Progress In Electromagnetics Research Letters, Vol. 27, 133–140, 2011

A WAVELET OPERATOR ON THE INTERVAL IN SOLVING MAXWELL'S EQUATIONS

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Abstract—In this paper, a differential wavelet-based operator defined on an interval is presented and used in evaluating the electromagnetic field described by Maxwell's curl equations, in time domain. The wavelet operator has been generated by using Daubechies wavelets with boundary functions. A spatial differential scheme has been performed and it has been applied in studying electromagnetic phenomena in a lossless medium. The proposed approach has been successfully tested on a bounded axial-symmetric cylindrical domain.

1. INTRODUCTION

Wavelets analysis has been applied in a very large field of science. A large employment of wavelets has been achieved due to their filtering capability; furthermore, differential operators have been modeled from the compactly supported wavelets. From the first Beylkin representation of operators based on compactly supported wavelets [1], various approaches have been created. Beylkin introduced the differential and the integral operators [2], by adopting real line wavelets. In the same year the corrective coefficients have been introduced [3, 4] in order to apply the multiresolution (MRA) analysis to the framework of functions defined on an interval. Subsequently, differential and integral operators for the wavelets on an interval

Received 5 September 2011, Accepted 2 November 2011, Scheduled 8 November 2011
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have been derived [5–7, 10, 11]. In order to solve the Maxwell's curl equations in time domain, various numerical techniques arisen in technical literature, and continually novel ones are presented. An interesting approach is addressed by means of wavelets analysis. Rubinacci et al. [12] proposed wavelets as interpolating functions, and Pinho et al. [13] used interpolating wavelets in generating adaptive finite difference scheme. In this paper, a wavelet differential operator has been applied to the Maxwell's curl equations, in order to simulate electromagnetic transient phenomena. In Section 2, a brief outline of wavelet analysis is presented and details on differential operator based on Daubechies wavelets are addressed. In Section 3 the time-dependent PDEs describing electromagnetic phenomena are approached and a bounded axial-symmetric cylindrical domain is simulated in order to assess the proposed model.

2. WAVELETS DIFFERENTIATION MATRIX

Wavelets are localized functions in time or space, suitable to analyse transient signals. In the following the Daubechies compactly supported wavelets, defined on $[0, 2M-1]$, with M number of vanishing moments are taken into account [11]. By considering $L = 2M$, the following two functions are usually referred as scaling and wavelet functions, respectively:

$$\phi(x) = \sqrt{2}\sum_{k=0}^{L-1} h_k \phi(2x-k) \quad (1)$$

$$\psi(x) = \sqrt{2}\sum_{k=0}^{L-1} g_k \phi(2x-k) \quad (2)$$

They are obtained by dilating and translating the same function $\phi(x)$. The coefficients $H = \{h_k\}_{k=0}^{L-1}$ and $G = \{g_k\}_{k=0}^{L-1}$ are related by means of:

$$g_k = (-1)^k h_{L-k} \quad (3)$$

The scaling and wavelet functions satisfy the following conditions:

$$\int_{-\infty}^{+\infty} \phi(x)dx = 1, \quad \int_{-\infty}^{+\infty} |\phi(x)|^2 dx = 1, \quad \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1 \quad (4)$$

The scaling functions $\phi(x)$ gives rise to a MRA of $L^2(R)$ defined as a sequence of subspaces $\{V_j\}_{J\in Z}$ of $L^2(R)$ satisfying the following properties:

a) $\ldots V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots$

b) $\overline{\cup}_{j \in Z} V_j = L^2(R)$

c) $\bigcap_{j \in Z} V_j = \{0\}$

d) $f(x) \in V_0 \Longleftrightarrow f(2^{-j}) \in V_j$

f) $f(x) \in V_0 \Longleftrightarrow f(x-k) \in V_0$

g) $\exists \phi(x) \in V_0 : \{\phi(x-k)\}_{k \in Z}$ is an orthogonal basis of V_0

By defining W_j as an orthogonal complement of V_j in V_{j-1} and $V_{j-1} = V_j \oplus W_j$, $L^2(R) = \oplus_{j \in Z} W_j$. The dilation and translation of the functions $\phi(x)$ and $\psi(x)$ at a resolution level j are expressed by means of:

$$\phi_k^j(x) = 2^{-\frac{j}{2}} \phi\left(2^{-j}x - k\right) \tag{5}$$

$$\psi_k^j(x) = 2^{-\frac{j}{2}} \psi\left(2^{-j}x - k\right) \tag{6}$$

The coefficients $\{h_k\}_{k=0}^{L-1}$ are chosen so that $\{\psi_j^k(x)\}$ is an orthonormal basis and the function $\psi(x)$ has M vanishing moments. Once chosen the level j, a matrix operator which projects the original function into a discrete sequence of values can be generated. Namely, given a function sampled into $\{f_k^0\}_{k=1}^{N=2^n}$, it is transformed as follows:

$$\begin{aligned} \hat{f} \;=\; & \Big(q_1^1, q_2^1, \ldots, q_{N/2}^1, f_1^1, f_2^1, \ldots, f_{N/2}^1, q_1^2, q_2^2, \ldots, q_{N/4}^2, f_1^2, f_2^2, \\ & \ldots, f_{N/4}^2, \ldots, q_1^N, f_1^N\Big) \end{aligned} \tag{7}$$

This operation can be performed by using N orthogonal mapping P_J converting the coefficients f_k^{j-1} into the coefficients $\{q_k^j, f_k^j\}$ [2]. In [3, 4] corrective coefficients have been obtained allowing the representation of functions on an interval. Wavelets on the interval in [0,1] are considered and also used to represent differential operators [3, 4, 6, 7]. Grid point values of the first derivative of a known tabulated function are generated by introducing a suitable differentiation matrix D. In [1], the set of non-zero coefficients, which allows to determine the spatial differential operator $\frac{d}{dx}$ as the solution of a system of linear algebraic equations, is obtained. By defining the autocorrelation coefficient of H as:

$$a_n = 2 \sum_{i=0}^{L-1-n} h_i h_{i+n} \tag{8}$$

where $n = 1, \ldots, L-1$, with $L = 2M$, the non-zero coefficients of the spatial differential operator D can be carried out [1]:

$$r_l = 2\left[r_{2l} + \frac{1}{2}\sum_{k=1}^{\frac{L}{2}} a_{2k-1}\left(r_{2l-2k+1} + r_{2l+2k-1}\right)\right] \tag{9}$$

and

$$\sum_l l r_l = -1 \tag{10}$$

When $M \geq 2$ the Equations (9) and (10) have an unique solution with a finite number of non-zero r_l for $-L+2 \leq l \leq L-2$ and [1]

$$r_l = -r_{-l} \tag{11}$$

In Table 1, the coefficients of the Daubechies wavelets differentiation matrix D on the real line referred to various vanishing moments, are reported [1].

By choosing $M = 2$, the matrix D employs only two non-zero coefficients, r_1 and r_2 respectively,as reported in (12). In order to generate the differentiation matrix on an interval D_I, the coefficients r_l have to be opportunely computed near the boundaries.

$$D = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{12} & 0 \\ -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} \\ \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{12} & -\frac{2}{3} & 0 \end{bmatrix} \tag{12}$$

The differentiation matrix on the interval D_I, for each level j, is obtained by using block matrices $\Delta^j_{p,q}$ $p, q = 1, \ldots, 3$, namely:

$$D_I = \begin{bmatrix} \Delta^j_{1,1} & \Delta^j_{1,2} & \Delta^j_{1,3} \\ \Delta^j_{2,1} & \Delta^j_{2,2} & \Delta^j_{2,3} \\ \Delta^j_{3,1} & \Delta^j_{3,2} & \Delta^j_{3,3} \end{bmatrix} \tag{13}$$

The central block $\Delta^j_{2,2}$ is the matrix D reported in Equation (12), and the blocks $\Delta^j_{1,3}$, $\Delta^j_{3,1}$ are blocks with all entries equal to zero. For each resolution level j the blocks $\Delta^j_{1,1}$, $\Delta^j_{3,3}$ are always the same, and

Table 1. Coefficients for numerical differentiation.

	$M = 2$	$M = 3$	$M = 4$
r_0	0	0	0
r_1	$-\frac{2}{3}$	$-\frac{272}{365}$	$-\frac{39296}{49553}$
r_2	$\frac{1}{12}$	$\frac{53}{365}$	$\frac{76113}{396424}$
r_3	0	$-\frac{16}{1095}$	$-\frac{1664}{49553}$
r_4	0	$-\frac{1}{2920}$	$\frac{2645}{1189272}$
r_5	0	0	$\frac{128}{743295}$
r_6	0	0	$-\frac{1}{1189272}$

also the non-zero values of the entries of D_I are always the same, by varying the resolution level j. Indeed, the other blocks $\Delta^j_{1,2} = -\Delta^j_{2,1}$ and $\Delta^j_{2,3} = -\Delta^j_{3,2}$ depend on j: namely, a number of zero elements is added so that the size of D_I is 2^j for a fixed resolution level j [7]. The blocks are generated by following the approach used in [1], by means of the boundary functions. For $M = 2$ the differentiation matrix on the interval at the resolution level $j = -3$, is:

$$D_I = \begin{bmatrix} -1.9038 & 0.9444 & -0.2565 & 0 & 0 & 0 & 0 & 0 \\ -1.5163 & -0.0430 & 0.6752 & -0.0832 & 0 & 0 & 0 & 0 \\ 0.2565 & -0.6752 & 0 & \frac{2}{3} & -\frac{1}{12} & 0 & 0 & 0 \\ 0 & 0.0832 & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -0.0765 & 0 \\ 0 & 0 & 0 & \frac{1}{12} & -\frac{2}{3} & 0 & 0.5825 & -0.0397 \\ 0 & 0 & 0 & 0 & 0.0765 & -0.5825 & 0.0899 & 0.3150 \\ 0 & 0 & 0 & 0 & 0 & 0.0397 & -0.7936 & 0.6369 \end{bmatrix} \tag{14}$$

The size of the matrix is equal to 2^j.

3. APPLICATION TO MAXWELL'S CURL EQUATIONS IN TIME DOMAIN

Let us consider the time-dependent Maxwell's curl equations in a lossless medium for a transverse electric (TE) field. By using a rectangular coordinates system, the following coupled partial differential equations hold:

$$\begin{aligned} \frac{\partial E_z}{\partial t} &= \frac{1}{\epsilon_r \epsilon_0}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right), \\ \frac{\partial H_y}{\partial t} &= \frac{1}{\mu_0}\left(\frac{\partial E_z}{\partial x}\right), \\ \frac{\partial H_x}{\partial t} &= -\frac{1}{\mu_0}\left(\frac{\partial E_z}{\partial y}\right) \end{aligned} \tag{15}$$

By performing the double wavelet transform, in both the variables x and y, the field functions $E_z(x, y, t)$, $H_x(x, y, t)$, $H_y(x, y, t)$ are transformed into matrices E^w_z, H^w_x, H^w_y at time t: the rows report the x-direction expansion whilst the columns report the y-direction expansion [10]. Equations (15) are so re-written:

$$\begin{aligned}\frac{dE_z^w}{dt} &= \frac{1}{\epsilon_r\epsilon_0}\left(D_{Ix}H_y^w - D_{Iy}H_x^w\right),\\ \frac{dH_y^w}{dt} &= \frac{1}{\mu_0}\left(D_{Ix}E_z^w\right),\\ \frac{dH_x^w}{dt} &= -\frac{1}{\mu_0}\left(D_{Iy}E_z^w\right)\end{aligned} \tag{16}$$

in which the spatial derivatives are approximated by using differentiation matrices of type (14). Finally, the time derivatives are approximated by using an explicit finite difference scheme [14]. In order to assess the validity of the proposed approach, an axial symmetrical cylindrical domain is considered with the following boundary and initial conditions:

$$E_z(x,y,0) = 1 - \frac{R^2}{R_0^2}, \quad E_z(x_0,y_0,t) = 0, \quad \frac{\partial E_z(x,y,t)}{\partial t}\,\Big|_{t=0} = 0 \tag{17}$$

$$R_0 = \sqrt{x_0^2 + y_0^2} = 0.1\,\mathrm{m}, \qquad R = \sqrt{x^2+y^2}, \qquad 0 \le R \le R_0. \tag{18}$$

The following analytical solution holds:

$$E_z(R,t) = 8\sum_{n=1}^{+\infty} \frac{J_0\left(\frac{\beta_n R}{R_0}\right)}{\beta_n^3 J_1(\beta_n)} \cos\left(\frac{\beta_n t}{R_0\sqrt{\epsilon_r\epsilon_0\mu_0}}\right) \tag{19}$$

where J_0 and J_1 are the Bessel functions of first kind of zero and first order respectively, β_n are the positive zeros of $J_0(\beta)$, $\epsilon_r\epsilon_0$ and μ_0 are the constitutive parameter of the medium ($\epsilon_r = 10$). In Fig. 1, the analytical and computed space profiles of the electric field E_z are compared for a radial direction at times 0.33 µs and 0.44 µs, with $M = 2$, $j = -3$. A good agreement has been reached.

The obtained relative error is:

$$\frac{\left\|E_z - \widetilde{E}_z\right\|_2}{\|E_z\|_2} = \left\{\begin{array}{ll} 2.06\cdot 10^{-2}, & t = 0.33\,\mu\mathrm{s} \\ 2.87\cdot 10^{-2}, & t = 0.44\,\mu\mathrm{s}\end{array}\right\} \tag{20}$$

where E_z and $\widetilde{E}_z$ are the analytical and approximated field components, respectively. By decreasing the resolution level j, improvement on the relative error is obtained. In fact for $M = 2$, $j = -4$ the following result holds:

$$\frac{\left\|E_z - \widetilde{E}_z\right\|_2}{\|E_z\|_2} = \left\{\begin{array}{ll} 8.13\cdot 10^{-3}, & t = 0.33\,\mu\mathrm{s} \\ 9.27\cdot 10^{-3}, & t = 0.44\,\mu\mathrm{s}\end{array}\right\} \tag{21}$$

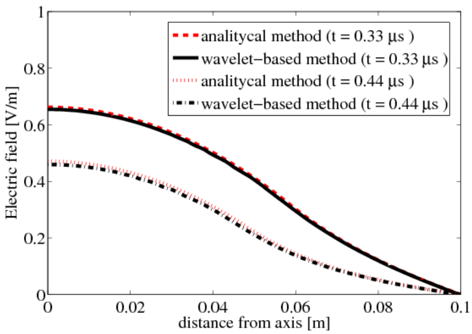

Figure 1. Analytical and computed space profiles of the electric field for $t = 0.33\,\mu\text{s}$ and t=0.44 µs, with $M = 2$, $j = -3$.

4. CONCLUSION

In this paper, a differential wavelet-based operator used to solve Maxwell's equations has been presented. A comparison between the proposed wavelet-based method and the analytical solution of a two-dimensional wave propagation problem in lossless medium has been traced out to validate the proposed approach. The comparison among computed and analytical results shows good agreement.

ACKNOWLEDGMENT

This work has been supported by Universita' degli Studi di Palermo.

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