l}\delta(\vec{r})\right\}, \qquad (2)$$

where †: $\rho(\vec{r},t)$ is the charge density, $\vec{j}(\vec{r},t)$ is the current density and

$$\overline{r^{2n}_{i_1\ \ldots i_l}}(t) = \frac{(-1)^l}{(2l-1)!!}\int d^3\xi\,\xi^{2l+2n+1}\rho\left(\vec{\xi},t\right)\partial_{i_1}\ldots\partial_{i_l}\frac{1}{\xi} \qquad (3)$$

$$\overline{\rho^{2n}_{i_1\ \ldots i_l}}(t) = \frac{(-1)^l}{c(l+1)}\int d^3\xi\,\xi^{2l+2n+1}\vec{j}\left(\vec{\xi},t\right)\left(\vec{\xi}\times\nabla\right)\partial_{i_1}\ldots\partial_{i_l}\frac{1}{\xi} \qquad (4)$$

$$\overline{R^{2n}_{i_1\ \ldots i_l}}(t) = \frac{(-1)^{l+1}}{2c(l+1)(n+1)(2l+2n+3)}\int d^3\xi\xi^{l+2n+1}\vec{j}\left(\vec{\xi},t\right)$$
$$\cdot\left[l(l+1)\frac{\vec{\xi}}{\xi}+(l+2n+3)\xi\nabla\right]\xi^{l+1}\partial_{i_1}\ldots\partial_{i_l}\frac{1}{\xi} \qquad (5)$$

are the Cartesian components of the electric, magnetic and toroid mean square radii of order n and multipolarity l, respectively, $\xi=|\vec{\xi}|$.

We start from the following formulas from Appendix E of [2], which represent the complete parametrization of the charge and current densities in terms of spherical differential operators and distributions ‡:

$$\rho(\vec{r},t) = \sum_{l,m,n}\frac{(2l+1)!!}{2^n n!(2l+2n+1)!!}\sqrt{\frac{4\pi}{2l+1}}\overline{r^{2n}_{lm}}(t)\Delta^n\delta_{lm}(\vec{r}), \qquad (6)$$

$$\vec{j}(\vec{r},t) = c\sum_{l,m,n}\frac{(2l+1)!!}{2^n n!(2l+2n+1)!!}\sqrt{\frac{4\pi}{2l+1}}\left\{\frac{1}{l}\overline{\rho^{2n}_{lm}}(t)(\vec{r}\times\nabla)\right.$$
$$\cdot\Delta^n\delta_{lm}(\vec{r})+\frac{1}{l}\left[\dot{\overline{r^0_{lm}}}(t)\delta_{n,0}\Delta^{-1}-\overline{R^{2n}_{lm}}(t)\right]\nabla\times(\vec{r}\times\nabla)$$
$$\left.\cdot\Delta^n\delta_{lm}(\vec{r})-\dot{\overline{r^{2n}_{lm}}}(t)\nabla\Delta^{n-1}\delta_{lm}(\vec{r})\right\}, \qquad (7)$$

where $\rho(\vec{r},t)$ is the charge density, $\vec{j}(\vec{r},t)$ is the current density, $\overline{r^{2n}_{lm}}(t)$, $\overline{\rho^{2n}_{lm}}(t)$ and $\overline{R^{2n}_{lm}}(t)$ are the electric, magnetic and toroid mean square

† The operator Δ^{-1} (inverse Laplacian) means: $\Delta^{-1}\delta(\vec{r})=-\frac{1}{4\pi}\frac{1}{r}$. Because formally $(\vec{r}\times\nabla)\delta(\vec{r})\equiv 0$, the summations for the terms containing $\overline{\rho^{2n}_{i_1\ \ldots i_l}}$ and $\overline{R^{2n}_{i_1\ \ldots i_l}}$ begin from $l=1$. Eqs. (3), (4), (5) represent the components of three totally symmetric traceless tensors of order l. Each of them has $2l+1$ independent components.

‡ There exist some differences in the notations from [1,2]. The quantities $Q^{(2n)}_{lm}$, $M^{(2n)}_{lm}$ $R^{(2n)}_{lm}$ from Appendix E of [2] are ours $\overline{r^{2n}_{lm}}$, $\overline{\rho^{2n}_{lm}}$, $\overline{R^{2n}_{lm}}$ from Eqs. (8)–(10).

radii of order n and multipolarity l respectively [7]:

$$\overline{r_{lm}^{2n}}(t) = \sqrt{\frac{4\pi}{2l+1}}\int r^{l+2n} Y_{lm}^{*}(\vec{n})\rho(\vec{r},t)d^3r, \tag{8}$$

$$\overline{\rho_{lm}^{2n}}(t) = -\frac{i}{c}\sqrt{\frac{4\pi l}{(l+1)(2l+1)}}\int r^{2n+l}\vec{Y}_{llm}^{*}(\vec{n})\cdot\vec{j}(\vec{r},t)d^3r, \tag{9}$$

$$\begin{aligned}\overline{R_{lm}^{2n}}(t) = &\frac{-1}{c(2l+1)}\sqrt{\frac{4\pi l}{l+1}}\int r^{l+2n+1}\left[\frac{\sqrt{l}}{2l+2n+3}\vec{Y}_{ll+1m}^{*}(\vec{n})\right.\\ &\left.+\frac{\sqrt{l+1}}{2(n+1)}\vec{Y}_{ll-1m}^{*}(\vec{n})\right]\cdot\vec{j}(\vec{r},t)d^3r,\end{aligned} \tag{10}$$

$r = |\vec{r}|$, $\vec{n} = \frac{\vec{r}}{r}$, $\delta_{lm}(\vec{r})$ are the spherical delta functions (see [2, 8, 9]) and $\vec{Y}_{ll'm}$ are the vector spherical harmonics ([10]). Let us first prove Eq. (1). Using the properties of the spherical gradient operator ([8, 9]), it is easy to show the following addition theorem:

$$\sum_m Y_{lm}^{*}(\vec{n})Y_{lm}(-\nabla) = \frac{2l+1}{4\pi}P_l\left(-\vec{n}\cdot\nabla\right), \tag{11}$$

where P_l are the Legendre polynomials. Next, using this addition theorem and the definition of the spherical delta functions:

$$\delta_{lm}(\vec{r}) = \frac{1}{(2l-1)!!}Y_{lm}(-\nabla)\delta(\vec{r}),$$

we can make the summation over the index m in Eq. (6) as follows:

$$\begin{aligned}&\sum_m \overline{r_{lm}^{2n}}\Delta^n\delta_{lm}(\vec{r})\\ &= \frac{1}{(2l-1)!!}\sqrt{\frac{2l+1}{4\pi}}\int d^3r' r'^{l+2n}\rho\left(\vec{r'},t\right)P_l\left(-\vec{n'}\cdot\nabla\right)\Delta^n\delta(\vec{r}),\end{aligned} \tag{12}$$

where $\vec{n'} = \frac{\vec{r'}}{r}$. From [11] we have §:

$$P_l\left(\hat{\vec{r}}\cdot\hat{\vec{s}}\right) = \frac{1}{l!}\mathcal{T}_l\,\hat{\vec{r}}^{l}\bullet l\bullet\hat{\vec{s}}^{l}, \tag{13}$$

where $\mathcal{T}_l$ is the detracer operator and $\bullet l\bullet$ means l-fold contraction. The detracer operator $\mathcal{T}_l$ which is defined in [11] transforms any totally symmetric l-th rank tensor to a totally traceless form. We have further:

$$\mathcal{T}_l\vec{n'}^{l} = (-1)^l r'^{l+1}\nabla'^{l}\frac{1}{r'}, \tag{14}$$

§ Between our spherical harmonics and the spherical harmonics from [11] there exists the relation: $Y_{lm}^{(Ref.\ [8])}(\theta,\phi) = (-1)^m\sqrt{\frac{4\pi(l+m)!}{(2l+1)(l-m)!}}Y_{lm}^{(this\ paper)}(\theta,\phi)$. The definitions of the Legendre polynomials are identical.

so it follows that:

$$P_l\left(-\vec{n'}\cdot\nabla\right)=\frac{1}{l!}r'^{l+1}\nabla'^l\frac{1}{r'}\bullet l\bullet\vec{\nabla}^l. \tag{15}$$

From Eqs. (3), (12), (15) it follows that:

$$\sum_m \overline{r_{lm}^{2n}}\Delta^n\delta_{lm}(\vec{r})=\sqrt{\frac{2l+1}{4\pi}}\frac{(-1)^l}{l!}\overline{r_{i_1\ \dots i_l}^{2n}}(t)\Delta^n\partial_{i_1}\dots\partial_{i_l}\delta(\vec{r}) \tag{16}$$

and from (16), (6) we easily obtain Eq. (1).

We prove now Eq. (2). We start from Eq. (7) where the part which contains the electric mean square radii is transformed using the same method described previously. We obtain:

$$\sum_m \overline{r_{lm}^{2n}}\nabla\Delta^{n-1}\delta_{lm}(\vec{r})=\sqrt{\frac{2l{+}1}{4\pi}}\frac{(-1)^l}{l!}\overline{r_{i_1\ \dots i_l}^{2n}}(t)\nabla\Delta^{n-1}\partial_{i_1}\dots\partial_{i_l}\delta(\vec{r}) \tag{17}$$

and

$$\sum_m \overline{r_{lm}^{0}}\Delta^{-1}\nabla\times(\vec{r}\times\nabla)\Delta^n\delta_{lm}(\vec{r}) = \sqrt{\frac{2l+1}{4\pi}}\frac{(-1)^l}{l!}\overline{r_{i_1\dots i_l}^{0}}(t)\Delta^{-1}\nabla \times(\vec{r}\times\nabla)\Delta^n\partial_{i_1}\dots\partial_{i_l}\delta(\vec{r}). \tag{18}$$

In order to transform the parts which contain the magnetic and toroid mean square radii, we need to evaluate the sums: $\sum_m \overline{\rho_{lm}^{2n}}(t)(\vec{r}\times\nabla)\Delta^n\delta_{lm}(\vec{r})$ and $\sum_m \overline{R_{lm}^{2n}}(t)\nabla\times(\vec{r}\times\nabla)\Delta^n\delta_{lm}(\vec{r})$. For this, we make in Eqs. (9), (10) the replacements (see for example [7]):

$$\begin{aligned}\vec{Y}^*_{llm}(\vec{n}) &= \frac{i(\vec{r}\times\nabla)}{\sqrt{l(l+1)}}Y^*_{lm}(\vec{n})\\ \vec{Y}^*_{ll-1m}(\vec{n}) &= \frac{1}{\sqrt{2l+1}}\left(\sqrt{l}\frac{\vec{r}}{r}+\frac{r}{\sqrt{l}}\nabla\right)Y^*_{lm}(\vec{n})\\ \vec{Y}^*_{ll+1m}(\vec{n}) &= \frac{1}{\sqrt{2l+1}}\left(-\sqrt{l+1}\frac{\vec{r}}{r}+\frac{r}{\sqrt{l+1}}\nabla\right)Y^*_{lm}(\vec{n}),\ \vec{n}=\frac{\vec{r}}{r}\end{aligned} \tag{19}$$

and then we use again Eq. (11). One obtains:

$$\sum_m \overline{\rho_{lm}^{2n}}(t)(\vec{r}\times\nabla)\Delta^n\delta_{lm}(\vec{r})=\frac{(-1)^l}{(2l-1)!!l!}\sqrt{\frac{2l+1}{4\pi}}\overline{\rho_{i_1\ \dots i_l}^{2n}}(t)(\vec{r}\times\nabla) \cdot\Delta^n\partial_{i_1}\dots\partial_{i_1}\delta(\vec{r}), \tag{20}$$

$$\sum_m \overline{R_{lm}^{2n}}(t)\nabla\times(\vec{r}\times\nabla)\Delta^n\delta_{lm}(\vec{r})=\frac{(-1)^l}{(2l-1)!!l!}\sqrt{\frac{2l+1}{4\pi}}\overline{R_{i_1\ \dots i_l}^{2n}}(t) \cdot\nabla\times(\vec{r}\times\nabla)\Delta^n\partial_{i_1}\dots\partial_{i_1}\delta(\vec{r}). \tag{21}$$

Introducing Eqs. (17), (18), (20), (21) into Eq. (7) one obtains Eq. (2).

We end this section with some observations about Eqs. (3)–(5). If we explicitly write Eq. (3) for ($l = 1$, $n = 0$) and for ($l = 1, n = 1$), we obtain the well known expressions of the electric dipole and of the first mean square radius of the electric dipole, respectively: $\overline{r_i^0}(t) = \int d^3\xi\xi_i\rho(\vec{\xi}, t)$, $\overline{r_i^2}(t) = \int d^3\xi\xi^2\xi_i\rho(\vec{\xi}, t)$. If we put in Eq. (5) ($l = 1, n = 0$) we obtain the expression of the toroid dipole: $\overline{R_i^0} \equiv t_i = \frac{1}{10c}\int d^3r[r_i(\vec{r}\cdot\vec{j}) - 2r^2 j_i]$. In the case of the magnetic dipole and of the higher order electric, magnetic and toroid mean square radii, some sign differences or even some different normalization constants appear as compared to the definitions which exist in the literature (e.g., [7, 12, 13]). These differences are physically irrelevant, but we have to pay attention to the conventions we are using during the calculation. The advantage of using the definitions (3)–(5) consists in the compact writing of the Cartesian components of the multipoles and of their relation with the spherical components for any multipolarity and order. The relation betwen the Cartesian components of any type of mean square radius (electric, magnetic or toroid), generically noted by $\overline{\Re^{2n}_{i_1\ldots i_l}}(t)$ and its spherical components $\overline{\Re^{2n}_{lm}}(t)$ is:

$$\overline{\Re^{2n}_{lm}}(t) = \frac{(-1)^m}{\sqrt{(l+m)!(l-m)!}}\sum_{k=0}^{m}(-i)^k\begin{pmatrix} m \\ k \end{pmatrix}\overline{\Re^{2n}_{\underbrace{1\ldots1}_{m-k}\underbrace{2\ldots2}_{k}\underbrace{3\ldots3}_{l-m}}}(t), \tag{22}$$

where $\begin{pmatrix} m \\ k \end{pmatrix} = \frac{m!}{k!(m-k)!}$. It can be obtained from the Cartesian expression of the spherical harmonics [14]:

$$\frac{1}{r^{l+1}}Y_{lm}(\theta,\phi) = (-1)^{l+m}\sqrt{\frac{2l+1}{4\pi}\frac{1}{(l+m)!(l-m)!}}\sum_{k=0}^{m} i^k \begin{pmatrix} m \\ k \end{pmatrix} \cdot \partial_1^{m-k}\partial_2^k\partial_3^{l-m}\frac{1}{r}. \tag{23}$$

We have written in this section the Cartesian multipole expansions for an arbitrary source of charges and currents $\rho(\vec{r}, t)$, $\vec{j}(\vec{r}, t)$ which satisfy the continuity equation. The multipole expansion of the field produced by this source can be obtained by using the equations:

$$\begin{aligned} \phi(\vec{r}, t) &= \int d^3\vec{r'}\frac{\rho\left(\vec{r'}, t - \frac{|\vec{r}-\vec{r'}|}{c}\right)}{|\vec{r}-\vec{r'}|}, \\ \vec{A}(\vec{r}, t) &= \frac{1}{c}\int d^3\vec{r'}\frac{\vec{j}\left(\vec{r'}, t - \frac{|\vec{r}-\vec{r'}|}{c}\right)}{|\vec{r}-\vec{r'}|}, \end{aligned} \tag{24}$$

$$\vec{E}(\vec{r},t) = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi(\vec{r},t), \qquad \vec{B}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t).$$

3. ANOTHER PROOF OF EQS. (1), (2)

We present in this section another method for proving Eqs. (1), (2). Having in mind the interesting tensorial decomposition Eq. (2.8) from [2], which illustrates for pedagogical purposes the correct reduction method of the Cartesian multipoles up to the order 3, we prove our Eqs. (1), (2) by transforming them into tensorial identities.

For this, we start from Eqs. (1), (2) (which are to be proved) and write them in the form of Cartesian tensorial identities. Next, we further transform these tensorial identities by using some combinatorics and the properties of the hypergeometric functions, till we get some obvious algebraic identities; then we can conclude that Eqs. (1), (2) from which we have started are correct. The calculation is quite long and it will be presented in short.

We start with the proof of the longer Eq. (2). We write the obvious identity:

$$j_\alpha(\vec{r},t) = \int d^3\xi j_\alpha\left(\vec{\xi},t\right)\delta\left(\vec{r}-\vec{\xi}\right) \tag{25}$$

and then we use the formal Taylor expansion of the function $\delta(\vec{r}-\vec{\xi})$:

$$\delta\left(\vec{r}-\vec{\xi}\right) = \sum_{L=0}^{\infty}\frac{(-1)^L}{L!}\xi_{i_1}\ldots\xi_{i_L}\partial_{i_1}\ldots\partial_{i_L}\delta(\vec{r}). \tag{26}$$

Introducing Eq. (26) into Eq. (25), one obtains:

$$j_\alpha(\vec{r},t) = \sum_{L=0}^{\infty}\frac{(-1)^L}{L!}\left(\int d^3\xi j_\alpha\left(\vec{\xi},t\right)\xi_{i_1}\ldots\xi_{i_L}\right)\partial_{i_1}\ldots\partial_{i_L}\delta(\vec{r}). \tag{27}$$

If we introduce Eqs. (3)–(5), (27) into Eq. (2), after some simple algebric manipulations, the identity we have to prove becomes ‖:

$$\sum_{L=0}^{\infty}\frac{(-1)^L}{L!}\left(\int d^3\xi j_\alpha\left(\vec{\xi},t\right)\xi_{i_1}\ldots\xi_{i_L}\right)\partial_{i_1}\ldots\partial_{i_L}\delta(\vec{r})$$
$$= \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\frac{-(2l+1)}{2^n n!(2l+2n+1)!!l(l+1)!}$$

‖ For abbreviation, we have used the notation $\partial_{i_1\ldots i_l}$ instead of $\partial_{i_1}\ldots\partial_{i_l}$ and we have omitted the argument of $\vec{j}(\vec{\xi},t)$ in the right-hand side. The notation $\partial^{(i_k,i_r)}_{i_1\ldots i_l}$ means that the indices i_k, i_r are missing from the set of the indices $(i_1\ldots i_l)$.

$$\cdot \int d^3\xi \xi^{2l+2n+1} \Bigg[-l^2 j_\alpha \partial_{i_1 \dots i_l} \frac{1}{\xi} + l \sum_{k=1}^{l} j_{i_k} \partial_\alpha \partial^{(i_k)}_{i_1 \dots i_l} \frac{1}{\xi}$$

$$+(l-1) \sum_{k=1}^{l} \left(\vec{j}\cdot\nabla\right) \delta_{\alpha i_k} \partial^{(i_k)}_{i_1 \dots i_l} \frac{1}{\xi} - \sum_{\substack{k,r=1\\ k\neq r}}^{l} \delta_{i_k i_r} (\vec{j}\cdot\nabla) \partial_\alpha \partial^{(i_k,i_r)}_{i_1 \dots i_l} \frac{1}{\xi} \Bigg] \partial_{i_1 \dots i_l} \Delta^n \delta(\vec{r})$$

$$+ \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} \frac{-(2l+1)}{2^{n+1}(n+1)!(2l+2n+3)!!(l+1)!}$$

$$\cdot \int d^3\xi \; \xi^{2l+2n+1} \left[(2l+2n+3)(l+1) \left(\vec{j}\cdot\vec{\xi}\right) + (l+2n+3)\xi^2 \left(\vec{j}\cdot\nabla\right) \right]$$

$$\cdot \partial_{i_1 \dots i_{l-1}} \partial_\alpha \frac{1}{\xi} \partial_{i_1 \dots i_{l-1}} \Delta^{n+1} \delta(\vec{r})$$

$$+ \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} \frac{(2l+1)}{2^{n+1}(n+1)!(2l+2n+3)!!(l+1)!(l+1)}$$

$$\cdot \int d^3\xi \; \xi^{2l+2n+1} \left[(2l+2n+3)(l+1) \left(\vec{j}\cdot\vec{\xi}\right) + (l+2n+3)\xi^2 \left(\vec{j}\cdot\nabla\right) \right]$$

$$\cdot \sum_{k=1}^{l+1} \delta_{\alpha i_k} \partial^{(i_k)}_{i_1 \dots i_{l+1}} \frac{1}{\xi} \partial_{i_1 \dots i_{l+1}} \Delta^n \delta(\vec{r})$$

$$+ \sum_{l=1}^{\infty} \frac{-(2l+1)}{l!(2l-1)!!} \int d^3\xi \; \xi^{2l-1} \left(\vec{j}\cdot\vec{\xi}\right) \partial_\alpha \partial_{i_1 \dots i_{l-1}} \frac{1}{\xi} \partial_{i_1 \dots i_{l-1}} \delta(\vec{r})$$

$$+ \sum_{l=1}^{\infty} \frac{-1}{l!(2l-1)!!} \int d^3\xi \; \xi^{2l+1} \left(\vec{j}\cdot\nabla\right) \partial_\alpha \partial_{i_1 \dots i_{l-1}} \frac{1}{\xi} \partial_{i_1 \dots i_{l-1}} \delta(\vec{r})$$

$$+ \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \Bigg[\frac{-(2l+1)}{2^n n!(2l+2n-1)!!(l+1)!} \int d^3\xi \; \xi^{2l+2n-1} \left(\vec{j}\cdot\vec{\xi}\right)$$

$$\cdot \sum_{k=1}^{l+1} \delta_{\alpha i_k} \partial^{(i_k)}_{i_1 \dots i_{l+1}} \frac{1}{\xi} - \frac{(2l+1)}{2^n n!(2l+2n+1)!!(l+1)!}$$

$$\cdot \int d^3\xi \xi^{2l+2n+1} \left(\vec{j}\cdot\nabla\right) \sum_{k=1}^{l+1} \delta_{\alpha i_k} \partial^{(i_k)}_{i_1 \dots i_{l+1}} \frac{1}{\xi} \Bigg] \Delta^{n-1} \partial_{i_1 \dots i_{l+1}} \delta(\vec{r}). \quad (28)$$

Next, we make some changes of the summation indices in the right hand side (r.h.s.) of the above equation, so that we have L derivatives of $\delta(\vec{r})$ in all the terms. So, in the first term of the r.h.s. we make the replacement: $(l, n) \to (L, n)$: $l + 2n = L$, in the second and in the third: $(l, n) \to (L, n)$: $l + 2n + 1 = L$, in the fourth and in the fifth: $l \to L$: $l - 1 = L$. In the last term we make the replacement $(l, n) \to (L, n)$: $l+2n-1 = L$, and after that $n \to N$: $n = N+1$. Then, we introduce Eq. (A1) in Eq. (28). It follows then the symmetrization in the derivative indices of $\delta(\vec{r})$, using Eq. (A2) from the Appendix. We emphasize that all these symmetrizations do not imply the change of the r.h.s. (by additions or subtractions), but simply the renotation of some summation indices.

We give an example: let us consider the quantity ¶ : $T \equiv \mathcal{P}(\delta^{\kappa+1}_{i_a i_b} \xi^{L-2n-2\kappa-2}_{i_c})_{i_1 \ldots i_{L-2n}} \partial_{i_1 \ldots i_{L-2n}} \Delta^n \delta(\vec{r})$, which can be written in the form: $T = \mathcal{P}(\delta^{\kappa+1}_{i_a i_b} \xi^{L-2n-2\kappa-2}_{i_c})_{i_1 \ldots i_{L-2n}} \delta_{i_{L-2n+1} i_{L-2n+2}} \ldots \delta_{i_{L-1} i_L} \partial_{i_1 \ldots i_L} \delta(\vec{r})$. According to the formula (A2) from Appendix, it contains $\frac{(L-2n)!}{2^{\kappa+1}(L-2n-2\kappa-2)!(\kappa+1)!}$ terms. On the other hand, the quantity:

$$T^{sym} = \mathcal{P}(\delta^{\kappa+n+1}_{i_a i_b} \xi^{L-2n-2\kappa-2}_{i_c})_{i_1 \ldots i_L} \partial_{i_1 \ldots i_L} \delta(\vec{r}).$$

contains $\frac{(L)!}{2^{\kappa+n+1}(L-2n-2\kappa-2)!(\kappa+n+1)!}$ terms. Therefore, we perform the symmetrization of all the derivative indices of the function $\delta(\vec{r})$ by putting:

$$\begin{aligned} &\mathcal{P}(\delta^{\kappa+1}_{i_a i_b} \xi^{L-2n-2\kappa-2}_{i_c})_{i_1 \ldots i_{L-2n}} \delta_{i_{L-2n+1} i_{L-2n+2}} \ldots \delta_{i_{L-1} i_L} \partial_{i_1 \ldots i_L} \delta(\vec{r}) \\ &= \frac{2^n (L-2n)!(\kappa+n+1)!}{L!(\kappa+1)!)} \mathcal{P}(\delta^{\kappa+n+1}_{i_a i_b} \xi^{L-2n-2\kappa-2}_{i_c})_{i_1 \ldots i_L} \partial_{i_1 \ldots i_L} \delta(\vec{r}) \,. \end{aligned}$$

After the symmetrization of all the terms from the r.h.s. of the Eq. (28) and after some simplifications, we see that the proof of the Eq. (2) is equivalent with the proof of the following tensorial identity:

$$j_\alpha \xi_{i_1} \ldots \xi_{i_L} = T_1 + T_2 + T_3 + T_4 + T_5, \tag{29}$$

where:

$$\begin{aligned} T_1 = \frac{(-1)}{(L+1)(2L+1)!!} &\left[\frac{\xi_\alpha \left(\vec{j} \cdot \vec{\xi}\right)}{\xi^2} \sum_{\kappa=0}^{\left[\frac{L-2}{2}\right]} (-1)^\kappa (2\kappa+2)\,(2L-2\kappa-1)!! \xi^{2\kappa+2} \right. \\ &\cdot \mathcal{P}\left(\delta^{\kappa+1}_{i_a i_b} \xi^{L-2\kappa-2}_{i_c}\right)_{i_1 \ldots i_L} \end{aligned}$$

¶ We use the notation: $\mathcal{P}(\delta^k_{i_a i_b} \xi^{L-2k}_{i_c})_{i_1 \ldots i_L} \equiv \mathcal{P}(\delta_{i_1 i_2} \ldots \delta_{i_{2k-1} i_{2k}} \xi_{i_{2k+1}} \cdots \xi_{i_L})_{i_1 \ldots i_L}$ which means all the distinct terms obtained by all the permutations of the indices $i_1 \ldots i_L$, which contain k Kronecker symbols and $L - k$ coordinates ξ. Obviously, the notation $\mathcal{P}(\delta^\kappa_{i_a i_b} \delta_{\alpha i_c} j_{i_d} \xi^{L-2\kappa-2}_{i_e})_{i_1 \ldots i_L} \equiv \mathcal{P}(\delta_{i_1 i_2} \ldots \delta_{i_{2\kappa-1} i_{2\kappa}} \delta_{\alpha i_{2\kappa+1}} j_{i_{2\kappa+2}} \xi_{i_{2\kappa+3}} \cdots \xi_{i_L})_{i_1 \ldots i_L}$ means all the distinct terms obtained by all the permutations of the indices $i_1 \ldots i_L$, which contain κ Kronecker symbols of the type $\delta_{i_k i_j}$, one Kronecker symbol of the type $\delta_{\alpha i_p}$, one component of the current j_{i_p} and $L - 2\kappa - 2$ coordinates ξ_i.

$$+\left(\vec{j}\cdot\vec{\xi}\right)\sum_{\kappa=0}^{\left[\frac{L-1}{2}\right]}(-1)^{\kappa}(2\kappa+2)(2L-2\kappa-1)!!\xi^{2\kappa}\mathcal{P}\left(\delta^{\kappa}_{i_a i_b}\delta_{\alpha i_c}\xi^{L-2\kappa-1}_{i_d}\right)_{i_1\ldots i_L}$$
$$-j_\alpha\sum_{\kappa=0}^{\left[\frac{L}{2}\right]}(-1)^{\kappa}(2L-2\kappa+1)!!\xi^{2\kappa}\mathcal{P}\left(\delta^{\kappa}_{i_a i_b}\xi^{L-2\kappa}_{i_c}\right)_{i_1\ldots i_L}$$
$$-\xi_\alpha\sum_{\kappa=0}^{\left[\frac{L-1}{2}\right]}(-1)^{\kappa}(2L-2\kappa+1)!!\xi^{2\kappa}\mathcal{P}\left(\delta^{\kappa}_{i_a i_b}j_{i_c}\xi^{L-2\kappa-1}_{i_d}\right)_{i_1\ldots i_L}$$
$$\left.+\xi^2\sum_{\kappa=0}^{\left[\frac{L-2}{2}\right]}(-1)^{\kappa}(2L-2\kappa-1)!!\xi^{2\kappa}\mathcal{P}\left(\delta^{\kappa}_{i_a i_b}\delta_{\alpha i_c}j_{i_d}\xi^{L-2\kappa-2}_{i_e}\right)_{i_1\ldots i_L}\right],$$

$$T_2=\sum_{n=0}^{\left[\frac{L-2}{2}\right]}\frac{(2L-4n-1)}{2(n+1)!(2L-2n+1)!!}\xi^{2n}\left[\left(\vec{j}\cdot\vec{\xi}\right)\sum_{\kappa=0}^{\left[\frac{L-2n-1}{2}\right]}(-1)^{\kappa}(2n+2\kappa+1)(2L-4n\right.$$
$$-2\kappa-3)!!\xi^{2\kappa}\frac{(\kappa+n)!}{\kappa!}\mathcal{P}\left(\delta^{\kappa+n}_{i_a i_b}\delta_{\alpha i_c}\xi^{L-2n-2\kappa-1}_{i_d}\right)_{i_1\ldots i_L}$$
$$+\xi^2\sum_{\kappa=0}^{\left[\frac{L-2n-2}{2}\right]}(-1)^{\kappa}(2L-4n-2\kappa-3)!!\xi^{2\kappa}\frac{(\kappa+n)!}{\kappa!}$$
$$\left.\cdot\mathcal{P}\left(\delta^{\kappa+n}_{i_a i_b}\delta_{\alpha i_c}j_{i_d}\xi^{L-2n-2\kappa-2}_{i_e}\right)_{i_1\ldots i_L}\right],$$

$$T_3=\sum_{n=0}^{\left[\frac{L-1}{2}\right]}\frac{-(2L-4n+1)}{n!(2L-2n+1)!!(L-2n+1)(L-2n)}\xi^{2n}$$
$$\cdot\left\{-j_\alpha\,(L-2n)^2\,(2L-4n-1)!!n!\mathcal{P}\left(\delta^{n}_{i_a i_b}\xi^{L-2n}_{i_c}\right)_{i_1\ldots i_L}\right.$$
$$+j_\alpha\xi^2\sum_{\kappa=0}^{\left[\frac{L-2n-2}{2}\right]}(-1)^{\kappa}\left[(L-2n)^2+2\kappa+2\right](2L-4n-2\kappa-3)!!\xi^{2\kappa}\frac{(\kappa+n+1)!}{(\kappa+1)!}$$
$$\cdot\mathcal{P}\left(\delta^{\kappa+n+1}_{i_a i_b}\xi^{L-2n-2\kappa-2}_{i_c}\right)_{i_1\ldots i_L}$$
$$+(L-2n)\xi_\alpha\sum_{\kappa=0}^{\left[\frac{L-2n-1}{2}\right]}(-1)^{\kappa}(2L-4n-2\kappa-1)!!\xi^{2\kappa}\frac{(\kappa+n)!}{\kappa!}$$
$$\cdot\mathcal{P}\left(\delta^{\kappa+n}_{i_a i_b}j_{i_c}\xi^{L-2n-2\kappa-1}_{i_d}\right)_{i_1\ldots i_L}$$
$$-(2L-4n-1)\xi^2\sum_{\kappa=0}^{\left[\frac{L-2n-2}{2}\right]}(-1)^{\kappa}(2L-4n-2\kappa-3)!!\xi^{2\kappa}\frac{(\kappa+n)!}{\kappa!}$$
$$\cdot\mathcal{P}\left(\delta^{\kappa+n}_{i_a i_b}\delta_{\alpha i_c}j_{i_d}\xi^{L-2n-2\kappa-2}_{i_e}\right)_{i_1\ldots i_L}$$
$$+(L-2n-1)\left(\vec{j}\cdot\vec{\xi}\right)\sum_{\kappa=0}^{\left[\frac{L-2n-1}{2}\right]}(-1)^{\kappa}(2L-4n-2\kappa-1)!!\xi^{2\kappa}\frac{(\kappa+n)!}{\kappa!}$$
$$\cdot\mathcal{P}\left(\delta^{\kappa+n}_{i_a i_b}\delta_{\alpha i_c}-\xi^{L-2n-2\kappa-1}_{i_d}\right)_{i_1\ldots i_L}$$

$$-2\left(\vec{j}\cdot\vec{\xi}\right)\xi_\alpha\sum_{\kappa=0}^{\left[\frac{L-2n-2}{2}\right]}(-1)^\kappa(2L-4n-2\kappa-1)!!\xi^{2\kappa}\frac{(\kappa+n+1)!}{\kappa!}$$
$$\cdot\mathcal{P}\left(\delta_{i_ai_b}^{\kappa+n+1}\xi_{i_c}^{L-2n-2\kappa-2}\right)_{i_1\ldots i_L}$$
$$+2\xi_\alpha\xi^2\sum_{\kappa=0}^{\left[\frac{L-2n-3}{2}\right]}(-1)^\kappa(2L-4n-2\kappa-3)!!\xi^{2\kappa}\frac{(\kappa+n+1)!}{\kappa!}$$
$$\cdot\mathcal{P}\left(\delta_{i_ai_b}^{\kappa+n+1}j_{i_c}\xi_{i_d}^{L-2n-2\kappa-3}\right)_{i_1\ldots i_L}$$
$$+2\left(\vec{j}\cdot\vec{\xi}\right)\xi^2\sum_{\kappa=0}^{\left[\frac{L-2n-3}{2}\right]}(-1)^\kappa(2L-4n-2\kappa-3)!!\xi^{2\kappa}\frac{(\kappa+n+1)!}{\kappa!}$$
$$\cdot\mathcal{P}\left(\delta_{i_ai_b}^{\kappa+n+1}\delta_{\alpha i_c}\xi_{i_d}^{L-2n-2\kappa-3}\right)_{i_1\ldots i_L}$$
$$-2\xi^4\sum_{\kappa=0}^{\left[\frac{L-2n-4}{2}\right]}(-1)^\kappa(2L-4n-2\kappa-5)!!\xi^{2\kappa}\frac{(\kappa+n+1)!}{\kappa!}$$
$$\left.\cdot\mathcal{P}\left(\delta_{i_ai_b}^{\kappa+n+1}\delta_{\alpha i_c}j_{i_d}\xi_{i_e}^{L-2n-2\kappa-4}\right)_{i_1\ldots i_L}\right\},$$

$$T_4=\sum_{n=0}^{\left[\frac{L-2}{2}\right]}\frac{-(2L-4n-1)}{(n+1)!(2L-2n+1)!!(L-2n)(L-2n-1)}\xi^{2n}\left\{\xi_\alpha(\vec{j}\cdot\vec{\xi})\sum_{\kappa=0}^{\left[\frac{L-2n-2}{2}\right]}(-1)^\kappa\right.$$
$$\cdot(2L-4n-2\kappa-3)!!\Big[-(2L-2n+1)(L-2n)+(L+2)(2L-4n-2\kappa-1)\Big]$$
$$\cdot\xi^{2\kappa}\frac{(\kappa+n+1)!}{\kappa!}\mathcal{P}\left(\delta_{i_ai_b}^{\kappa+n+1}\xi_{i_c}^{L-2n-2\kappa-2}\right)_{i_1\ldots i_L}$$
$$+\left(\vec{j}\cdot\vec{\xi}\right)\xi^2\sum_{\kappa=0}^{\left[\frac{L-2n-3}{2}\right]}(-1)^\kappa(2L-4n-2\kappa-5)!!\Big[(2L-2n+1)(L-2n)$$
$$-(L+2)(2L-4n-2\kappa-3)\Big]\xi^{2\kappa}\frac{(\kappa+n+1)!}{\kappa!}\mathcal{P}\left(\delta_{i_ai_b}^{\kappa+n+1}\delta_{\alpha i_c}\xi_{i_d}^{L-2n-2\kappa-3}\right)_{i_1\ldots i_L}$$
$$-(L+2)j_\alpha\xi^2\sum_{\kappa=0}^{\left[\frac{L-2n-2}{2}\right]}(-1)^\kappa(2L-4n-2\kappa-3)!!\xi^{2\kappa}\frac{(\kappa+n+1)!}{\kappa!}$$
$$\cdot\mathcal{P}\left(\delta_{i_ai_b}^{\kappa+n+1}\xi_{i_c}^{L-2n-2\kappa-2}\right)_{i_1\ldots i_L}$$
$$-(L+2)\xi_\alpha\xi^2\sum_{\kappa=0}^{\left[\frac{L-2n-3}{2}\right]}(-1)^\kappa(2L-4n-2\kappa-3)!!\xi^{2\kappa}\frac{(\kappa+n+1)!}{\kappa!}$$
$$\cdot\mathcal{P}\left(\delta_{i_ai_b}^{\kappa+n+1}j_{i_c}\xi_{i_d}^{L-2n-2\kappa-3}\right)_{i_1\ldots i_L}$$
$$+(L+2)\xi^4\sum_{\kappa=0}^{\left[\frac{L-2n-4}{2}\right]}(-1)^\kappa(2L-4n-2\kappa-5)!!\xi^{2\kappa}\frac{(\kappa+n+1)!}{\kappa!}$$
$$\left.\cdot\mathcal{P}\left(\delta_{i_ai_b}^{\kappa+n+1}\delta_{\alpha i_c}j_{i_d}\xi_{i_e}^{L-2n-2\kappa-4}\right)_{i_1\ldots i_L}\right\}$$

and

$$T_5 = \sum_{n=0}^{\left[\frac{L-2}{2}\right]} \frac{(2L-4n-1)}{2(n+1)!(2L-2n+1)!!(L-2n)} \xi^{2n} \left\{ (\vec{j}\cdot\vec{\xi}) \sum_{\kappa=0}^{\left[\frac{L-2n-1}{2}\right]} (-1)^\kappa \right.$$
$$\cdot(2L-4n-2\kappa-3)!! \Big[-(2L-2n+1)(L-2n) + (L+2)$$
$$\cdot(2L-4n-2\kappa-1) \Big] \xi^{2\kappa} \tfrac{(\kappa+n)!}{\kappa!} \mathcal{P}(\delta_{i_a i_b}^{\kappa+n} \delta_{\alpha i_c} \xi_{i_d}^{L-2n-2\kappa-1})_{i_1\ldots i_L}$$
$$-(L+2)\xi^2 \sum_{\kappa=0}^{\left[\frac{L-2n-2}{2}\right]} (-1)^\kappa (2L-4n-2\kappa-3)!! \xi^{2\kappa} \tfrac{(\kappa+n)!}{\kappa!}$$
$$\left. \cdot\mathcal{P}(\delta_{i_a i_b}^{\kappa+n} \delta_{\alpha i_c} j_{i_d} \xi_{i_e}^{L-2n-2\kappa-2})_{i_1\ldots i_L} \right\}.$$

If we put $L = 2$ in Eq. (29), we obtain the identity (2.8) from [2].

We now proceed to the proof of Eq. (29). First, we note that a part of T_1 and a part of T_3 give exactly the left hand side:

$$\frac{(-1)}{(L+1)(2L+1)!!} \left[-j_\alpha \sum_{\kappa=0}^{\left[\frac{L}{2}\right]} (-1)^\kappa (2L-2\kappa+1)!! \xi^{2\kappa} \mathcal{P}\left(\delta_{i_a i_b}^{\kappa} \xi_{i_c}^{L-2\kappa}\right)_{i_1\ldots i_L} \right]_{|\kappa=0}$$
$$+ \sum_{n=0}^{\left[\frac{L-1}{2}\right]} \frac{-(2L-4n+1)}{n!(2L-2n+1)!!(L-2n+1)(L-2n)} \xi^{2n} \Big[-j_\alpha (L-2n)^2$$
$$\cdot(2L-4n-1)!! n! \mathcal{P}(\delta_{i_a i_b}^{n} \xi_{i_c}^{L-2n})_{i_1\ldots i_L} \Big]_{|n=0} = j_\alpha \xi_{i_1} \ldots \xi_{i_L}. \qquad (30)$$

Then we notice that in the r.h.s. of Eq. (29) there exist five categories of terms: $j_\alpha(\ldots)$, $\xi_\alpha(\ldots)$, $\xi^2(\ldots)$, $(\vec{j}\cdot\vec{\xi})(\ldots)$, and $\xi_\alpha(\vec{j}\cdot\vec{\xi})(\ldots)$. We are analysing them one by one. For example, let us analyse the coefficient of ξ_α from the r.h.s.:

$$C(\xi_\alpha) = \sum_{n=0}^{\left[\frac{L-1}{2}\right]} \sum_{\kappa=0}^{\left[\frac{L-2n-1}{2}\right]} \frac{-(2L-4n+1)}{n!(2L-2n+1)!!(L-2n+1)} (-1)^\kappa (2L-4n-2\kappa-1)!!$$
$$\cdot\xi^{2n+2\kappa} \frac{(\kappa+n)!}{\kappa!} \mathcal{P}\left(\delta_{i_a i_b}^{\kappa+n} j_{i_c} \xi_{i_d}^{L-2n-2\kappa-1}\right)_{i_1\ldots i_L}$$
$$+ \sum_{n=0}^{\left[\frac{L-3}{2}\right]} \sum_{\kappa=0}^{\left[\frac{L-2n-3}{2}\right]} \frac{-2(2L-4n+1)}{n!(2L-2n+1)!!(L-2n+1)(L-2n)} (-1)^\kappa$$
$$\cdot(2L-4n-2\kappa-3)!! \xi^{2n+2\kappa+2} \frac{(\kappa+n+1)!}{\kappa!} \mathcal{P}\left(\delta_{i_a i_b}^{\kappa+n+1} j_{i_c} \xi_{i_d}^{L-2n-2\kappa-3}\right)_{i_1\ldots i_L}$$

$$+\sum_{n=0}^{\left[\frac{L-3}{2}\right]}\sum_{\kappa=0}^{\left[\frac{L-2n-3}{2}\right]}\frac{(2L-4n-1)(L+2)}{(n+1)!(2L-2n+1)!!(L-2n-1)(L-2n)}(-1)^{\kappa}$$
$$\cdot(2L-4n-2\kappa-3)!!\xi^{2n+2\kappa+2}\frac{(\kappa+n+1)!}{\kappa!}\mathcal{P}\left(\delta_{i_ai_b}^{\kappa+n+1}j_{i_c}\xi_{i_d}^{L-2n-2\kappa-3}\right)_{i_1\ldots i_L}$$
$$+\frac{1}{(L+1)(2L+1)!!}\sum_{\kappa=0}^{\left[\frac{L-1}{2}\right]}(-1)^{\kappa}(2L-2\kappa+1)!!\xi^{2\kappa}\mathcal{P}\left(\delta_{i_ai_b}^{\kappa}j_{i_c}\xi_{i_d}^{L-2\kappa-1}\right)_{i_1\ldots i_L}.$$

We shall prove that $C(\xi_\alpha)=0$. We group the terms now according to the powers of $\delta_{i_ai_b}$. For that, we make some changes of the summation indices. In the first term we put: $(n,\kappa)\to(K,n)$: $\kappa+n=K$, in the second and the third: $(n,\kappa)\to(K,n)$: $\kappa+n+1=K$, and in the last term we simply rename: $\kappa\to K$. After some simplifications, one obtains:

$$C(\xi_\alpha) = \left[\sum_{K=1}^{\left[\frac{L-1}{2}\right]}\sum_{n=0}^{K}\frac{(-1)^{K+n+1}(2L-4n+1)}{n!(2L-2n+1)!!(L-2n+1)}\right.$$
$$\cdot(2L-2n-2K-1)!!\frac{K!}{(K-n)!}$$
$$+\sum_{K=1}^{\left[\frac{L-1}{2}\right]}\sum_{n=0}^{K-1}\frac{2(-1)^{K+n}(2L-4n+1)}{n!(2L-2n+1)!!(L-2n+1)(L-2n)}$$
$$\cdot(2L-2n-2K-1)!!\frac{K!}{(K-n-1)!}$$
$$+\sum_{K=1}^{\left[\frac{L-1}{2}\right]}\sum_{n=0}^{K-1}\frac{(-1)^{K+n+1}(2L-4n-1)(L+2)}{(n+1)!(2L-2n+1)!!(L-2n-1)(L-2n)}$$
$$\cdot(2L-2n-2K-1)\frac{K!}{(K-n-1)!}$$
$$\left.+\sum_{K=1}^{\left[\frac{L-1}{2}\right]}\frac{(-1)^{K}}{(L+1)(2L+1)}(2L-2K+1)!!\right]\int d^3\xi\,\xi^{2k}\mathcal{P}(\delta_{i_ai_b}^{K}j_{i_c}\xi_{i_d}^{L-2K-1})_{i_1\ldots i_L}.$$

Now, let K be fixed in the interval $\overline{1,\left[\frac{L-1}{2}\right]}$. Decomposing in partial fractions all the terms and rearranging them, one obtains:

$$C(\xi_\alpha; K = fixed) = \left[-2(-1)^K K! \sum_{n=0}^{K} \frac{(-1)^n}{n!(K-n)!} \frac{(2L-2n-2K-1)!!}{(2L-2n+1)!!} \right.$$

$$+ \frac{(-1)^K K!}{(L+1)} \sum_{n=0}^{K} \frac{(-1)^n}{n!(K-n)!} \frac{1}{(1-\frac{2}{L+1}n)} \frac{(2L-2n-2K-1)!!}{(2L-2n+1)!!}$$

$$+ \frac{2(-1)^K K!}{(L+1)} \sum_{n=0}^{K-1} \frac{(-1)^n}{n!(K-n-1)!} \frac{1}{(1-\frac{2}{(L+1)}n)} \frac{(2L-2n-2K-1)!!}{(2L-2n+1)!!}$$

$$+ \frac{2(-1)^K K!}{L} \sum_{n=0}^{K-1} \frac{(-1)^n}{n!(K-n-1)!} \frac{1}{(1-\frac{2}{L}n)} \frac{(2L-2n-2K-1)!!}{(2L-2n+1)!!}$$

$$+ (-1)^K K! \sum_{n=0}^{K} \frac{(-1)^n}{n!(K-n)!} \frac{1}{(1-\frac{2}{L+2}n)} \frac{(2L-2n-2K+1)!!}{(2L-2n+3)!!}$$

$$\left. + \frac{(L+2)(-1)^K K!}{(L+1)} \sum_{n=0}^{K} \frac{(-1)^n}{n!(K-n)!} \frac{1}{(1-\frac{2}{L+1}n)} \frac{(2L-2n-2K+1)!!}{(2L-2n+3)!!} \right]$$

$$\int d^3\xi \, \xi^{2K} \mathcal{P}(\delta^K_{i_a i_b} j_{i_c} \xi^{L-2K-1}_{i_d})_{i_1 \ldots i_L}.$$

We notice that, the calculation of $C(\xi_\alpha; K = fixed)$ reduces to three sums, which can be easily evaluated by the means of the formulas (A7)–(A15) from Appendix:

$$\sum_{n=0}^{K} \frac{(-1)^n}{n!(K-n)!} \frac{1}{1-cn} \frac{(2L-2n-2K-1)!!}{(2L-2n+1)!!}$$
$$= \begin{cases} \frac{(-1)^K}{2^{K+1}} \frac{\Gamma(L-2K+\frac{1}{2})}{\Gamma(L+\frac{3}{2})} \begin{pmatrix} 2K \\ K \end{pmatrix}, \; c = 0 \\ \frac{1}{K!2^{K+1}} \frac{\Gamma(L-K+\frac{1}{2})}{\Gamma(L+\frac{3}{2})} \; {}_3F_2\left(\left. \begin{matrix} -K, -L-\frac{1}{2}, -\frac{1}{c} \\ -L+K+\frac{1}{2}, 1-\frac{1}{c}, \end{matrix} \right| 1 \right), \; c \neq 0 \end{cases} \tag{31}$$

$$\sum_{n=0}^{K-1} \frac{(-1)^n}{n!(K-n-1)!} \frac{1}{1-cn} \frac{(2L-2n-2K-1)!!}{(2L-2n+1)!!}$$
$$= \begin{cases} \frac{(-1)^{K-1}}{2^{K+1}} \frac{\Gamma\left(L-2K+\frac{3}{2}\right)}{\Gamma\left(L+\frac{3}{2}\right)} \begin{pmatrix} 2K-1 \\ K-1 \end{pmatrix}, \; c = 0 \\ \frac{1}{(K-1)!2^{K+1}} \frac{\Gamma\left(L-K+\frac{1}{2}\right)}{\Gamma\left(L+\frac{3}{2}\right)} \; {}_3F_2\left(\left. \begin{matrix} -K+1, -L-\frac{1}{2}, -\frac{1}{c} \\ -L+K+\frac{1}{2}, 1-\frac{1}{c} \end{matrix} \right| 1 \right), \; c \neq 0 \end{cases} \tag{32}$$

$$\sum_{n=0}^{K}\frac{(-1)^n}{n!(K-n)!}\frac{1}{1-cn}\frac{(2L-2n-2K+1)!!}{(2L-2n+3)!!}$$
$$=\begin{cases}\frac{(-1)^K}{2^{K+1}}\frac{\Gamma\left(L-2K+\frac{3}{2}\right)}{\Gamma\left(L+\frac{5}{2}\right)}\binom{2K}{K},\ c=0\\ \frac{1}{(K)!2^{K+1}}\frac{\Gamma\left(L-K+\frac{3}{2}\right)}{\Gamma\left(L+\frac{5}{2}\right)}\,{}_3F_2\left(\begin{matrix}-K,-L-\frac{3}{2},-\frac{1}{c}\\ -L+K-\frac{1}{2},1-\frac{1}{c}\end{matrix}\middle|\,1\right),\ c\neq 0\end{cases}\tag{33}$$

Using (31), (32), and (33) in the expression of $C(\xi_\alpha;K=fixed)$, one obtains (leaving aside the common factors):

$$\begin{aligned}C(\xi_\alpha;K=fixed)&\sim 2(-1)^{K+1}\Gamma\left(L-2K+\frac{1}{2}\right)\frac{(2K)!}{K!}\\
&+\frac{1}{L+1}\Gamma\left(L-K+\frac{1}{2}\right){}_3F_2\left(\begin{matrix}-K,-L-\frac{1}{2},-\frac{L}{2}-\frac{1}{2}\\ -L+K+\frac{1}{2},-\frac{L}{2}+\frac{1}{2}\end{matrix}\middle|\,1\right)\\
&+\frac{2K}{L+1}\Gamma\left(L-K+\frac{1}{2}\right){}_3F_2\left(\begin{matrix}-K+1,-L-\frac{1}{2},-\frac{L}{2}-\frac{1}{2}\\ -L+K+\frac{1}{2},-\frac{L}{2}+\frac{1}{2}\end{matrix}\middle|\,1\right)\\
&+\frac{2K}{L}\Gamma\left(L-K+\frac{1}{2}\right){}_3F_2\left(\begin{matrix}-K+1,-L-\frac{1}{2},-\frac{L}{2}\\ -L+K+\frac{1}{2},1-\frac{L}{2}\end{matrix}\middle|\,1\right)\\
&+\frac{1}{L+\frac{3}{2}}\Gamma\left(L-K+\frac{3}{2}\right){}_3F_2\left(\begin{matrix}-K,-L-\frac{3}{2},-\frac{L}{2}-1\\ -L+K-\frac{1}{2},-\frac{L}{2}\end{matrix}\middle|\,1\right)\\
&+\frac{L+2}{(L+1)(l+\frac{3}{2})}\Gamma\left(L-K+\frac{3}{2}\right){}_3F_2\left(\begin{matrix}-K,-L-\frac{3}{2},-\frac{L}{2}-\frac{1}{2}\\ -L+K-\frac{1}{2},-\frac{L}{2}+\frac{1}{2}\end{matrix}\middle|\,1\right).\end{aligned}\tag{34}$$

Now, using the formula (A3) from Appendix for the following two sets of parameters:

$$\begin{cases}a\to -K+1\\ b\to -L-\frac{1}{2}\\ c\to -\frac{L}{2}\\ d\to -L+K+\frac{1}{2}\\ e\to 1-\frac{L}{2}\end{cases}\quad\text{and}\quad\begin{cases}a\to -K\\ b\to -L-\frac{3}{2}\\ c\to -\frac{L}{2}-1\\ d\to -L+K-\frac{1}{2}\\ e\to -\frac{L}{2}\end{cases}$$

one obtains:

$$C(\xi_\alpha; K = fixed) \sim 2(-1)^{K+1}\Gamma\left(L-2K+\frac{1}{2}\right)\frac{(2K)!}{K!}$$
$$+\frac{1}{L+1}\Gamma\left(L-K+\frac{1}{2}\right){}_3F_2\left(\begin{array}{c} -K,-L-\frac{1}{2},-\frac{L}{2}-\frac{1}{2} \\ -L+K+\frac{1}{2},-\frac{L}{2}+\frac{1}{2}\end{array}\middle| 1\right)$$
$$+\frac{2K}{L+1}\Gamma\left(L-K+\frac{1}{2}\right){}_3F_2\left(\begin{array}{c} -K+1,-L-\frac{1}{2},-\frac{L}{2}-\frac{1}{2} \\ -L+K+\frac{1}{2},-\frac{L}{2}+\frac{1}{2}\end{array}\middle| 1\right)$$
$$-2\frac{\left(-L+K-\frac{1}{2}\right)}{L+1}\Gamma\left(L-K+\frac{1}{2}\right){}_3F_2\left(\begin{array}{c} -K,-L-\frac{1}{2},-\frac{L}{2}-\frac{1}{2} \\ -L+K-\frac{1}{2},-\frac{L}{2}+\frac{1}{2}\end{array}\middle| 1\right). \quad (35)$$

Next, we use Eq. (A4) from Appendix, for the parameters:

$$\begin{cases} p \to 3 & a_1 \to -\frac{L}{2}-\frac{1}{2} \\ q \to 2 & a_2 \to -K \\ \rho \to -\frac{L}{2}-\frac{1}{2} & b_1 \to -L+K+\frac{1}{2} \\ \sigma \to -L+K-\frac{1}{2} & \end{cases}$$

and, thereafter, Eq. (A5) for ${}_2F_1\left(\begin{array}{c} -K,-L-\frac{1}{2} \\ -L+K+\frac{1}{2}\end{array}\middle| 1\right)$. One easily obtains $C(\xi_\alpha) = 0$.

In the same way, one analyse the coefficients for j_α, ξ^2, $(\vec{j}\cdot\vec{\xi})$ and $\xi_\alpha(\vec{j}\cdot\vec{\xi})$ excepting the terms from Eq. (30). One obtains that they are all zero, and thus the identity (29) is proved.

The procedure for the multipole expansion of the charge is much simpler than that for the current. If we use the obvious identity: $\rho(\vec{r},t) = \int d^3\xi\ \rho\left(\vec{\xi},t\right)\delta(\vec{r}-\vec{\xi})$ and the formal Taylor expansion of the delta function Eq. (26) it follows:

$$\rho(\vec{r},t) = \sum_{L=0}^{\infty}\frac{(-1)^L}{L!}\left(\int d^3\xi\ \rho\left(\vec{\xi},t\right)\xi_{i_1}\dots\xi_{i_L}\right)\partial_{i_1}\dots\partial_{i_L}\delta(\vec{r}) \quad (36)$$

Introducing Eq. (36) in Eq. (1), one obtains:

$$\sum_{L=0}^{\infty}\frac{(-1)^L}{L!}\left(\int d^3\xi\ \rho(\vec{\xi},t)\xi_{i_1}\dots\xi_{i_L}\right)\partial_{i_1}\dots\partial_{i_L}\delta(\vec{r})$$
$$= \sum_{l,n}\frac{(-1)^l(2l+1)!!}{2^n n!l!(2l+2n+1)!!}\overline{r^{2n}_{i_1\dots i_l}}(t)\Delta^n\partial_{i_1}\dots\partial_{i_l}\delta(\vec{r}), \quad (37)$$

which represents another form of the equation to be proved. We change the summation indices $(l, n) \to (L, n)$: $l + 2n = L$. One obtains:

$$\sum_{L=0}^{\infty} \frac{(-1)^L}{L!} \left(\int d^3\xi\, \rho\left(\vec{\xi}, t\right) \xi_{i_1} \dots \xi_{i_L} \right) \partial_{i_1} \dots \partial_{i_L} \delta(\vec{r})$$
$$= \sum_{L=0}^{\infty} \sum_{n=0}^{\left[\frac{L}{2}\right]} \frac{(-1)^L (2L-4n+1)!!}{2^n n!(L-2n)!(2L-2n+1)!!} \overline{r^{2n}_{i_1 \dots i_{L-2n}}}(t) \Delta^n \partial_{i_1} \dots \partial_{i_{L-2n}} \delta(\vec{r}). \quad (38)$$

In the r.h.s. of Eq. (38) we replace the charge mean square radius $\overline{r^{2n}_{i_1 \dots i_{L-2n}}}(t)$ according to the definition Eq. (3). We obtain:

$$\sum_{L=0}^{\infty} \frac{(-1)^L}{L!} \left(\int d^3\xi\, \rho\left(\vec{\xi}, t\right) \xi_{i_1} \dots \xi_{i_L} \right) \partial_{i_1} \dots \partial_{i_L} \delta(\vec{r})$$
$$= \sum_{L=0}^{\infty} \sum_{n=0}^{\left[\frac{L}{2}\right]} \frac{(2L-4n+1)}{2^n n!(L-2n)!(2L-2n+1)!!}$$
$$\cdot \left(\int d^3\xi\, \xi^{2L-2n+1} \rho(\vec{\xi}, t) \partial_{i_1} \dots \partial_{i_{L-2n}} \frac{1}{\xi} \right) \Delta^n \partial_{i_1} \dots \partial_{i_{L-2n}} \delta(\vec{r}). \quad (39)$$

Then we transform the r.h.s. of Eq. (39) by applying the formula (A1) from Appendix. It follows:

$$r.h.s. = \sum_{L=0}^{\infty} \sum_{n=0}^{\left[\frac{L}{2}\right]} \frac{(2L-4n+1)(-1)^L}{2^n n!(L-2n)!(2L-2n+1)!!} \xi^{2n}$$
$$\cdot \left(\int d^3\xi \rho\left(\vec{\xi}, t\right) \sum_{k=0}^{\left[\frac{L-2n}{2}\right]} (-1)^k (2L-4n-2k-1)!! \xi^{2k} \mathcal{P}\left(\delta^k_{i_a i_b} \xi^{L-2n-2k}_{i_c}\right)_{i_1 \dots i_{L-2n}} \right)$$
$$\cdot \Delta^n \partial_{i_1} \dots \partial_{i_{L-2n}} \delta(\vec{r}). \quad (40)$$

Now we are going to make some symmetrizations by using Eq. (A2) from Appendix:

$$\mathcal{P}\left(\delta^k_{i_a i_b} \xi^{L-2n-2k}_{i_c}\right)_{i_1 \dots i_{L-2n}} \delta_{i_{L-2n+1} i_{L-2n+2}} \dots \delta_{i_{L-1} i_L} \partial_{i_1} \dots \partial_{i_L} \delta(\vec{r})$$
$$= \frac{2^n (L-2n)!(n+k)!}{k! L!} \mathcal{P}\left(\delta^{k+n}_{i_a i_b} \xi^{L-2n-2k}_{i_c}\right)_{i_1 \dots i_L} \partial_{i_1} \dots \partial_{i_L} \delta(\vec{r}) \quad (41)$$

Coming back to the Eq. (39), the identity we have to prove becomes:

$$\xi_{i_1}\dots\xi_{i_L}=\sum_{n=0}^{\left[\frac{L}{2}\right]}\sum_{k=0}^{\left[\frac{L-2n}{2}\right]}(-1)^k\frac{(2L-4n+1)(n+k)!(2L-4n-2k-1)!!}{n!k!(2L-2n+1)!!}\xi^{2n+2k}$$
$$\cdot\mathcal{P}(\delta^{n+k}_{i_ai_b}\xi^{L-2n-2k}_{i_c})_{i_1\dots i_L}. \tag{42}$$

This is just another form of the decomposition Eq. (1). We change the summation indices: $(n,k)\to(n,K):n+k=K$, $K=\overline{0,\left[\frac{L}{2}\right]}$, $n=\overline{0,K}$:

$$\xi_{i_1}\dots\xi_{i_L} = \sum_{K=0}^{\left[\frac{L}{2}\right]}\sum_{n=0}^{K}(-1)^{K-n}\frac{(2L-4n+1)K!(2L-2n-2K-1)!!}{n!(K-n)!(2L-2n+1)!!}\xi^{2K}$$
$$\cdot\mathcal{P}(\delta^{K}_{i_ai_b}\xi^{L-2K}_{i_c})_{i_1\dots i_L}. \tag{43}$$

We can easily notice that for $K=0$ it follows $n=0$, and the r.h.s. of the above equation is equal to $\xi_{i_1}\dots\xi_{i_L}$.

Now, let $K>0$. We want to show that, in this case, the r.h.s. of Eq. (43) is zero. After some simple manipulations, one obtains:

$$\sum_{K=1}^{\left[\frac{L}{2}\right]}\sum_{n=0}^{K}(-1)^{K-n}\frac{(2L-4n+1)K!(2L-2n-2K-1)!!}{n!(K-n)!(2L-2n+1)!!}\xi^{2K}\mathcal{P}\Big(\delta^{K}_{i_ai_b}\xi^{L-2K}_{i_c}\Big)_{i_1\dots i_L}$$
$$=\sum_{K=1}^{\left[\frac{L}{2}\right]}\xi^{2K}\mathcal{P}(\delta^{K}_{i_ai_b}\xi^{L-2K}_{i_c})_{i_1\dots i_L}\Bigg[(2L+1)\sum_{n=0}^{K}(-1)^{K-n}$$
$$\cdot\frac{K!(2L-2n-2K-1)!!}{n!(K-n)!(2L-2n+1)!!}-4\sum_{n=0}^{K}(-1)^{K-n}\frac{nK!(2L-2n-2K-1)!!}{n!(K-n)!(2L-2n+1)!!}\Bigg].$$

The two summations with respect to n can be easily performed using the formula (A15) from Appendix. One finds:

$$(2L+1)\sum_{n=0}^{K}(-1)^{K-n}\frac{K!(2L-2n-2K-1)!!}{n!(K-n)!(2L-2n+1)!!}$$
$$=4\sum_{n=0}^{K}(-1)^{K-n}\frac{nK!(2L-2n-2K-1)!!}{n!(K-n)!(2L-2n+1)!!}=\frac{(2L+1)(2K)!\Gamma\big(L-2K+\frac{1}{2}\big)}{2^{K+1}K!\Gamma\big(L+\frac{3}{2}\big)}.$$

and now, the proof of the expansion (1) is finished.

4. CONCLUSIONS

We have established the Cartesian multipole expansions of an arbitrary distribution of charges and currents by using two methods. The first method starts with the formulas from [2] and consists in the use of the properties of the spherical gradient operator and of the detracer operator. The second method is based on some combinatorics and on the properties of the hypergeometric functions.

The exact and compact expansions Eqs. (1), (2) allow the correct consideration of all the multipole contributions up to an arbitrary given order. If we explicitly write these expansions up to the second order derivative of the delta function, we obtain:

$$\begin{aligned}\rho(\vec{r},t) &= \overline{r^0}(t)\delta(\vec{r}) - \overline{r_i^0}(t)\partial_i\delta(\vec{r}) + \frac{1}{2}\overline{r_{i_1i_2}^0}(t)\partial_{i_1}\partial_{i_2}\delta(\vec{r}) \\ &\quad + \frac{1}{6}\overline{r^2}(t)\Delta\delta(\vec{r}) + \mathcal{O}(3),\end{aligned} \tag{44}$$

$$\begin{aligned}j_i(\vec{r},t) &= \dot{\overline{r_i^0}}(t)\delta(\vec{r}) - c\varepsilon_{ii_1i_2}\overline{\rho_{i_2}^0}(t)\partial_{i_1}\delta(\vec{r}) - \frac{1}{2}\dot{\overline{r_{ii_1}^0}}(t)\partial_{i_1}\delta(\vec{r}) \\ &\quad - \frac{1}{6}\dot{\overline{r^2}}(t)\partial_i\delta(\vec{r}) + \frac{c}{6}\varepsilon_{ii_1i_2}\overline{\rho_{i_2i_3}^0}(t)\partial_{i_1}\partial_{i_3}\delta(\vec{r}) \\ &\quad + \frac{1}{6}\dot{\overline{r_{i\,i_1i_2}^0}}(t)\partial_{i_1}\partial_{i_2}\delta(\vec{r}) + c\varepsilon_{ii_1i_2}\varepsilon_{i_2i_3i_4}\overline{R_{i_4}^0}(t)\partial_{i_1}\partial_{i_3}\delta(\vec{r}) \\ &\quad + \frac{1}{10}\dot{\overline{r_{i_1}^2}}(t)\partial_i\partial_{i_1}\delta(\vec{r}) + \mathcal{O}(3),\end{aligned} \tag{45}$$

where: $\overline{r^0}(t) = \int d^3\xi\rho(\vec{\xi},t)$ is the total charge, $\overline{r^2}(t) = \int d^3\xi\xi^2\rho(\vec{\xi},t)$ is the first mean square radius of the total charge, $\overline{r_i^0}(t) = \int d^3\xi\xi_i\rho(\vec{\xi},t)$ is the charge dipole, $\overline{r_i^2}(t) = \int d^3\xi\xi^2\xi_i\rho(\vec{\xi},t)$ is the first mean square radius of the charge dipole, $\overline{r_{i_1i_2}^0}(t) = \int d^3\xi(\xi_{i_1}\xi_{i_2} - \frac{1}{3}\delta_{i_1i_2}\xi^2)\rho(\vec{\xi},t)$ is the charge quadrupole, $\overline{r_{i_1i_2i_3}^0}(t) = \int d^3\xi[\xi_{i_1}\xi_{i_2}\xi_{i_3} - \frac{1}{5}\xi^2(\xi_{i_1}\delta_{i_2i_3} + \xi_{i_2}\delta_{i_1i_3} + \xi_{i_3}\delta_{i_1i_2})]\rho(\vec{\xi},t)$ is the charge octupole, $\overline{\rho_i^0}(t) = \frac{1}{2c}\int d^3\xi(\vec{j}\times\vec{\xi})_i$ is the magnetic dipole, $\overline{\rho_{i_1i_2}^0}(t) = \frac{1}{c}\int d^3\xi[(\vec{j}\times\vec{\xi})_{i_1}\xi_{i_2} + (\vec{j}\times\vec{\xi})_{i_2}\xi_{i_1}]$ is the magnetic quadrupole, $\overline{R_i^0}(t) = \frac{1}{10c}\int d^3\xi[\xi_i(\vec{\xi}\cdot\vec{j}) - 2\xi^2 j_i]$ is the toroid dipole, ε_{ijk} is the Levi-Civita symbol and overdot means time derivative. If we go further on with the current density expansion up two the third order derivative of the delta function, we find that the following multipoles appear on the same footing: first mean square radius of the magnetic dipole, toroid quadrupole, magnetic octupole, charge hexadecapole, first mean square radius of the charge quadrupole and second mean square radius of the total charge.

From Eqs. (44), (45) and (24) one can easily obtain the first multipolar terms of the electromagnetic field. A detailed study of the electromagnetic field thus obtained as compared with the electromagnetic field given in the usual textbooks (e.g., [12, 13]) can be found in [7], Section VII$^+$. Here we only mention that, the toroid dipole contribution to the radiation was found to be on the same footing with other "usual" magnetic , electric or interference contributions.

ACKNOWLEDGMENT

One of the authors (G.V.) is indebted to Professor M. Apostol for many useful discussions.

APPENDIX A. USEFUL FORMULAS

- The following formulas can be found in [11, 14]:

$$\partial_{i_1}\ldots\partial_{i_l}\frac{1}{r}$$
$$=\frac{(-1)^l}{r^{2l+1}}\sum_{k=0}^{\left[\frac{l}{2}\right]}(-1)^k(2l-2k-1)!!r^{2k}\mathcal{P}\left(\delta_{i_1i_2}\ldots\delta_{i_{2k-1}i_{2k}}r_{i_{2k+1}}\ldots r_{i_l}\right)$$
$$\equiv\frac{(-1)^l}{r^{2l+1}}\sum_{k=0}^{\left[\frac{l}{2}\right]}(-1)^k(2l-2k-1)!!r^{2k}\mathcal{P}\left(\delta^k_{i_ai_b}r^{l-2k}_{i_c}\right)_{i_1\ldots i_l} \quad \text{(A1)}$$

where $\mathcal{P}$ means all the permutations of the indices $i_1\ldots i_l$ which give distinct terms.

$$\mathcal{P}\left(\delta^\kappa_{i_ai_b}r^{n-2\kappa}_{i_c}\right)_{i_1\ldots i_n}\ \text{has}\ \frac{n!}{2^\kappa(n-2\kappa)!\kappa!}\ \text{terms.} \quad \text{(A2)}$$

This number represents the number of ways of selecting κ distinct pair of objects from n distinct objects.

- We have used the following properties of the hypergeometric functions [15]:

$${}_3F_2\left(\begin{array}{c}a,b,c\\ d,e\end{array}\middle|\ 1\right)=\frac{\Gamma(d)\Gamma(e)\Gamma(1-a)\Gamma(c-b)}{\Gamma(d-b)\Gamma(e-b)\Gamma(1+b-a)\Gamma(c)}$$
$$\cdot\,{}_3F_2\left(\begin{array}{c}b,1+b-d,1+b-e\\ 1+b-a,1+b-c\end{array}\middle|\ 1\right)$$

$^+$ We remember that in [7], there exist some different conventions in the definition of the first Cartesian multipoles, as we have noted at the end of Section 2.

$$+\frac{\Gamma(d)\Gamma(e)\Gamma(1-a)\Gamma(b-c)}{\Gamma(d-c)\Gamma(e-c)\Gamma(1+c-a)\Gamma(b)}$$
$$\cdot\, {}_3F_2\left(\begin{array}{c} c, 1+c-d, 1+c-e \\ \\ 1+c-a, 1+c-b \end{array}\middle|\; 1\right) \tag{A3}$$

$$\sigma\, {}_pF_q\left(\begin{array}{cc} (a_{p-1}), & \rho \\ \\ (b_{q-1}), & \sigma \end{array}\middle|\; z\right) - \rho\, {}_pF_q\left(\begin{array}{cc} (a_{p-1}), & \rho+1 \\ \\ (b_{q-1}), & \sigma+1 \end{array}\middle|\; z\right)$$
$$= (\sigma-\rho)\, {}_pF_q\left(\begin{array}{cc} (a_{p-1}), & \rho \\ \\ (b_{q-1}), & \sigma+1 \end{array}\middle|\; z\right) \tag{A4}$$

$${}_2F_1\left(\begin{array}{cc} -n, & b \\ \\ c, & \end{array}\middle|\; 1\right) = \frac{(c-b)_n}{(c)_n} \tag{A5}$$

$$\prod_{j=1}^{q} b_j \left[{}_pF_q\left(\begin{array}{c} (a_{p-1}), \sigma \\ \\ (b_q) \end{array}\middle|\; z\right) - {}_pF_q\left(\begin{array}{c} (a_{p-1}), \sigma+1 \\ \\ (b_q) \end{array}\middle|\; z\right)\right]$$
$$+z\prod_{j=1}^{p-1} a_j\; {}_pF_q\left(\begin{array}{c} (a_{p-1})+1, \sigma+1 \\ \\ (b_q)+1 \end{array}\middle|\; z\right) = 0. \tag{A6}$$

and the the following properties of the Pochhammer symbols [15]:

$$(a)_k \equiv a(a+1)(a+2)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \tag{A7}$$

$$\left(\begin{array}{c} a \\ k \end{array}\right) = (-1)^k \frac{(-a)_k}{k!} \tag{A8}$$

$$(a+m)_k = \frac{(a)_k (a+k)_m}{(a)_m} \tag{A9}$$

$$(1)_n = n! \tag{A10}$$

$$(a)_{n+k} = (a)_n (a+n)_k \tag{A11}$$

$$(a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k} \tag{A12}$$

$$(2n-1)!! = \frac{2^n}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right) \quad \Gamma(1-a-k) = (-1)^k \frac{\Gamma(1-a)}{(a)_k} \tag{A13}$$

$$\Gamma(a+k) = \Gamma(a)(a)_k \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)} \tag{A14}$$

- We have used in Section 3 the following summation formula:

$$\sum_k \frac{1}{1-ck}\binom{a}{k}\binom{b}{n-k} = \begin{cases} = \binom{a+b}{n}, \ c=0 \ (Vandermonde) \\ = \frac{(-1)^n(-b)_n}{n!}\ {}_3F_2\left(\begin{array}{c} -n,-a,-\frac{1}{c} \\ 1+b-n, 1-\frac{1}{c} \end{array} \middle| \ 1 \right), \ c \neq 0 \end{cases} \tag{A15}$$

The first part of Eq. (A15) is well known as Vandermonde's identity or Vandermonde's convolution. We shall prove here the second part ($c \neq 0$):

$$\begin{aligned} S &\equiv \sum_k \frac{1}{1-ck}\binom{a}{k}\binom{b}{n-k} \overset{(A8)}{=} \\ &= -\frac{1}{c}\sum_k \frac{(-1)^n}{k-\frac{1}{c}}\frac{(-a)_k}{k!}\frac{(-b)_{n-k}}{(n-k)!} \overset{(A9)}{=} \\ &= \sum_k \frac{(-1)^n(-\frac{1}{c})_k}{(1-\frac{1}{c})_k}\frac{(-a)_k}{k!}\frac{(-b)_{n-k}}{(n-k)!} \overset{(A10),(A12)}{=} \\ &= \frac{(-1)^n(-b)_n}{n!}\sum_k \frac{(-n)_k(-a)_k(-\frac{1}{c})_k}{(1+b-n)_k(1-\frac{1}{c})_k k!} = \\ &= \frac{(-1)^n(-b)_n}{n!}\ {}_3F_2\left(\begin{array}{c} -n,-a,-\frac{1}{c} \\ 1+b-n, 1-\frac{1}{c} \end{array} \middle| \ 1 \right). \end{aligned}$$

REFERENCES

1. Dubovik, V. M. and A. A. Tcheshkov, "Multipolnoe razlojenie v classichescoi i v kvantovoi torii polia i izluchenie," *Fiz. Elem. Chastits At. Yadra*, Vol. 5, 791–837, 1974; *Sov. J. Part. Nucl.*, Vol. 5, 318, 1974.
2. Dubovik, V. M. and V. V. Tugushev, "Toroid moments in electrodynamics and solid-state physics," *Phys. Rep.*, Vol. 187, 145–202, 1990.
3. Kaelberer, T., V. A. Fedotov, N. Papasimakis, D. P. Tsai, and N. I. Zheludev, "Toroidal dipolar response in a metamaterial," *Science*, Vol. 330, 1510–1512, 2010.

4. Petschulat, J., "Understanding the electric and magnetic response of isolated metaatoms by means of a multipolar field decomposition," *Optics Express*, Vol. 18, 14455–14466, 2010.
5. Kuo, C. Y., R. L. Chern, and C. C. Chang, "Multipole expansion of electromagnetic scattering wave by a small cylindrical pore on a perfect conducting semi-infinite half space," *Progress In Electromagnetics Research B*, Vol. 26, 179–212, 2010.
6. Andrews, D. L. and W. A. Ghoul, "Irreducible fourth-rank Cartesian tensors," *Phys. Rev. A*, Vol. 25, 2647–2657, 1982.
7. Radescu, E. E. and G. Vaman, "Exact calculation of the angular momentum loss, recoil force, and radiation intensity for an arbitrary source in terms of electric, magnetic and toroid multipoles," *Phys. Rev. E*, Vol. 65, 046609-1–47, 2002.
8. Rowe, E. G. P., "Spherical delta functions and multipole expansions," *J. Math. Phys.*, Vol. 19, 1962–1968, 1978.
9. Weniger, E. J., "Addition Theorems as three-dimensional Taylor expansions," *Int. J. Quant. Chem.*, Vol. 76, 280–295, 2000.
10. Edmonds, A., *Deformations of Atomic Nuclei*, Publishing House for Foreign Literature, Moscow, 1958 (in Russian).
11. Applequist, J., "Traceless cartesian tensor forms for spherical harmonic functions: New theorems and applications to electrostatics of dielectric media," *J. Phys. A*, Vol. 22, 4303–4330, 1989.
12. Landau, L. D. and E. M. Lifschitz, *Course of Theoretical Physics*, Pergamon Press, New York, 1993.
13. Jackson, J. D., *Classical Electrodynamics*, Wiley, New York, 1999.
14. Hobson, E. W., *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, Cambridge, 1931.
15. Prudnikov, A. P., Y. A. Brychkov, and O. I. Marichev, *Integrals and Series*, Nauka, Moscow, 1986 (in Russian).