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CALCULATION OF THE MUTUAL INDUCTANCE AND THE MAGNETIC FORCE BETWEEN A THICK CIRCULAR COIL OF THE RECTANGULAR CROSS SECTION AND A THIN WALL SOLENOID (INTEGRO-DIFFERENTIAL APPROACH)

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Abstract—Systems that employ stimulating and implantable monitoring devices utilize inductive links, such as external and implanted coils. The calculation of the mutual inductance and the magnetic force between these coils is important for optimizing power transfer. This paper deals with an efficient and new approach for determining the mutual inductance and the magnetic force between two coaxial coils in air. The setup is comprised of a thick circular coil of the rectangular cross section and a thin wall solenoid. We use an integro-differential approach to calculate these electrical parameters. The mutual inductance and the magnetic force are obtained using the complete elliptic integrals of the first and second kind, Heuman's Lambda function and one term that has to be solved numerically. All possible regular and singular cases were solved. The results of the presented work have been verified with the filament method and previously published data.

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The advantage of these proposed formulas for mutual inductance or for the magnetic force is that they give the solution in the analytical and the semi-analytical form either for regular cases or singular cases. It is not case with already known methods in which it is necessary to take particular care of these cases of consideration.

1. INTRODUCTION

Magnetically coupled coils are important in magnetically controllable devices and sensors; in modern medicine and telemetric systems applied in biomedical engineering (long–term implantable devices, such as pacemakers, cochlear implants, defibrillators, and instrumented orthopedic implants); and in conventional medical MRI systems, superconducting coils, and tokamaks. In each of these applications, it is necessary to calculate or measure the mutual inductance of magnetically coupled coils. Mutual inductance is a fundamental electrical engineering parameter for a coil that can be computed by applying Neumann's formula directly or by using alternate methods [1–44]. The purpose of this paper is to present an integro-differential solution for calculating the mutual inductance and the magnetic force between two coaxial coils: one coil is a thin wall solenoid, and the other is a thick circular coil of the rectangular cross section. This calculation leads to accurate and new formulas expressed as a combination of the complete elliptic integrals of the first and second kind, Heuman's Lambda function and one term with no analytical solution. This term must be solved numerically, and in this paper, we use Gaussian numerical integration because the kernel function of this integral is a continuous function in all intervals of integration. The formulas for mutual inductance and magnetic force presented in this paper are suitable and easily applicable for calculations. They are excellent alternatives to numerical methods because of their analytical nature. Mutual inductance and magnetic force values computed using the proposed approach closely match expected values from previously published data. For the presented coil configuration, the mutual inductance and the magnetic force have been calculated using an integral approach [4]. Both approaches give the same formula for the mutual inductance, but the formulas for the magnetic force differ. We numerically show that these magnetic force formulas lead to the same results in both regular and singular cases, despite their completely different analytical expressions. This fact can be useful for calculating the mutual inductance and the magnetic force between all possible coil configurations (circular coils with rectangular cross sections, thin wall solenoids, and pancakes), allowing selection between the integro-differential approach and the integral approach based on the approach that best reduces all mathematical procedures and computation time.

2. BASIC EXPRESSIONS

From the basic formula [4] the mutual inductance of a system: thin wall solenoid-thick circular coil with rectangular cross section, (see Fig. 1) can be given by,

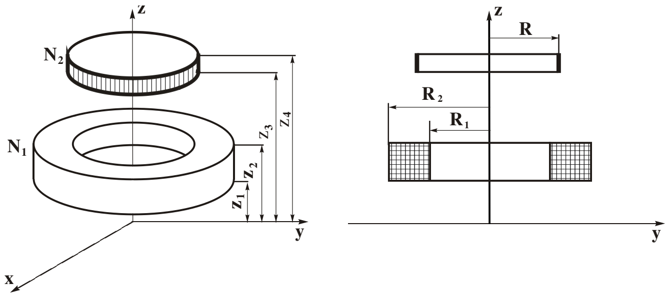

Figure 1. Thin wall solenoid-Thick circular coil with rectangular cross section.

$$M = \frac{\mu_0 N_1 N_2 R}{(R_2 - R_1)(z_2 - z_1)(z_4 - z_3)} \int_0^{\pi} \int_{R_1}^{R_2} \int_{z_1}^{z_2} \int_{z_3}^{z_4} \frac{\cos\theta r dr dz_I dz_{II} d\theta}{r_0} \tag{1}$$

where $\mu_0 = 4\pi \times 10^{-7}$ H/m — the permeability of free space (vacuum), r, θ, z — the cylindrical coordinates,

$$r_0 = \sqrt{(z_{II} - z_I)^2 + R^2 + r^2 - 2Rr\cos\theta}$$

and N_1, N_2 are the total number of turns. The coils are in air. The calculation of the magnetic force between these coils is closely related to the calculation of their mutual inductance [1]. Because their mutual energy is equal to the product of their mutual inductance and the currents in their coils, the component of the magnetic force of attraction or repulsion in any direction is equal to the product of the currents multiplied by the differential coefficient of the mutual inductance. The differential coefficient is taken with respect to that coordinate. As is evident in [29–40], the magnetic force may be calculated by simple differentiation in cases where a general formula for the mutual inductance is available. This formula exists in our case, and the magnetic force between the coils can be calculated with the following expression:

$$F = I_1 I_2 \frac{\partial M}{\partial z} \tag{2}$$

where I_1 and I_2 are the coils' currents, M is their mutual inductance and z is the generalized coordinate. The magnetic force has only an axial component because coils are coaxial. The generalized coordinate z represents the axial coordinate, and the mutual inductance has to be expressed in the function of this coordinate so that formula (2) can be applied.

3. CALCULATION METHOD

In (1), the integrations are made over z_I, z_{II}, r and θ. The mutual inductance of the mentioned system is [4],

$$M = \frac{2\mu_0 N_1 N_2 R^4}{(R_2 - R_1)(z_2 - z_1)(z_4 - z_3)} \sum_{n=1}^{n=8} (-1)^{n-1} \Phi_n = M_0 \sum_{n=1}^{n=8} (-1)^{n-1} \Phi_n \tag{3}$$

where

$$M_0 = \frac{2\mu_0 N_1 N_2 R^4}{(R_2 - R_1)(z_2 - z_1)(z_4 - z_3)} \quad (\text{in}\, H)$$

$$\begin{aligned}
\Phi_n = & -\frac{\rho_n\sqrt{\rho_n}}{48k_n}\left[4\rho_n^2+13-6t_n^2\right]E(k_n)+\frac{k_n\sqrt{\rho_n}}{192}\Big[4\rho_n^4-11\rho_n^2+7-10t_n^4 \\
& -t_n^2\left(10\rho_n^2-33\right)\Big]K(k_n)+\frac{k_n t_n^2}{24(\rho_n+1)}\sqrt{\rho_n}(\rho_n-1)\left(\rho_n^2-3\right)K(k_n) \\
& +\frac{k_n\sqrt{\rho_n}}{96}\sqrt{t_n^2+1}\left(2t_n^2-13\right)\left(\sqrt{t_n^2+1}-1\right)K(k_n)+\frac{\pi\left|t_n\right|\rho_n}{24}\left(\rho_n^2-3\right) \\
& \mathrm{sgn}(\rho_n-1)[1-\Lambda_0(\varepsilon_n,k_n)]+\frac{\pi\left|t_n\right|}{192}\sqrt{t_n^2+1}\left(2t_n^2-13\right)[1-\Lambda_0(\nu_n, \\
& k_n)]+\frac{\pi\left|t_n\right|}{192}\sqrt{t_n^2+1}\left(2t_n^2-13\right)\mathrm{sgn}\left(\rho_n-\sqrt{t_n^2+1}\right)[1-\Lambda_0(\xi_n,k_n)] \\
& +\frac{1-4t_n^2}{16}J_n
\end{aligned}$$

$$k_n^2 = \frac{4\rho_n}{(\rho_n+1)^2+t_n^2}, \quad h_n = \frac{4\rho_n}{(\rho_n+1)^2}, \quad m_n = \frac{2}{\sqrt{t_n^2+1}+1}$$

$$\varepsilon_n = \arcsin\sqrt{\frac{1-h_n}{1-k_n^2}}, \ \nu_n = \arcsin\frac{|t_n|}{\sqrt{t_n^2+1}+1}, \ \xi_n = \arcsin\sqrt{\frac{1-m_n}{1-k_n^2}}$$

$$J_n = \int_0^{\pi/2} \operatorname{arcsinh} \frac{\rho_n + \cos 2\beta}{\sqrt{\sin^2(2\beta) + t_n^2}} d\beta$$

$$\rho_1 = \rho_2 = \rho_5 = \rho_6 = \frac{R_1}{R}; \quad \rho_3 = \rho_4 = \rho_7 = \rho_8 = \frac{R_2}{R}$$

$$t_1 = t_4 = \frac{z_4 - z_1}{R}; \; t_2 = t_3 = \frac{z_4 - z_2}{R}; \; t_5 = t_8 = \frac{z_3 - z_2}{R}; \; t_6 = t_7 = \frac{z_3 - z_1}{R}$$

In [4], the expression for mutual inductance is given differently than (3) in this paper. By inspection, it is possible to show that both lead to the same analytical expression.

Expression (3) is directly applicable for each case of the calculation except for the singular cases that appears when $t_n = 0$, $\rho_n \neq 1$ or $t_n = 0$, $\rho_n = 1$ $(k_n^2 = 1)$.

We have to distinguish two possible singular cases.

a) $t_n = 0$, $\rho_n \neq 1\,(k_n^2 \neq 1)$

In this case for Φ_n we obtain,

$$\Phi_n = -\frac{\rho_n\sqrt{\rho_n}}{48k_n}\left[4\rho_n^2 + 13\right]E(k_n) + \frac{k_n\sqrt{\rho_n}}{192}\left[4\rho_n^4 - 11\rho_n^2 + 7\right]K(k_n) + \frac{1}{16}J_{n0} \quad (4)$$

where

$$J_{n0} = \int_0^{\pi/2} \log\left[\rho_n + \cos(2\beta) + \sqrt{\rho_n^2 + 1 + 2\rho_n\cos(2\beta)}\right]d\beta + \frac{\pi}{2}\log(2), \text{ if } \rho_n > 1$$

$$J_{n0} = \frac{1}{2}\int_0^{\pi/2} \log\left[\rho_n + \cos(\beta) + \sqrt{\rho_n^2 + 1 + 2\rho_n\cos(\beta)}\right]d\beta$$

$$-\frac{1}{2}\int_0^{\pi/2} \log\left[-\rho_n + \sin(\beta) + \sqrt{\rho_n^2 + 1 - 2\rho_n\sin(\beta)}\right]d\beta \text{ if } \rho_n < 1$$

b) $t_n = 0$, $\rho_n = 1\,(k_n^2 = 1)$

In this case for Φ_n we obtain,

$$\Phi_n = -\frac{17}{48} + \frac{G}{8} \quad (5)$$

where $G = 0.9159655941772411\ldots$, is Catalina's constant.

Applying (2) in the expression for the mutual inductance (3) the magnetic force between two mentioned coils can be obtained as follows,

$$F = F_0 \sum_{n=1}^{n=8} (-1)^n \Psi_n \quad (6)$$

where

$$F_0 = \frac{2\mu_0 N_1 N_2 I_1 I_2 R^3}{(R_2 - R_1)(z_2 - z_1)(z_4 - z_3)} \quad (\text{in } N)$$

$$\Psi_n = \frac{t_n \rho_n \sqrt{\rho_n}}{4k_n} E(k_n) - \frac{t_n k_n^3}{768 k_n'^2 \sqrt{\rho_n}} \Big\{ 4\rho_n^4 - 11\rho_n^2 - 19 - 6t_n^4 + 3t_n^2 - 2\rho_n^2 t_n^2$$
$$-16\rho_n t_n^2 + \frac{32t_n^2}{\rho_n+1} - \left(4t_n^2 - 26\right)\sqrt{t_n^2+1} \Big\} E(k_n) + \frac{t_n k_n \sqrt{\rho_n}}{192} \Big\{ -8\rho_n^2$$
$$-18t_n^2 - 32\rho_n + 5 + \frac{64}{\rho_n+1} + \frac{6 - 12t_n^2}{\sqrt{t_n^2+1}+1} \Big\} K(k_n) + \frac{\pi}{24}\operatorname{sgn}(t_n)$$
$$\times \Big\{ \operatorname{sgn}(\rho_n - 1)\rho_n \left(\rho_n^2 - 3\right)[1 - \Lambda_0(\varepsilon_n, k_n)] + \sqrt{t_n^2+1}\left(t_n^2 - 2\right)[1 -$$
$$\Lambda_0(\nu_n, k_n)] + \sqrt{t_n^2+1}\left(t_n^2 - 2\right)\operatorname{sgn}\left(\rho_n - \sqrt{t_n^2+1}\right)[1 - \Lambda_0(\xi_n, k_n)] \Big\}$$
$$-\frac{t_n |t_n| k_n^2}{768\rho_n} \Big\{ \frac{8\operatorname{sgn}(\rho_n - 1)\rho_n \left(\rho_n^2 - 3\right)}{\Delta_1}\sin(2\varepsilon_n) + \frac{\sqrt{t_n^2+1}\left(2t_n^2 - 13\right)}{\Delta_2}$$
$$\sin(2\nu_n) + \frac{\sqrt{t_n^2+1}\left(2t_n^2 - 13\right)\operatorname{sgn}\left(\rho_n - \sqrt{t_n^2+1}\right)}{\Delta_3}\sin(2\zeta_n) \Big\} [K(k_n)$$
$$-E(k_n)] + \frac{t_n}{384} \Big\{ \frac{8k_n^3(\rho_n - 1)\left(\rho_n^2 - 3\right)}{\Delta_1 k_n'^2} \Big[E(k_n) - k_n'^2 K(k_n)$$
$$\sin^2 \varepsilon_n \Big] - \frac{2\sqrt{m_n}\left(2t_n^2 - 13\right)}{\Delta_2} \Big[E(k_n) - k_n'^2 K(k_n)\sin^2 \nu_n \Big] -$$
$$\frac{k_n \sqrt{m_n}\left(2t_n^2 - 13\right)}{\Delta_3 k_n'^2 \sqrt{\rho_n}\left(\rho_n - \sqrt{t_n^2+1}\right)} \times \Big[(\rho_n - 1)^2 + t_n^2 - t_n^2 k_n^2 \sqrt{t_n^2+1} \Big] [E(k_n)$$
$$-k_n'^2 K(k_n)\sin^2 \zeta_n \Big] \Big\} - \frac{t_n}{2} J_n$$

$$\Delta_1 = \sqrt{1 - k_n'^2 \sin^2(\varepsilon_n)}, \quad \Delta_2 = \sqrt{1 - k_n'^2 \sin^2(\nu_n)},$$
$$\Delta_3 = \sqrt{1 - k_n'^2 \sin^2(\zeta_n)}, \quad k_n'^2 = 1 - k_n^2$$

If we compare the expression for magnetic force (6) obtained by differentiation to the expression obtained by direct integration [4], we notice two completely different expressions. Because it is difficult to show mathematically that they lead to the same expression, we will analyze the numerical exactness of these expressions to show that they lead to the same results. We will confirm this statement with a few examples to show that magnetic force can be calculated with two differently derived expressions from the expression of the corresponding mutual inductance. The previous expression can also be used to calculate the magnetic force in the singular case that appears for $t_n = 0$, $\rho_n \neq 1$ or $t_n = 0$, $\rho_n = 1$ ($k_n^2 = 1$).

We give final expressions for four possible singular cases:

a) $z_2 = z_3$ and $R > R_2 > R_1$ or $R_2 > R_1 > R$

$$\Psi_5 = \Psi_8 = 0 \tag{7}$$

b) $z_2 = z_3$ and $R = R_1$

$$\Psi_5 = -\frac{\pi}{3}, \quad \Psi_8 = -\frac{\pi}{6} \tag{8}$$

c) $z_2 = z_3$ and $R_1 < R < R_2$

$$\Psi_5 = -\frac{\pi}{3}, \quad \Psi_8 = 0 \tag{9}$$

d) $z_2 = z_3$ and $R = R_1$

$$\Psi_5 = -\frac{\pi}{6}, \quad \Psi_8 = -\frac{\pi}{3} \tag{10}$$

In (3) and (6), Φ_n and Ψ_n are dimensionless. K and E are complete elliptic integrals of the first and the second kind [45, 46]. Λ_0 is Heuman's Lambda function [45, 46].

4. EXAMPLES

To verify the validity of the presented expressions, we solve the following example problems.

4.1. Example 1

Consider a thick circular coil with a rectangular cross section (the first coil) and a thin wall solenoid (the second coil) for which the mutual inductance will be calculated using the presented method and the filament method [4]. The coil dimensions and number of turns are as follows:

Presented approach:

First coil: $R_1 = 30\,\text{cm}$, $R_2 = 50\,\text{cm}$, $z_1 = -40\,\text{cm}$, $z_2 = 40\,\text{cm}$, $N_1 = 100$,

Second coil: $R = 10\,\text{cm}$, $z_3 = -80\,\text{cm}$, $z_4 = 80\,\text{cm}$, $N_2 = 1000$.

Filament method [4]:

First coil: $R_{\mathrm{I}} = R = 10\,\mathrm{cm}$, $a = (z_4 - z_3)/2 = 160\,\mathrm{cm}$, $N_2 = 1000$.

Second coil: $R_{II} = (R_2 + R_1)/2 = 40\,\mathrm{cm}$, $h_{II} = (R_2 - R_1)/2 = 20\,\mathrm{cm}$, $b = (z_2 - z_1)/2 = 80\,\mathrm{cm}$, $N_1 = 100$.

Coils are coaxial ($c = 0\,\mathrm{cm}$).

Calculate the mutual inductance between these coils.

The presented approach (3) gives the mutual inductance,

$$M_{\text{This Work}} = 2.15868775\,\mathrm{mH}$$

Execution time was 0.1099 seconds.

The mutual inductance has been obtained with Gaussian numerical integration. Using the filament method [4], values for the mutual inductance are given in Table 1. In addition, the corresponding computational time and the discrepancy between the two calculations are given (see Table 1). From Table 1, one can conclude that all results obtained by the two approaches are in a good agreement. However, to get the proposed approach's result for mutual inductance using the filament method, one has to make more subdivisions. These subdivisions increase the computation cost. Also, we compared the mutual inductance value obtained using the presented method with the mutual inductance value obtained from the finite element calculation (Software Flux, Cedrat).

To achieve high accuracy for these and other cases using FEM, it is necessary to create a mesh in the model using many elements and nodes that can increase computation time. Pre-processing time to generate 16291 nodes and 8080 elements was 15 minutes, and the computation time about 35 seconds. The mutual inductance was,

$$M_{\text{Flux}} = 2.1586\,\mathrm{mH}$$

Table 1. Comparison of computational efficiency.

$K/m/n$ Subdivisions	M_{Filament} (mH)	Computational Time (Seconds)	Discrepancy (%)
5/5/5	2.16129158	0.073721	0.12062
15/15/15	2.15901922	1.718260	0.01536
30/30/30	2.15877344	13.014610	0.00397
40/40/40	2.15873636	29.918979	0.00225
55/55/55	2.15871363	80.338788	0.00120
70/70/70	2.15870379	164.762007	0.00074
85/85/85	2.15869866	312.806815	0.00051

Thus, this value perfectly corresponds to the one obtained by the presented method and the filament method, but the advantage of the presented method is obvious regarding the computation time.

4.2. Example 2

In this example, we consider a thin wall solenoid and a thick circular coil of the rectangular cross section. The thin wall solenoid touches the thick circular coil; therefore, the distance between coils is zero. The radius of the thin wall solenoid is equal to the inner radius of the thick coil. (In the singular case, the radius of the thin wall is equal to the outer radius of the thick coil.) The coil dimensions and number of turns are as follows:

First coil: $R_1 = 30\,\text{cm}$, $R_2 = 50\,\text{cm}$, $z_1 = -40\,\text{cm}$, $z_2 = 40\,\text{cm}$, $N_1 = 100$,

Second coil: $R = 30\,\text{cm}$, $z_3 = 40\,\text{cm}$, $z_4 = 120\,\text{cm}$, $N_2 = 100$.

By inspection we obtain $t_5 = 0$, $\rho_5 = 1$ ($k_5^2 = 1$) because $R = R_1 = 30\,\text{cm}$ and $z_2 = z_3 = 40\,\text{cm}$. Applying (3) with the modification (5), the presented approach gives,

$$M = 0.64126676\,\text{mH}$$

By the filament method [4], the mutual inductance is,

$$M = 0.64116332\,\text{mH}$$

If the second coil is given by $R = 50\,\text{cm}$, $z_3 = 40\,\text{cm}$, $z_4 = 120\,\text{cm}$ and $N_2 = 100$, by inspection we obtain $t_8 = 0$ and $\rho_8 = 1\,(k_8^2 = 1)$. Applying (3) with the modification in (5), the presented approach gives,

$$M = 1.26240266\,\text{mH}$$

By the filament method [4], the mutual inductance is,

$$M = 1.262243210\,\text{mH}$$

If the second coil is given by $R = 40\,\text{cm}$, $z_3 = 40\,\text{cm}$, $z_4 = 120\,\text{cm}$, and $N_2 = 100$, by inspection we find $t_5 = t_8 = 0$, $\rho_5 \neq 1$ and $\rho_8 \neq 1$. Applying (3) with the modification in (4), the presented approach gives,

$$M = 0.99701346\,\text{mH}$$

By the filament method [4], the mutual inductance is,

$$M = 0.99682801\,\text{mH}$$

All results match closely, showing that formula (3) can be considered a general formula for calculating the mutual inductance of the proposed configuration, along with modifications (4) and (5) for regular or singular cases.

4.3. Example 3

Calculate the magnetic force between a thin wall solenoid and a thick circular coil of the rectangular cross section. Coils have different radiuses and lengths.

The coil dimensions and number of turns are as follows:

First coil: $R_1 = 30\,\text{cm}$, $R_2 = 50\,\text{cm}$, $z_1 = -10\,\text{cm}$, $z_2 = 10\,\text{cm}$, $N_1 = 100$, $I_1 = 1\,\text{A}$,

Second coil: $R = 10\,\text{cm}$, $z_3 = -20\,\text{cm}$, $z_4 = 20\,\text{cm}$, $N_2 = 1000$, $I_2 = 1\,\text{A}$.

The distance between coil axes is $c = (0 - 30)\,\text{cm}$.

From Table 2, we can see that formula (6), presented in this paper, and formula [4] give the same results. Thus we confirm the validity of both formulas for the magnetic force calculation obtained in two different ways.

4.4. Example 4

In this example, we calculate the magnetic force if coils are in contact at their bases. The magnetic force has been calculated along the

Table 2. Comparison of computational efficiency.

c (m)	$F_{\text{This Work}}$ (mN)	F [4] (mN)	Discrepancy (%)
0	0.00	0.00	-
0.02	1.08730357	1.08730357	0.00
0.04	2.15709787	2.15709787	0.00
0.06	3.19195682	3.19195682	0.00
0.08	4.17471591	4.17471591	0.00
0.10	5.08881123	5.08881123	0.00
0.12	5.91879279	5.91879279	0.00
0.14	6.65096959	6.65096959	0.00
0.16	7.27410015	7.27410015	0.00
0.18	7.78002703	7.78002703	0.00
0.20	8.16416584	8.16416584	0.00
0.22	8.42578622	8.42578622	0.00
0.24	8.56805054	8.56805054	0.00
0.26	8.59779762	8.59779762	0.00
0.28	8.52507684	8.52507684	0.00
0.30	8.36246095	8.36246095	0.00

line where $z_2 = z_3 = 0.2\,\mathrm{cm}$ and for r between 2 and 5 cm. Coils are in contact on this line, and at these points we use formula (6) taking into consideration singular cases (7)–(10). For the comparison, we used the formula for calculating the magnetic force obtained by integration [4] and the method presented in [22]. All results match closely (see Table 3).

First coil: $R_1 = 3\,\mathrm{cm}$, $R_2 = 4\,\mathrm{cm}$, $z_1 = -2\,\mathrm{cm}$, $z_2 = 2\,\mathrm{cm}$, $N_1 = 100$, $I_1 = 1\,\mathrm{A}$.

Second coil: $R = r\,\mathrm{cm}$, $z_3 = 2\,\mathrm{cm}$, $z_4 = 6\,\mathrm{cm}$, $N_2 = 100$, $I_2 = 1\,\mathrm{A}$.

Thus we confirm the validity of both formulas for the magnetic force calculation obtained in two different ways for the treatment of the singular cases. The formula (6) presented in this paper gives the same results as those obtained from [4] (see Table 3).

4.5. Example 5

In the superconducting tokamak JT-60SC, we calculate the mutual inductance between poloidal field coils (CS1 and EF3) for plasma shaping, distributed around the vacuum vessel [47]. The coil dimensions and number of turns are as follows:

(CS1): $R_c = 0.901\,\mathrm{m}$, $z_c = 1.686\,\mathrm{m}$, $\Delta R = 0.381\,\mathrm{m}$, $\Delta z = 1.124\,\mathrm{m}$, $N_1 = 418$,

(EF3): $R_c = 1.730\,\mathrm{m}$, $z_c = 3.346\,\mathrm{m}$, $\Delta R = 0.668\,\mathrm{m}$, $\Delta z = 0.576\,\mathrm{m}$, $N_2 = 432$.

Table 3. Comparison of computational efficiency.

r (m)	$F_{\text{This Work}}$ (mN)	F [4] (mN)	F [22] (mN)
0.02	3.06163703	3.06163703	3.06163704
0.022	3.74119475	3.74119475	3.74119454
0.025	4.90868246	4.90868246	4.90868214
0.028	6.26378148	6.26378148	6.26378147
0.03	7.27660372	7.27660372	7.27693072
0.032	8.22700727	8.22700727	8.22706529
0.036	9.36772613	9.36772613	9.36841472
0.038	9.48577057	9.48577057	9.48643696
0.04	9.25276973	9.25276973	9.25352213
0.045	8.22746449	8.22746449	8.22746449
0.048	7.64324523	7.64324523	7.64324464
0.05	7.26952745	7.26952745	7.26952721

If we interpret this configuration as a system consisting of a thin wall solenoid with negligible cross section and a thick circular coil with a rectangular cross section (the same configuration presented in this paper), we have:

Thick coil of rectangular cross section (EF3): $R_1 = 1.396\,\mathrm{m}$, $R_2 = 2.064\,\mathrm{m}$, $z_1 = 3.058\,\mathrm{m}$, $z_2 = 3.634\,\mathrm{m}$, $N_1 = 432$.

Thin wall solenoid (CS1): $R = 0.901\,\mathrm{m}$, $z_3 = 1.124\,\mathrm{m}$, $z_4 = 2.248\,\mathrm{m}$, $N_2 = 418$.

This assumption can be justified on the basis that the ratio of the lateral over the radial length is more pronounced for coil (CS1) than for coil (EF3).

Applying formula (3) of this paper, the mutual inductance obtained is,

$$M = 60.6159\,\mathrm{mH}$$

With finite element calculations (Software Flux, Cedrat), we obtain,

$$M = 60.6225\,\mathrm{mH}$$

All of these results match, so we consider that the presented method is valid, fast and accurate for calculating the mutual inductance in either small or large coils.

5. CONCLUSION

New and accurate mutual inductance and magnetic force expressions for a system comprised of a thin wall solenoid and a thick circular coil of the rectangular cross section, in air, are derived and presented in this paper. We propose new formulas for the magnetic force calculation by differentiating the corresponding expression for mutual inductance. All cases, regular and singular, have been incorporated into this new formula. This novel approach can provide a suitable alternative to modern numerical methods, such as the finite element method or the boundary element method, because of its rapidity and accuracy. The results are obtained over complete elliptic integrals of the first and second kind. The results also involve Heuman's Lambda function and a single non-analytically solvable term. We solve this term using Gaussian numerical integration because the kernel function is a smooth function on all intervals of integration. The proposed method can be used for a large scale of practical applications, such as millimeter and submillimeter sized biomedical telemetric systems (e.g., for implanted, injected, or ingested devices) and superconducting coils. Several examples confirm the validity of the presented method.

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Erratum to CALCULATION OF THE MUTUAL INDUCTANCE AND THE MAGNETIC FORCE BETWEEN A THICK CIRCULAR COIL OF THE RECTANGULAR CROSS SECTION AND A THIN WALL SOLENOID (INTEGRO-DIFFERENTIAL APPROACH) by S. Babic, C. Akyel, F. Sirois, G. Lemarquand, R. Ravaud, and V. Lemarquand, in *Progress In Electromagnetics Research B*, Vol. 33, pp. 221–237, 2011

Equation (6) in this paper was presented incorrectly and the term involving Ψ_n should be given by

$$\Psi_n = \frac{t_n\rho_n\sqrt{\rho_n}}{4k_n}E(k_n) - \frac{t_n k_n^3}{768k_n'^2\sqrt{\rho_n}}\Bigg\{4\rho_n^4 - 11\rho_n^2 - 19 - 6t_n^4 + 3t_n^2 - 2\rho_n^2 t_n^2$$

$$-16\rho_n t_n^2 + \frac{32t_n^2}{\rho_n+1} - \left(4t_n^2 - 26\right)\sqrt{t_n^2+1}\Bigg\}E(k_n) + \frac{t_n k_n\sqrt{\rho_n}}{192}\Bigg\{-8\rho_n^2$$

$$-18t_n^2 - 32\rho_n + 5 + \frac{64}{\rho_n+1} + \frac{6-12t_n^2}{\sqrt{t_n^2+1}+1}\Bigg\}K(k_n) + \frac{\pi}{24}\mathrm{sgn}(t_n)$$

$$\times\Bigg\{\mathrm{sgn}(\rho_n-1)\rho_n\left(\rho_n^2-3\right)\left[1-\Lambda_0(\varepsilon_n,k_n)\right] + \sqrt{t_n^2+1}\left(t_n^2-2\right)\Big[1-$$

$$\Lambda_0(\nu_n,k_n)\Big] + \sqrt{t_n^2+1}\left(t_n^2-2\right)\mathrm{sgn}\left(\rho_n - \sqrt{t_n^2+1}\right)\left[1-\Lambda_0\left(\xi_n,k_n\right)\right]\Bigg\}$$

$$-\frac{t_n\left|t_n\right|k_n^2}{768\rho_n}\Bigg\{\frac{8\mathrm{sgn}(\rho_n-1)\rho_n\left(\rho_n^2-3\right)}{\Delta_1}\sin(2\varepsilon_n) + \frac{\sqrt{t_n^2+1}\left(2t_n^2-13\right)}{\Delta_2}$$

$$\sin(2\nu_n) + \frac{\sqrt{t_n^2+1}\left(2t_n^2-13\right)\mathrm{sgn}\left(\rho_n-\sqrt{t_n^2+1}\right)}{\Delta_3}\sin(2\xi_n)\Bigg\}\Big[K(k_n)$$

$$-E(k_n)\Big] + \frac{t_n}{384}\Bigg\{\frac{8k_n^3(\rho_n-1)\left(\rho_n^2-3\right)}{\Delta_1 k_n'^2}\Big[E(k_n) - k_n'^2K(k_n)$$

$$\sin^2\varepsilon_n\Big] - \frac{2\sqrt{m_n}\left(2t_n^2-13\right)}{\Delta_2}\left[E(k_n) - k_n'^2K(k_n)\sin^2\nu_n\right]$$

$$-\frac{k_n\sqrt{m_n}\left(2t_n^2-13\right)}{\Delta_3 k_n'^2\sqrt{\rho_n}\left(\rho_n-\sqrt{t_n^2+1}\right)}\times\left[(\rho_n-1)^2+t_n^2-t_n^2k_n^2\sqrt{t_n^2+1}\right][E(k_n)$$

$$\left.-k_n'^2K(k_n)\sin^2\xi_n\right]\Bigg\}-\frac{t_n}{2}J_n$$

where

$$\Delta_1=\sqrt{1-k_n'^2\sin^2(\varepsilon_n)},\quad \Delta_2=\sqrt{1-k_n'^2\sin^2(\nu_n)},$$
$$\Delta_3=\sqrt{1-k_n'^2\sin^2(\xi_n)},\quad k_n'^2=1-k_n^2$$

In equations (7) to (10) the expressions Ψ_5 and Ψ_8 were presented incorrectly. They should be given by,

a) $z_2 = z_3$ and $R_2 > R_1 > R$ or $R > R_2 > R_1$

$$\Psi_5 = \Psi_8 = 0 \tag{7}$$

b) $z_2 = z_3$ and $R = R_1 < R_2$

$$\Psi_5 = -\frac{\pi}{6},\quad \Psi_8 = -\frac{\pi}{3} \tag{8}$$

c) $z_2 = z_3$ and $R_1 < R < R_2$

$$\Psi_5 = 0,\quad \Psi_8 = -\frac{\pi}{3} \tag{9}$$

d) $z_2 = z_3$ and $R_1 < R = R_2$

$$\Psi_5 = 0,\quad \Psi_8 = -\frac{\pi}{6} \tag{10}$$