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POTENTIAL GENERATED BY ROTATING CHARGED CYLINDERS

T.-C. Toh*

Lexmark International Inc., 740 W New Circle Rd., Lexington, KY 40550, USA

Abstract—The potential field generated by two charged cylindrical perfect electrical conductors sandwiching a dielectric plane of finite thickness, and the influence of the dielectric plane on the field, is analysed. In particular, the field profile is examined when the cylinders are (i) rotating at some constant angular velocities, and (ii) surrounded, respectively, by uniform dielectric tubes of finite thickness.

1. INTRODUCTION

A toy model is developed to study the potential field generated by two, infinitely long, charged cylindrical perfect electrical conductors (PEC) sandwiching an infinite dielectric plane of finite thickness. This has obvious applications in the colour laser printer industry. An immediate application of this work is to determine the force exerted on small charged particles between a charged cylinder and the dielectric plane; more precisely, the dynamics of transferring charged particles from a rotating cylinder onto a moving dielectric plane, where the dielectrics are assumed to be imperfect. In particular, it provides a foundational framework for studying toner transfer on a photoconductor onto a belt in a laser printer — this work is pursued elsewhere.

The paper is organized as follows. Section 2 examines the potential generated by charged PEC cylinders in the absence of a tubular dielectric of finite thickness surrounding the PEC cylinders. Charged cylinders having constant angular velocities are investigated in Section 3, along with the presence of dielectric tubes of finite thickness surrounding the charged cylinders. Section 4 concludes the paper.

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* Corresponding author: Tze-Chuen Toh (ttoh@lexmark.com).

2. ELECTROSTATIC TOY MODEL

In this section, the charged cylinders are assumed to be PEC surrounded by an infinitesimally thin insulation. The cylinders and the dielectric plane are assumed to be surrounded by air. The dielectric plane separating the two cylinders is assumed to be a perfect dielectric of thickness h and electric permittivity ε. The axes of the cylinders are parallel to the z-axis. Since the cylinders and dielectric are of infinite extent, the 3D Dirichlet problem of determining the potential reduces to a 2D Dirichlet problem. Thus in this paper, Laplace's equation is solved in 2D. Set $D = \mathbf{R} \times [-\frac{1}{2}h, \frac{1}{2}h]$, $\tilde{\mathbf{R}}^2_+ = \left\{(x, y) \in \mathbf{R}^2 : y > \frac{1}{2}h\right\}$ and $\tilde{\mathbf{R}}^2_- = \left\{(x, y) \in \mathbf{R}^2 : y < -\frac{1}{2}h\right\}$. *Cf.* Fig. 1 below.

2.1. Proposition

Let $C_\pm = \left\{(x, y) \in \tilde{\mathbf{R}}^2_\pm : x^2 + (y - \delta_\pm \mp \frac{1}{2}h)^2 \le a^2_\pm\right\}$ be PEC disks of radii $a_\pm$ respectively, for some fixed $\delta_- < 0 < \delta_+$, where $|\delta_\pm| > a_\pm$. Given $(\tilde{\mathbf{R}}^2_\pm, \varepsilon_0)$ and the strip (D, ε), if the potential φ satisfies the following boundary condition

$$\varphi = \begin{cases} \varphi_+ \text{ on } \partial C_+, \\ \varphi_- \text{ on } \partial C_-, \end{cases} \tag{1}$$

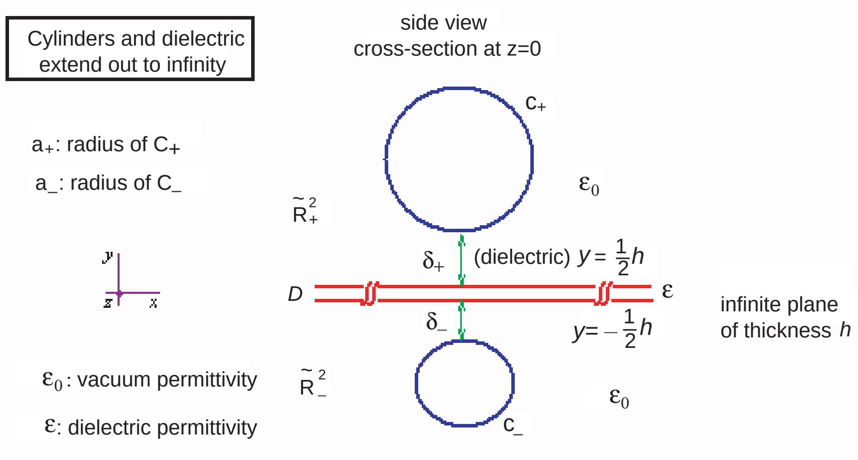

Figure 1. Two charged cylinders separated by a dielectric medium D.

then, φ on $\Omega = \mathbf{R}^2 - (C_+ \cup C_-)$ is given by

$$\varphi = \begin{cases} -\frac{1}{2\pi\varepsilon_0}\tilde{\lambda}_+ \ln\sqrt{x^2+(y-\tilde{y}_+)^2} - \frac{1}{2\pi\varepsilon_0}\tilde{\lambda}_+^{(1)} \ln\sqrt{x^2+(y+\tilde{y}_+-h)^2} + \\ -\frac{1}{2\pi\varepsilon_0}\tilde{\lambda}_+^{(2)} \ln\sqrt{x^2+(y+\tilde{y}_++h)^2} + \\ -\frac{1}{2\pi\varepsilon_0}\tilde{\lambda}'_- \ln\sqrt{x^2+(y-\tilde{y}_-)^2} - \frac{1}{2\pi\varepsilon_0}\tilde{\lambda}''_- \ln\sqrt{x^2+(y-\tilde{y}_-)^2}, \text{on}\, \Omega_+, \\ -\frac{1}{2\pi\varepsilon}\Big\{\tilde{\lambda}'_+ \ln\sqrt{x^2+(y-\tilde{y}_+)^2} + \tilde{\lambda}_+ \ln\sqrt{x^2+(y-\tilde{y}_+)^2} \\ +\tilde{\lambda}_+^{(2)} \ln\sqrt{x^2+(y+\tilde{y}_++h)^2}\Big\} + \\ -\frac{1}{2\pi\varepsilon}\Big\{\tilde{\lambda}'_- \ln\sqrt{x^2+(y-\tilde{y}_-)^2} + \tilde{\lambda}_- \ln\sqrt{x^2+(y-\tilde{y}_-)^2} \\ +\tilde{\lambda}_-^{(2)} \ln\sqrt{x^2+(y+\tilde{y}_--h)^2}\Big\},\ \text{on}\, D, \\ -\frac{1}{2\pi\varepsilon_0}\tilde{\lambda}'_+ \ln\sqrt{x^2+(y-\tilde{y}_+)^2} - \frac{1}{2\pi\varepsilon_0}\tilde{\lambda}''_+ \ln\sqrt{x^2+(y-\tilde{y}_+)^2} + \\ -\frac{1}{2\pi\varepsilon_0}\tilde{\lambda}_- \ln\sqrt{x^2+(y-\tilde{y}_-)^2} - \frac{1}{2\pi\varepsilon_0}\tilde{\lambda}_-^{(1)} \ln\sqrt{x^2+(y+\tilde{y}_-+h)^2} + \\ -\frac{1}{2\pi\varepsilon_0}\tilde{\lambda}_-^{(2)} \ln\sqrt{x^2+(y+\tilde{y}_--h)^2}, \quad \text{on}\, \Omega_-. \end{cases} \tag{2}$$

where $\Omega_\pm = \tilde{\mathbf{R}}^2_\pm - C_\pm$, $y_\pm = \pm\sqrt{\delta_\pm^2 - a_\pm^2}$, and $\tilde{y}_\pm = y_\pm \pm \frac{1}{2}h$ are the y-coordinates for the equivalent charges $\{\tilde{\lambda}_\pm, \tilde{\lambda}'_\pm, \tilde{\lambda}''_\pm, \tilde{\lambda}^{(1)}_\pm, \tilde{\lambda}^{(2)}_\pm\}$ defined by

$$\tilde{\lambda}_+^{(2)} = \frac{2\varepsilon}{\varepsilon+\varepsilon_0}\left\{\frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0} + \frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}\frac{\ln\left|\tilde{y}_+ + \frac{3}{2}h\right|}{\ln\left|\tilde{y}_+ - \frac{1}{2}h\right|}\right\}^{-1}\tilde{\lambda}_+ \equiv \tilde{\alpha}_+\tilde{\lambda}_+,$$

$$\tilde{\lambda}'_+ = \left(\frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\alpha}_+ - 1\right)\tilde{\lambda}_+ = -\tilde{\lambda}_+^{(1)} \quad \text{and}$$

$$\tilde{\lambda}''_+ = (1-\tilde{\alpha}_+)\tilde{\lambda}_+,$$

$$\tilde{\lambda}_-^{(2)} = \frac{2\varepsilon}{\varepsilon+\varepsilon_0}\left\{\frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0} + \frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}\frac{\ln\left|\tilde{y}_- - \frac{3}{2}h\right|}{\ln\left|\tilde{y}_- + \frac{1}{2}h\right|}\right\}^{-1}\tilde{\lambda}_- \equiv \tilde{\alpha}_-\tilde{\lambda}_-,$$

$$\tilde{\lambda}'_- = \left(\frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\alpha}_- - 1\right)\tilde{\lambda}_- = -\tilde{\lambda}_-^{(1)} \quad \text{and} \quad \tilde{\lambda}''_- = (1-\tilde{\alpha}_-)\tilde{\lambda}_-,$$

$$\tilde{\lambda}_+ = \left(\varphi_- - \frac{\tilde{\gamma}_-}{\tilde{\gamma}_+}\varphi_+\right)\left(\tilde{\beta}_- - \frac{\tilde{\gamma}_-}{\tilde{\gamma}_+}\tilde{\beta}_+\right)^{-1} \text{and}$$

$$\tilde{\lambda}_- = \frac{1}{\tilde{\gamma}_+}\left(\varphi_+ - \tilde{\beta}_+\tilde{\lambda}_+\right), \text{ with}$$

$$\tilde{\beta}_+ = -\frac{1}{2\pi\varepsilon_0}\Big\{\ln|a_+| + \left(1 - \frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\alpha}_+\right)\ln|2y_+ - a_+| \\ +\tilde{\alpha}_+ \ln|2(y_+ + h) - a_+|\Big\},$$

$$\begin{aligned}
\tilde{\gamma}_{+} &= -\tfrac{1}{2\pi\varepsilon_0}\tfrac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\alpha}_{+}\ln|y_{+}-y_{-}+h-a_{+}| \quad \text{and} \\
\tilde{\beta}_{-} &= -\tfrac{1}{2\pi\varepsilon_0}\tfrac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\alpha}_{-}\ln|y_{-}-y_{+}-h+a_{-}|\,, \\
\tilde{\gamma}_{-} &= -\tfrac{1}{2\pi\varepsilon_0}\Big\{\ln|a_{-}| + \Big(1-\tfrac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\alpha}_{-}\Big)\ln|2y_{-}+a_{-}| \\
&\quad +\tilde{\alpha}_{-}\ln|2(y_{-}-h)+a_{-}|\Big\}\,.
\end{aligned}$$

Proof. Set $\partial D_{\pm} = \left\{(x,y) : y = \pm\frac{1}{2}h\right\}$, and let $\delta_{\pm}$ be the distance of the center of $C_{\pm}$ above $\partial D_{\pm}$ respectively. Then, potential φ satisfies Laplace's equation $\Delta\varphi = 0$ on $\Omega = \mathbf{R}^2 - (C_{+}\cup C_{-})$, together with the following boundary conditions: (1) and

$$\varepsilon_0\partial_y\varphi\big|_{y=-\frac{1}{2}h^-} = \varepsilon\partial_y\varphi\big|_{y=-\frac{1}{2}h^+}, \tag{3}$$

$$\varepsilon\partial_y\varphi\big|_{y=\ \frac{1}{2}h^-} = \varepsilon_0\partial_y\varphi\big|_{y=\ \frac{1}{2}h^+}, \tag{4}$$

$$\varphi\big|_{y=-\frac{1}{2}h^-} = \varphi\big|_{y=-\frac{1}{2}h^+}, \tag{5}$$

$$\varphi\big|_{y=\ \frac{1}{2}h^-} = \varphi\big|_{y=\ \frac{1}{2}h^+}. \tag{6}$$

Now, recall that a charged cylinder of radius a_{+}, a distance δ_{+} away from the origin along the y-axis can be represented by a line charge $\tilde{\lambda}_{+}$ at a distance $y_{+} = \sqrt{\delta_{+}^2 - a_{+}^2}$ away from the origin along the y-axis [1, 2]. The image $\tilde{\lambda}_{+}^{(1)}$ of $\tilde{\lambda}_{+}$ on ∂D_{+} on is considered first, and then followed by the image $\tilde{\lambda}_{+}^{(2)}$ corresponding to the reflection of $\tilde{\lambda}_{+}$ on ∂D_{-}. By the method of images, the potential φ^{+} resulting from $\tilde{\lambda}_{+}$ is:

$$\begin{aligned}
\varphi^{+} &= -\tfrac{1}{2\pi\varepsilon_0}\tilde{\lambda}_{+}\ln\sqrt{x^2+(y-\tilde{y}_{+})^2} - \tfrac{1}{2\pi\varepsilon_0}\tilde{\lambda}_{+}^{(1)}\ln\sqrt{x^2+(y+\tilde{y}_{+}-h)^2} + \\
&\quad -\tfrac{1}{2\pi\varepsilon_0}\tilde{\lambda}_{+}^{(2)}\ln\sqrt{\mathrm{x}^2+(y+\tilde{y}_{+}+h)^2} \text{ on } \Omega_{+}, \\
\varphi^{+} &= -\tfrac{1}{2\pi\varepsilon}\tilde{\lambda}'_{+}\ln\sqrt{x^2+(y-\tilde{y}_{+})^2} - \tfrac{1}{2\pi\varepsilon_0}\tilde{\lambda}_{+}\ln\sqrt{x^2+(y-\tilde{y}_{+})^2} + \\
&\quad -\tfrac{1}{2\pi\varepsilon_0}\tilde{\lambda}_{+}^{(2)}\ln\sqrt{x^2+(y+\tilde{y}_{+}+h)^2} \quad \text{on } D, \\
\varphi^{+} &= -\tfrac{1}{2\pi\varepsilon_0}\tilde{\lambda}'_{+}\ln\sqrt{x^2+(y-\tilde{y}_{+})^2} - \tfrac{1}{2\pi\varepsilon_0}\tilde{\lambda}''_{+}\ln\sqrt{x^2+(y-\tilde{y}_{+})^2} \quad \text{on } \tilde{\mathbf{R}}_{-}^{2},
\end{aligned}$$

where $\tilde{\lambda}'_{+}, \tilde{\lambda}''_{+}, \tilde{\lambda}_{+}^{(1)}, \tilde{\lambda}_{+}^{(2)}$ are determined as a function of $\tilde{\lambda}_{+}$ via the boundary conditions on $\partial D_{\pm}$, and $\tilde{y}_{+} = y_{+} + \frac{1}{2}h$ is the y-coordinate of the equivalent charge $\tilde{\lambda}_{+}$. The corresponding y-coordinates of the image charges $\tilde{\lambda}_{+}^{(1)}(\tilde{\lambda}_{+}^{(2)})$ relative to $\partial D_{+}(\partial D_{-})$ is $y_{+} - h(y_{+} + h)$, respectively.

Applying condition (3) yields $\tilde{\lambda}_+^{(2)} = \tilde{\lambda}_+ - \tilde{\lambda''}_+$ whilst condition (4) yields $\tilde{\lambda}_+^{(1)} = -\tilde{\lambda'}_+$. The boundary condition (5) gives $\tilde{\lambda}_+^{(2)} = -\tilde{\lambda}_+ + \frac{\varepsilon}{\varepsilon_0}\tilde{\lambda''}_+ + \left(\frac{\varepsilon}{\varepsilon_0} - 1\right)\frac{\ln|y_+ + h|}{\ln|y_+|}\tilde{\lambda'}_+$, and $\tilde{\lambda}'_+ = -\tilde{\lambda}_+ + \frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\lambda}_+^{(2)}$. Furthermore, (6) gives $\tilde{\lambda}'_+ = \frac{\varepsilon-\varepsilon_0}{\varepsilon_0}\tilde{\lambda}_+ + \frac{\varepsilon}{\varepsilon_0}\tilde{\lambda}_+^{(1)} + \frac{\varepsilon-\varepsilon_0}{\varepsilon_0}\frac{\ln|\tilde{y}_+ + \frac{3}{2}h|}{\ln|\tilde{y}_+ - \frac{1}{2}h|}\tilde{\lambda}_+^{(2)}$. Whence, substituting $\tilde{\lambda}_+^{(1)} = \tilde{\lambda}_+ - \tilde{\lambda}'_+$ yields $\tilde{\lambda}'_+ = \frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}\tilde{\lambda}_+ + \frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}\frac{\ln|\tilde{y}_+ + \frac{3}{2}h|}{\ln|\tilde{y}_+ - \frac{1}{2}h|}\tilde{\lambda}_+^{(2)}$. In particular, $\tilde{\lambda}_+^{(2)} = \frac{2\varepsilon}{\varepsilon+\varepsilon_0}\left\{\frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0} + \frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}\frac{\ln|\tilde{y}_+ + \frac{3}{2}h|}{\ln|\tilde{y}_+ - \frac{1}{2}h|}\right\}^{-1}\tilde{\lambda}_+ \equiv \tilde{\alpha}_+\tilde{\lambda}_+$, $\tilde{\lambda}'_+ = (\frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\alpha}_+ - 1)\tilde{\lambda}_+ = -\tilde{\lambda}_+^{(1)}$, $\tilde{\lambda}''_+ = (1-\tilde{\alpha}_+)\tilde{\lambda}_+$.

To complete the proof, φ is solved for $\tilde{\lambda}_-$ in the absence of $\tilde{\lambda}_+$. Clearly, the proof follows that of $\tilde{\lambda}_+$ *mutatis mutandis*. In particular, with a little bit of thought, it can be seen from the symmetry that $\varphi^+ \to \varphi^-$ under the following transformation:

$$\lambda_+ \to \lambda_-, \quad y_+ \to y_- \quad \text{and} \quad h \to -h.$$

The complete solution is obtained by superposing the respective potentials from the line charges:

$$\varphi = \begin{cases} \varphi^+|\Omega_+ + \varphi^-|\Omega_+ & \text{on}\,\Omega_+, \\ \varphi^+|D + \varphi^-|D & \text{on}\,D, \\ \varphi^+|\Omega_- + \varphi^-|\Omega_- & \text{on}\,\Omega_-, \end{cases}$$

where $f|X$ denotes the restriction of a function f to the space X.

Lastly, the boundary condition (1) is required to express $\tilde{\lambda}_\pm$ as functions of $\varphi_\pm$. Indeed, it will suffice to consider the point $(0, y_+ + \frac{1}{2}h - a_+) \in C_+$ as $\varphi_\pm$ are constants on $\partial C_\pm$. From (1), $\varphi(0, y_+ + \frac{1}{2}h - a_+) = \varphi_+$. Hence, solving it gives $\varphi_+ = \tilde{\beta}_+\tilde{\lambda}_+ + \tilde{\gamma}_+\tilde{\lambda}_-$, where

$$\tilde{\beta}_+ = -\frac{1}{2\pi\varepsilon_0}\left\{\ln|a_+| + \left(1 - \frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\alpha}_+\right)\ln|2y_+ - a_+| + \tilde{\alpha}_+\ln|2(y_+ + h) - a_+|\right\},$$

$$\tilde{\gamma}_+ = -\frac{1}{2\pi\varepsilon_0}\frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\alpha}_+\ln|y_+ - y_- + h - a_+|.$$

Likewise, at the point $(0, y_- - \frac{1}{2}h + a_-) \in C_-$, $\varphi(0, y_- - \frac{1}{2}h + a_-) = \varphi_-$. Hence, solving for the expression gives $\varphi_- = \tilde{\beta}_-\tilde{\lambda}_+ + \tilde{\gamma}_-\tilde{\lambda}_-$, where

$$\tilde{\beta}_- = -\frac{1}{2\pi\varepsilon_0}\frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\alpha}_-\ln|y_- - y_+ - h + a_-|,$$

$$\tilde{\gamma}_- = -\frac{1}{2\pi\varepsilon_0}\left\{\ln|a_-| + \left(1 - \frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\tilde{\alpha}_-\right)\ln|2y_- + a_-| + \tilde{\alpha}_-\ln|2(y_- - h) + a_-|\right\}.$$

Thus, solving for the pair of equations yield $\tilde{\lambda}_+ = (\varphi_- - \frac{\tilde{\gamma}_-}{\tilde{\gamma}_+}\varphi_+)(\tilde{\beta}_- - \frac{\tilde{\gamma}_-}{\tilde{\gamma}_+}\tilde{\beta}_+)^{-1}$ and $\tilde{\lambda}_- = \frac{1}{\tilde{\gamma}_+}(\varphi_+ - \tilde{\beta}_+\tilde{\lambda}_+)$, as desired.

A plot of the above potential for $a_+ = 30\,\mathrm{mm}$, $a_- = 15\,\mathrm{mm}$, $h = 0.5\,\mathrm{mm}$, $\delta_\pm = \pm 500\,\mu\mathrm{m}$, $\varphi_+ = 500\,\mathrm{V}$ and $\varphi_- = -1000\,\mathrm{V}$, where $\varepsilon = 2\varepsilon_0$, is presented in Fig. 2 below.

2.2. Corollary

Given conditions described in Proposition 2.1, the potential difference across D at $y = \pm\frac{1}{2}h$ is given by

$$\begin{aligned}\delta\varphi = &-\frac{1}{2\pi\varepsilon}\left\{\frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\ln\sqrt{x^2+y_+^2} + \ln\sqrt{x^2+(y_+ + 2h)^2} - \frac{2\varepsilon}{\varepsilon-\varepsilon_0}\right.\\ &\left.\times \ln\sqrt{x^2+(y_+ + h)^2}\right\}\tilde{\alpha}_+\tilde{\lambda}_+ + \frac{1}{2\pi\varepsilon}\left\{\frac{2\varepsilon}{\varepsilon-\varepsilon_0}\ln\sqrt{x^2+(y_- - h)^2}\right.\\ &\left. + -\frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\ln\sqrt{x^2+y_+^2} - \ln\sqrt{x^2+(y_+ + 2h)^2}\right\}\tilde{\alpha}_-\tilde{\lambda}_-,\end{aligned}$$

where the coefficients were defined in Proposition 2.1. In particular, $\lim\limits_{x\to\infty} \delta\varphi \to 0$.

Corollary 2.2 is evident, and the last statement follows immediately from $\frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0} + 1 = \frac{2\varepsilon}{\varepsilon-\varepsilon_0}$.

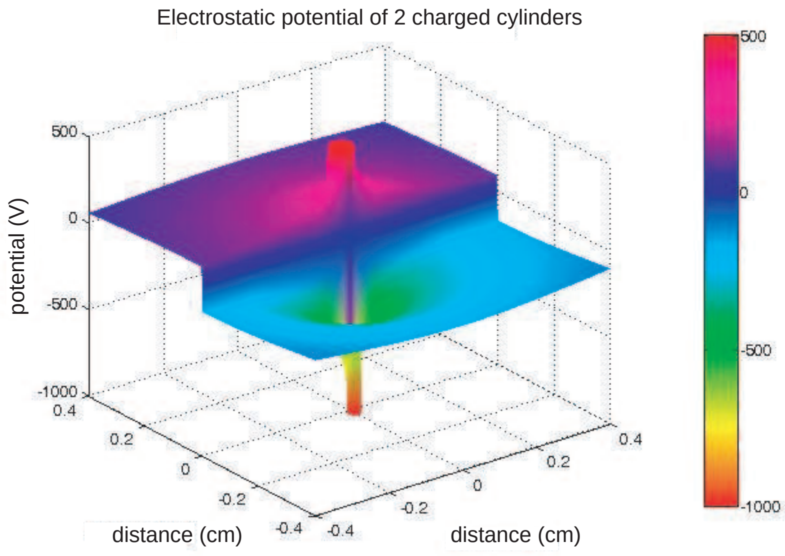

Figure 2. Potential resulting from two infinite charged cylinders.

2.3. Corollary

Suppose D is moving at a constant velocity $\mathbf{v} = v\mathbf{e}_x$. Then, the potential φ induces a displacement current density through D given by $j(x) = \frac{\varepsilon}{h} v\partial_x \delta\varphi(x)$. Explicitly,

$$\begin{aligned} j(x) = & -\frac{1}{2\pi}\frac{x}{h} v\tilde{\lambda}_+\tilde{\alpha}_+ \left\{ \frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\frac{1}{x^2+y_+^2} + \frac{1}{x^2+(y_++2h)^2} - \frac{2\varepsilon}{\varepsilon-\varepsilon_0}\frac{1}{x^2+(y_++h)^2} \right\} + \\ & -\frac{1}{2\pi}\frac{x}{h} v\tilde{\lambda}_-\tilde{\alpha}_- \left\{ \frac{2\varepsilon}{\varepsilon-\varepsilon_0}\frac{1}{x^2+(y_-+h)^2} - \frac{\varepsilon+\varepsilon_0}{\varepsilon-\varepsilon_0}\frac{1}{x^2+y_-^2} - \frac{1}{x^2+(y_-+2h)^2} \right\}. \end{aligned}$$

Proof. From $Q = CV$, where Q, C, V are charge, capacitance and voltage respectively, let $\tilde{Q} = \tilde{C}V$ denote the charge density (charge per unit area), where $\tilde{C}$ is the capacitance per unit area on D. Then, $j = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{Q} = -\tilde{C}\int_{-\frac{1}{2}h}^{\frac{1}{2}h} \partial_t \mathbf{E}\cdot d\mathbf{l}$ is the current density across D. Now, observe that $\tilde{C} = \frac{\varepsilon}{h}$ and $\partial_t\mathbf{E} = \partial_x\mathbf{E}\frac{\mathrm{d}x}{\mathrm{d}t} = -v\partial_x\nabla\varphi$. Hence,

$$-\int_{-\frac{1}{2}h}^{\frac{1}{2}h}\partial_t\mathbf{E}\cdot \mathrm{d}\mathbf{l} = v\int_{-\frac{1}{2}h}^{\frac{1}{2}h}\partial_x\partial_{\bar{y}}\hat{\varphi}(x,\bar{y})\mathrm{d}\bar{y} = v\partial_x(\varphi(x,\tfrac{1}{2}h)-\varphi(x,-\tfrac{1}{2}h)) = v\partial_x\delta\varphi,$$

and the result thus follows from Corollary 2.2.

3. EXTENSION OF THE MODEL

There are two parts to this section: the incorporation of (i) constant rotation about the axes of the cylinders, and (ii) surrounding each cylinder with a uniform tubular dielectric of finite thickness. Suppose that $C_\pm$ has an angular velocity of $\omega_\pm$. Let $\lambda_\pm$ denote the surface charge density of $C_\pm$. Then, the axial rotation generates a surface current density given by

$$\mathbf{J}_\pm = \lambda_\pm\ \mathbf{v} = \rho_\pm\omega_\pm a_\pm \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

Consequently, the vector potential [3, p. 238] or [4] is

$$A_\pm = \begin{cases} \frac{1}{2}\mu_0\lambda_\pm\omega_\pm a_\pm r \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} & \text{for } r \le a_\pm, \\ \frac{1}{2}\mu_0\lambda_\pm\omega_\pm \frac{a_\pm^3}{r} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} & \text{for } r > a_\pm, \end{cases} \tag{7}$$

at any point (r,θ). From $\mathbf{B}_\pm = \nabla\times\mathbf{A}_\pm$, in cylindrical coordinates, $\mathbf{B}_\pm = \mathbf{e}_z\frac{1}{r}\partial_r(rA_{\pm,\theta})$ gives

$$B_\pm = \begin{cases} \mu_0\lambda_\pm\omega_\pm a_\pm \text{ on } x^2 + (y-\delta_\pm-\frac{1}{2}h)^2 \le a_\pm^2, \\ 0 \quad \text{otherwise.} \end{cases} \tag{8}$$

Thus, as expected, there is an absence of magnetic field in Ω since the rotating, charged, infinite cylinder is equivalent to an infinite solenoid carrying a constant current. Hence, the electric field remains the same as the non-rotating case: $\partial_t A = 0$. That is, $\partial_t B_\pm = 0 \Rightarrow \nabla \times \mathbf{E} = 0 \Rightarrow \mathbf{E} = -\nabla\varphi$ and the following trivial result is established.

3.1. Proposition

Given the conditions stated in Proposition 2.1, suppose $C_\pm$ rotate about their respective centers at constant angular velocities $\omega_\pm$. Then, the potential φ on Ω is independent of $\omega_\pm$ and is defined by (2). In particular, the electric field remains the same also.

Note that if the surface charge density is not uniform, that is, if $\rho_+ = \rho_+(\theta)$, as will be the case if $C_\pm$ were surrounded by a uniform tubular dielectric, then the above results do not hold. So, suppose that $\partial C_\pm$ are surrounded by a tubular layer of dielectric of thickness $d_\pm$ with electric permittivity $\varepsilon_\pm$ respectively, and set $\hat{a}_\pm = a_\pm + d_\pm$. Next, define $\hat{C}_\pm = \left\{(x,y) : x^2 + (y - \hat{\delta}_\pm - \frac{1}{2}h)^2 \le \hat{a}_\pm^2\right\}$, where $\hat{\delta}_- < 0 < \hat{\delta}_+$ and $|\hat{\delta}_\pm|$ is the distance of the center of $\hat{C}_\pm$ to the plane $\partial D_\pm$. The resultant potential field as a result of the dielectric layer is given below.

3.2. Lemma

Under the conditions given in Proposition 2.1 wherein $C_\pm$ is surrounded by some dielectric medium $(\hat{C}_\pm - C_\pm, \varepsilon_\pm)$, the solution $\hat{\varphi}$ on $\hat{C}_+ \cup \hat{C}_-$ is given by

$$\hat{\varphi} = \begin{cases} -\frac{1}{2\pi\varepsilon_+}\hat{\lambda}_+ \ln\sqrt{x^2+(y-\hat{y}_+)^2} - \frac{1}{2\pi\varepsilon_+}\hat{\lambda}_+^{(1)}\ln\sqrt{x^2+(y+\hat{y}_+-h)^2} + \\ -\frac{1}{2\pi\varepsilon_+}\hat{\lambda}_+^{(2)}\ln\sqrt{x^2+(y+\hat{y}_++h)^2} + \\ -\frac{1}{2\pi\varepsilon_-}\hat{\lambda}'_-\ln\sqrt{x^2+(y-\hat{y}_-)^2} - \frac{1}{2\pi\varepsilon_-}\hat{\lambda}''_-\ln\sqrt{x^2+(y-\hat{y}_-)^2} \text{ on } \hat{C}_+, \\ -\frac{1}{2\pi\varepsilon_+}\hat{\lambda}'_+\ln\sqrt{x^2+(y-\hat{y}_+)^2} - \frac{1}{2\pi\varepsilon_+}\hat{\lambda}''_+\ln\sqrt{x^2+(y-\hat{y}_+)^2} + \\ -\frac{1}{2\pi\varepsilon_-}\hat{\lambda}_-\ln\sqrt{x^2+(y-\hat{y}_-)^2} - \frac{1}{2\pi\varepsilon_-}\hat{\lambda}_-^{(1)}\ln\sqrt{x^2+(y+\hat{y}_-+h)^2} + \\ -\frac{1}{2\pi\varepsilon_-}\hat{\lambda}_-^{(2)}\ln\sqrt{x^2+(y+\hat{y}_--h)^2} \text{ on } \hat{C}_-, \end{cases} \tag{9}$$

where $\hat{y}_\pm = w_\pm \pm \frac{1}{2}h$, $w_\pm = \pm\sqrt{\hat{\delta}_\pm^2 - \hat{a}_\pm^2}$, and $\hat{a}_\pm$ is the radius of $\hat{C}_\pm$. Finally, $\hat{\lambda}_\pm^{(2)} = \hat{\alpha}_\pm\hat{\lambda}_\pm$, $\hat{\lambda}'_\pm = \left(\frac{\varepsilon+\varepsilon_\pm}{\varepsilon-\varepsilon_\pm}\hat{\alpha}_\pm - 1\right)\hat{\lambda}_\pm = -\hat{\lambda}_\pm^{(1)}$ and

$\hat{\lambda}''_{\pm} = (1-\hat{\alpha}_{\pm})\hat{\lambda}_{\pm}$, where $\hat{\alpha}_{+} = \frac{2\varepsilon}{\varepsilon+\varepsilon_{+}}\left(\frac{\varepsilon+\varepsilon_{+}}{\varepsilon-\varepsilon_{+}} + \frac{\varepsilon-\varepsilon_{+}}{\varepsilon+\varepsilon_{+}}\frac{\ln\left|\hat{y}_{+}+\frac{3}{2}h\right|}{\ln\left|\hat{y}_{+}-\frac{1}{2}h\right|}\right)^{-1}$ and

$\hat{\alpha}_{-} = \frac{2\varepsilon}{\varepsilon+\varepsilon_{-}}\left(\frac{\varepsilon+\varepsilon_{-}}{\varepsilon-\varepsilon_{-}} + \frac{\varepsilon-\varepsilon_{-}}{\varepsilon+\varepsilon_{-}}\frac{\ln\left|\hat{y}_{-}-\frac{3}{2}h\right|}{\ln\left|\hat{y}_{-}+\frac{1}{2}h\right|}\right)^{-1}$.

Proof. Without loss of generality, it may be assumed initially that the spaces $\Omega_{\pm}$ have electric permittivities $\varepsilon_{\pm}$ respectively. Then, the solution φ of Proposition 2.1 on $\hat{C}_{\pm}$ becomes $\hat{\varphi}$ with the following replacement: $\alpha_{\pm} \to \hat{\alpha}_{\pm} = \alpha_{\pm}|_{\varepsilon_0=\varepsilon_{\pm}}$, $\delta_{\pm} \to \hat{\delta}_{\pm}$, $a_{\pm} \to \hat{a}_{\pm}$ and $h \to -h$.

The next result involves solving the Laplace equation in an inhomogeneous dielectric medium. The proof relies implicitly on the fact that the axes of the two cylinders are symmetric about the x-axis. The ploy is to conformally transform the difficult problem in the original coordinate system into a simpler problem in a new coordinate system whereby the two circles in the original system form two concentric circles in the new coordinate system such that the x-axis is transformed into a concentric circle lying in the annulus. The Dirichlet problem then becomes a simple matter to solve when $\varepsilon_{+} \neq \varepsilon_{-}$.

The proof of Lemma 3.3 is an extension of various works found in the literature wherein the potential are induced by either a point charge outside of two dielectric spheres [5] or some uniform electric field at infinity outside of two dielectric spheres via bispherical coordinates [6, 7]. Thus whilst solutions exist for solid spheres, it is difficult to find explicit solutions for cylinders.

3.3. Lemma

Let $\hat{\Omega} = \hat{\Omega}_{+} \cup \hat{\Omega}_{-}$, where the pair of spaces $(\hat{\Omega}_{\pm}, \varepsilon_{\pm})$ are defined by $\hat{\Omega}_{+} = \left\{(x,y) \in \mathbf{R}^2 - \hat{C}_{+} : y > \frac{1}{2}h\right\} - \hat{C}_{+}$ and $\hat{\Omega}_{-} = \left\{(x,y) \in \mathbf{R}^2 - \hat{C}_{-} : y \leq \frac{1}{2}h\right\} - \hat{C}_{-}$. Suppose further that $\sqrt{\hat{\delta}_{+}^2 - \hat{a}_{+}^2} = \sqrt{\hat{\delta}_{-}^2 - \hat{a}_{-}^2}$ is satisfied, where $\hat{\delta}_{\pm}$ are the respective distances from the centres of $\hat{C}_{\pm}$ to the points $(0, \pm\frac{1}{2}h)$. Then, the solution of Laplace's equation $\Delta\varphi = 0$ on $\hat{\Omega}$ satisfying the following boundary conditions

$$\varphi = \begin{cases} f(\theta) \text{ on } \partial\hat{C}_{+}, \\ g(\theta) \text{ on } \partial\hat{C}_{-}, \end{cases} \tag{10}$$

$$\varepsilon_{+}\partial_y\varphi\big|_{y=\frac{1}{2}h^{+}} = \varepsilon_{-}\partial_y\varphi\big|_{y=\frac{1}{2}h^{-}}, \tag{11}$$

$$\lim_{y\to\frac{1}{2}h^+}\varphi=\lim_{y\to\frac{1}{2}h^-}\varphi, \tag{12}$$

$$\varphi\to 0 \quad \text{as} \quad |r| \to\infty, \tag{13}$$

is given by

$$\varphi=\begin{cases} -\frac{1}{2\pi\varepsilon_+}\left\{a_0+b_0\ln r'+\sum_{n>0}\left(a_n r'^n+b_n r'^{-n}\right)\cos n\theta'\right. \\ \left.+\left(c_n r'^n+d_n r'^{-n}\right)\sin n\theta'\right\} \quad \text{on } \hat{\Omega}_+, \\ -\frac{1}{2\pi\varepsilon_-}\left\{a'_0+b'_0\ln r'+\sum_{n>0}\left(a'_n r'^n+b'_n r'^{-n}\right)\cos n\theta'\right. \\ \left.+\left(c'_n r'^n+d'_n r'^{-n}\right)\sin n\theta'\right\} \quad \text{on } \hat{\Omega}_-, \end{cases} \tag{14}$$

where

$$\begin{aligned}
(r',\theta') &= \left(\sqrt{\left(\frac{2r_0^2}{r}\right)^2+r_0^2-\frac{4r_0^3}{r}\cos\theta},\arccos\frac{(2r_0^2/r)\cos\theta-r_0}{\sqrt{(2r_0^2/r)^2+r_0^2-(4r_0^3/r)\cos\theta}}\right),\\
b_0 &= \left\{\varepsilon_+\int_0^{2\pi}f(\theta)\mathrm{d}\theta-\varepsilon_-\int_0^{2\pi}g(\theta)\mathrm{d}\theta\right\}\left\{\left(\frac{2\varepsilon_+}{\varepsilon_-}-1\right)\ln r'_+ +\ln r'_-\right\}^{-1},\\
a_0 &= -\varepsilon_+\int_0^{2\pi}f(\theta)\mathrm{d}\theta-b_0\ln r'_+,\\
a'_n &= \frac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+}+1\right)a_n+\frac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+}-1\right)b_n r'^{-2n}_+ \text{ and}\\
b'_n &= \frac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+}-1\right)a_n r'^{2n}_+ +\frac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+}+1\right)b_n,\\
c'_n &= \frac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+}+1\right)c_n+\frac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+}-1\right)d_n r'^{-2n}_+ \text{ and}\\
d'_n &= \frac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+}-1\right)c_n r'^{2n}_+ +\frac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+}+1\right)d_n,\\
a_n &= \left\{G_1-\left(\beta r'^n_- +\alpha r'^{-n}_-\right)F_1 r'^n_+\right\}\left\{\alpha\left(r'^n_- -r'^{-n}_- r'^{2n}_+\right)\right.\\
&\quad \left.+\beta\left(r'^{-n}_- -r'^n_- r'^{2n}_+\right)\right\}^{-1},\\
b_n &= F_1 r'^n_+ -a_n r'^{2n}_+,\\
c_n &= \left\{G_2-\left(\beta r'^n_- +\alpha r'^{-n}_-\right)F_2 r'^n_+\right\}\left\{\alpha\left(r'^n_- -r'^{-n}_- r'^{2n}_+\right)+\beta(r'^{-n}_-\right.\\
&\quad \left.-r'^n_- r'^{2n}_+)\right\}^{-1},
\end{aligned}$$

$$\begin{aligned}
d_n &= F_2 r'^n_+ -c_n r'^{2n}_+,\\
F_1 &= -2\varepsilon_+\int_0^{2\pi}f(\theta)\cos n\theta\mathrm{d}\theta \text{ and } F_2=-2\varepsilon_+\int_0^{2\pi}f(\theta)\sin n\theta\mathrm{d}\theta,\\
G_1 &= -2\varepsilon_-\int_0^{2\pi}g(\theta)\cos n\theta\mathrm{d}\theta \text{ and } G_2=-2\varepsilon_-\int_0^{2\pi}g(\theta)\sin n\theta\mathrm{d}\theta.
\end{aligned}$$

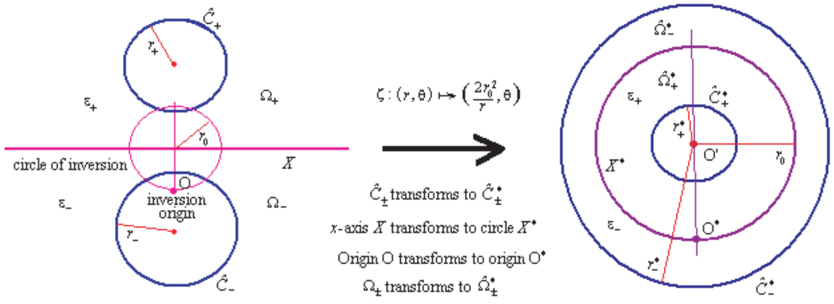

Figure 3. Transformation under the inversion of a circle in polar coordinates.

Proof. Using the inversion of a circle transformation [5, Section 2], $\zeta : (r, \theta) \mapsto (\frac{c}{r}, \theta)$, the Dirichlet problem can be converted to a simpler one; *cf.* Fig. 3 below, where $r'_\pm$ is the new radii of $\hat{C}_\pm$ in the new coordinate system under the ζ-transformation, and the x-axis X is the translated axis defined by $X = \{(x, \frac{1}{2}h) : -\infty < x < \infty\}$.

The constant $c = 2r_0^2$, where $r_0 \equiv \sqrt{\hat{\delta}_\pm^2 - \hat{a}_\pm^2}$ is the radius of the circle of inversion. This defines the radius of the x-axis in the new coordinate system. Now, by choosing O as the origin for the center of inversion where the circle of radius r_0 intersects the y-axis,[†] the two circles are transformed into concentric circles in the new frame — *cf.* Fig. 3, where O is the origin of (r, θ) whilst O^* is the origin of the ζ-space (r^*, θ), and the center O' of the concentric circles to O^* is of length r_0, with $r^* = \frac{c}{r}$. Here, $\hat{C}^*_\pm = \zeta(\hat{C}_\pm)$ and in particular, $\hat{\Omega}^*_\pm = \zeta(\hat{\Omega}_\pm)$. Under the transformation, $r^*_+ = r_0 \frac{\hat{\delta}_+ + \hat{a}_+ - r_0}{\hat{\delta}_+ + \hat{a}_+ + r_0}$ and $r^*_- = r_0 \frac{\hat{\delta}_- + \hat{a}_- + r_0}{\hat{\delta}_- + \hat{a}_- - r_0}$. Finally, translate the origin O^* to O' under $\tau : (r^*, \theta) \mapsto (r', \theta')$ defined by [5, p. 1162] $r' = \sqrt{r^{*2} + r_0^2 - 2r_0 r^* \cos\theta}$ and $\cos\theta' = \frac{r^* \cos\theta - r_0}{\sqrt{r^{*2} + r_0^2 - 2r_0 r^* \cos\theta}}$.

Note that the translation τ preserves distance; hence, $r'_\pm = r^*_\pm$ is the radius of $\hat{C}'_\pm (= \hat{C}^*_\pm)$ as can be easily verified. In particular, it is a conformal transformation as angles are also preserved.

[†] The choice of the origin O determines the outer circle of the annulus in the new coordinate system and it depends on which circle the origin O lies in. In the above proof, the origin lies in the lower circle and hence, the lower circle forms the outer boundary of the annulus: see Fig. 3 below.

Under the conformal transformation $\tau \circ \zeta$, the equivalent Dirichlet problem is:

$$\Delta'\psi' = 0 \quad \text{on} \quad \hat{\Omega}' \equiv \tau \circ \zeta(\hat{\Omega}) = \hat{\Omega}'_+ \cup \hat{\Omega}'_-$$

subject to

$$\psi' = \begin{cases} f(\theta) & \text{for} \quad r' = r'_+, \\ g(\theta) & \text{for} \quad r' = r'_-, \end{cases}$$

$$\varepsilon_+ \partial_{r'} \psi'|_{r \to (R')^-} = \varepsilon_- \partial_{r'} \psi'|_{r' \to (R')^+} \text{ and } \lim_{r \to R'^+} \psi' = \lim_{r \to R'^-} \psi',$$

where $\Delta' = \partial^2_{r'} + \frac{1}{r'}\partial_{r'} + \left(\frac{1}{r'}\right)^2 \partial^2_\theta$ is the Laplacian with respect to the new coordinate system defined by $\tau \circ \zeta$.

The solution for the annulus is well-known; *cf.* [8, p. 273] or [9, p. 203]:

$$\psi' = \begin{cases} -\frac{1}{2\pi\varepsilon_+} \left\{a_0 + b_0 \ln r' + \sum_{n>0} \left(a_n r'^n + b_n r'^{-n}\right) \cos n\theta \right. \\ \left. + \left(c_n r'^n + d_n r'^{-n}\right) \sin n\theta \right\} \text{ on } \hat{\Omega}'_+, \\ -\frac{1}{2\pi\varepsilon_-} \left\{a'_0 + b'_0 \ln r' + \sum_{n>0} \left(a'_n r'^n + b'_n r'^{-n}\right) \cos n\theta \right. \\ \left. + \left(c'_n r'^n + d'_n r'^{-n}\right) \sin n\theta \right\} \text{ on } \hat{\Omega}'_-, \end{cases}$$

where, for $n = 0$,

$$H_1 \equiv -\varepsilon_+ \int_0^{2\pi} f(\theta) \mathrm{d}\theta = a_0 + b_0 \ln r'_+ \quad \text{and}$$

$$H_2 \equiv -\varepsilon_- \int_0^{2\pi} g(\theta) \mathrm{d}\theta = a'_0 + b'_0 \ln r'_-,$$

and for $n > 0$,

$$F_1 \equiv -2\varepsilon_+ \int_0^{2\pi} f(\theta) \cos n\theta \mathrm{d}\theta = a_n (r'_+)^n + b_n (r'_+)^{-n},$$

$$F_2 \equiv -2\varepsilon_+ \int_0^{2\pi} f(\theta) \sin n\theta \mathrm{d}\theta = c_n (r'_+)^n + d_n (r'_+)^{-n},$$

$$G_1 \equiv -2\varepsilon_- \int_0^{2\pi} g(\theta) \cos n\theta \mathrm{d}\theta = a'_n (r'_-)^n + b'_n (r'_-)^{-n},$$

$$G_2 \equiv -2\varepsilon_- \int_0^{2\pi} g(\theta) \sin n\theta \mathrm{d}\theta = c'_n (r'_-)^n + d'_n (r'_-)^{-n}.$$

The condition $\varepsilon_+ \partial_{r'} \psi|_{r' \to (R')^-} = \varepsilon_- \partial_{r'} \psi|_{r' \to (R')^+}$ yields $b_0 = b'_0$,

$$a_n {r'}_+^{n-1} - b_n {r'}_+^{-(n+1)} = {a'}_n {r'}_+^{n-1} - b'_n {r'}_+^{-(n+1)}, \tag{15a}$$

$$c_n {r'}_+^{n-1} - d_n {r'}_+^{-(n+1)} = c'_n {r'}_+^{n-1} - d'_n {r'}_+^{-(n+1)}, \tag{15b}$$

and the continuity requirement yields $a_0 + b_0 \ln r'_+ = \frac{\varepsilon_+}{\varepsilon_-}(a'_0 + b'_0 \ln r'_+)$,

$$a'_n r'^n_+ + b'_n r'^{-n}_+ = \frac{\varepsilon_+}{\varepsilon_-}\left(a'_n r'^n_+ + b'_n r'^{-n}_+\right), \tag{16a}$$

$$c'_n r'^n_+ + d'_n r'^{-n}_+ = \frac{\varepsilon_+}{\varepsilon_-}\left(c'_n r'^n_+ + d'_n r'^{-n}_+\right). \tag{16b}$$

Whence, substituting $a'_0 = \frac{\varepsilon_-}{\varepsilon_+} a_0 + \left(\frac{\varepsilon_-}{\varepsilon_+} - 1\right) b_0 \ln r'_+$ into H_2, together with H_1, give

$$\begin{aligned} b_0 &= \left\{\varepsilon_+ \int_0^{2\pi} f(\theta)\mathrm{d}\theta - \varepsilon_- \int_0^{2\pi} g(\theta)\mathrm{d}\theta\right\}\left\{\left(\frac{2\varepsilon_+}{\varepsilon_-} - 1\right)\ln r'_+ + \ln r'_-\right\}^{-1}, \\ a_0 &= -\varepsilon_+ \int_0^{2\pi} f(\theta)\mathrm{d}\theta - b_0 \ln r'_+; \end{aligned}$$

whilst the two pairs (15a), (16a) and (15b), (16b) give:

$$\begin{aligned} a'_n &= \tfrac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+} + 1\right) a_n + \tfrac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+} - 1\right) b_n r'^{-2n}_+ \quad \text{and} \\ b'_n &= \tfrac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+} - 1\right) a_n r'^{2n}_+ + \tfrac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+} + 1\right) b_n, \\ c'_n &= \tfrac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+} + 1\right) c_n + \tfrac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+} - 1\right) d_n r'^{-2n}_+ \quad \text{and} \\ d'_n &= \tfrac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+} - 1\right) c_n r'^{2n}_+ + \tfrac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+} + 1\right) d_n. \end{aligned}$$

Finally, substituting $a'_n, \ldots, d'_n$ into the equations for G_1, G_2 and using F_1, F_2 yield

$$\begin{aligned} G_1 &= \left(\alpha r'^n_- + \beta r'^{-n}_- - \left(\beta r'^n_- + \alpha r'^{-n}_-\right) r'^{2n}_+\right) a_n + \left(\beta r'^n_- + \alpha r'^{-n}_-\right) r'^n_+ F_1, \\ G_2 &= \left(\alpha r'^n_- + \beta r'^{-n}_- - \left(\beta r'^n_- + \alpha r'^{-n}_-\right) r'^{2n}_+\right) c_n + \left(\beta r'^n_- + \alpha r'^{-n}_-\right) r'^n_+ F_2, \end{aligned}$$

where $\alpha = \frac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+} + 1\right)$ and $\beta = \frac{1}{2}\left(\frac{\varepsilon_-}{\varepsilon_+} - 1\right) r'^{-2n}_+$. Hence,

$$\begin{aligned} a_n &= \left\{G_1 - \left(\beta r'^n_- + \alpha r'^{-n}_-\right) F_1 r'^n_+\right\}\left\{\alpha\left(r'^n_- - r'^{-n}_- r'^{2n}_+\right) + \beta\left(r'^{-n}_- - r'^n_- r'^{2n}_+\right)\right\}^{-1}, \\ b_n &= F_1 r'^n_+ - a_n r'^{2n}_+, \\ c_n &= \left\{G_2 - \left(\beta r'^n_- + \alpha r'^{-n}_-\right) F_2 r'^n_+\right\}\left\{\alpha\left(r'^n_- - r'^{-n}_- r'^{2n}_+\right) + \beta\left(r'^{-n}_- - r'^n_- r'^{2n}_+\right)\right\}^{-1}, \\ d_n &= F_2 r'^n_+ - c_n r'^{2n}_+. \end{aligned}$$

To complete the proof, it remains to translate the transformed solution back to the original problem. That is, given (r', θ'), find (r^*, θ);

then, the solution on Ω is given by the composition $\psi = \psi' \circ \tau \circ \zeta$. Explicitly,

$$\tau \circ \zeta : (r, \theta) \mapsto (r', \theta') =$$
$$\left(\sqrt{\left(\frac{2r_0^2}{r}\right)^2 + r_0^2 - \frac{4r_0^3}{r} \cos\theta}, \arccos \frac{\left(2r_0^2/r\right)\cos\theta - r_0}{\sqrt{\left(2r_0^2/r\right)^2 + r_0^2 - \left(4r_0^3/r\right)\cos\theta}} \right).$$

So, define $\psi(r,\theta) = \psi'(r'(r,\theta), \theta'(r,\theta))$ under the transformation $\tau \circ \zeta$, and the proof is complete.

3.4. Lemma

Consider the infinite strip $D = \{(x,y) \in \mathbf{R}^2 : -\frac{1}{2}h \le y \le \frac{1}{2}h\}$, and suppose that $\Delta\psi = 0$ on D satisfies the following boundary conditions:

$$\psi = \begin{cases} h_+(x) \text{ on } \partial D_+ = D \cap \{y = \frac{1}{2}h\}, \\ h_-(x) \text{ on } \partial D_- = D \cap \{y = -\frac{1}{2}h\}, \end{cases} \tag{17}$$

where $\lim_{x\to\pm\infty} h_\pm(x) = 0$ and $h_\pm \in C(\partial D_\pm)$. Then, $\forall \delta > 0$, $\exists \ell > 0$ and ψ_ℓ on D satisfying $\Delta\psi_\ell = 0$ on $D_\ell = [-\frac{1}{2}\ell, \frac{1}{2}\ell] \times [-\frac{1}{2}h, \frac{1}{2}h]$ such that $|\psi - \psi_\ell| < \delta$ on D.

Proof. Fix some $\ell >> h$, and impose the periodic boundary condition $\psi_\ell(-\frac{1}{2}\ell, y) = 0 = \psi(\frac{1}{2}\ell, y)$ for any $y \in [-\frac{1}{2}h, \frac{1}{2}h]$. Then, the general solution on D_ℓ is given by

$$\psi_\ell = \sum\nolimits_{n>0} (a_n \cos\alpha x + b_n \sin\alpha x)(c_n \cosh\alpha y + d_n \sinh\alpha y),$$

where α is some constant to be determined. Since $\psi(\frac{1}{2}\ell, y) = 0$, it will suffice to set $a_n = 0$ and $\frac{1}{2}\ell\alpha = n\pi$ $\forall n = 1, 2, \ldots$. So, absorbing the constants b_n into (c_n, d_n), define

$$\psi_\ell = \begin{cases} \sum_{n>0} \sin\frac{2n\pi}{\ell} x (c_n \cosh\frac{2n\pi}{\ell} y + d_n \sinh\frac{2n\pi}{\ell} y) \text{ on } D_\ell, \\ 0 \quad \text{on } D - D_\ell. \end{cases}$$

Then, $H_\pm = c_n \cosh\frac{n\pi}{\ell} h \pm d_n \sinh\frac{n\pi}{\ell} h$, where $H_\pm = \frac{2}{\ell}\int_{-\ell/2}^{\ell/2} h_\pm(x) \sin\frac{2n\pi}{\ell} x dx$ and hence, $c_n = \frac{H_- + H_+}{2\cosh(n\pi h/\ell)}$ and $d_n = \frac{H_+ - H_-}{2\sinh(n\pi h/\ell)}$ for all $n = 1, 2, \ldots$. To complete the proof, for any $\delta > 0$, $\lim_{x\to\pm\infty} h_\pm(x) = 0$ implies $\exists x_0 > 0$ such that $x > \frac{1}{2}x_0 \Rightarrow \max|h_\pm(x)| < \delta$ and $x < -\frac{1}{2}x_0 \Rightarrow \max|h_\pm(x)| < \delta$. Choose $\ell = 2x_0$. Then, by the maximum moduli principle, $|\psi| < \delta$ whenever $(x, y) \in D - D_\ell$ and $|\psi - \psi_\ell| < \delta$ on D_ℓ by construction.

As a side remark, observe that on setting $\omega_n = \frac{2\pi n}{\ell}$ and $\delta\omega = \omega_{n+1} - \omega_n = \frac{2\pi}{\ell}$ $\forall n$, and noting that the pair (c_n, d_n) can be written as $(c_n, d_n) = \frac{2\pi}{\ell}(\tilde{c}_n, \tilde{d}_n)$ from the above proof, then, at an informal level, it can be seen intuitively that $\lim_{\ell\to\infty} \psi_\ell \to \tilde{\psi} = \int_0^\infty \sin\omega x(\tilde{c}_n \cosh\omega y + \tilde{d}_n \sinh\omega y)\mathrm{d}\omega$ on D. Furthermore, as $\Delta\tilde{c} = 0$ and $\Delta\tilde{d} = 0$, it is clear that $\Delta\tilde{\psi} = 0$ on D. However, it is not at all obvious whether $\tilde{\psi}$ satisfies the Dirichlet boundary condition on ∂D.

3.5. Theorem

Let $\hat{C}_\pm$ be disks in $\tilde{\mathbf{R}}^2_\pm$ of radii $\hat{a}_\pm$ respectively such that $\hat{C}_\pm = \left\{(x,y) \in \tilde{\mathbf{R}}^2_\pm : x^2 + (y-\hat{y}_\pm)^2 \le \hat{a}^2_\pm\right\}$, for some fixed $\hat{\delta}_- < 0 < \hat{\delta}_+$, where $\left|\hat{\delta}_\pm\right| > \hat{a}_\pm$, $\hat{y}_\pm = \hat{\delta}_\pm \pm \frac{1}{2}h$, $\tilde{\mathbf{R}}^2_+ = \left\{(x,y) \in \mathbf{R}^2 : y > \frac{1}{2}h\right\}$ and $\tilde{\mathbf{R}}^2_- = \left\{(x,y) \in \mathbf{R}^2 : y < -\frac{1}{2}h\right\}$. Moreover, set $\hat{\Omega}_\pm = \tilde{\mathbf{R}}^2_+ - \hat{C}_\pm$. Given the pair of spaces $(\tilde{\mathbf{R}}^2_\pm, \varepsilon_0)$ and (D, ε), if the potential φ satisfies $\Delta\varphi = 0$ on $\hat{\Omega} = \hat{\Omega}_+ \cup D \cup \hat{\Omega}_-$ together with the following boundary conditions:

$$\varphi = \begin{cases} f(\theta) \text{ on } \partial\hat{C}_+, \\ g(\theta) \text{ on } \partial\hat{C}_-, \end{cases} \tag{18}$$

$$\varepsilon_0 \partial_y \varphi\big|_{y=\frac{1}{2}h^+} = \varepsilon \partial_y \varphi\big|_{y=\frac{1}{2}h^-}, \tag{19}$$

$$\varepsilon \partial_y \varphi\big|_{y=-\frac{1}{2}h^+} = \varepsilon_0 \partial_y \varphi\big|_{y=-\frac{1}{2}h^-}, \tag{20}$$

$$\varphi \quad \text{is continuous on } \partial\mathrm{D}_\pm, \tag{21}$$

$$\varphi \to 0 \quad \text{as} \quad |r| \to \infty, \tag{22}$$

where $\partial D_\pm = \{(x,y) : y = \pm\frac{1}{2}h\}$, then the solution in $\hat{\Omega}$ is given by

$$\varphi = \begin{cases} -\frac{1}{2\pi\varepsilon_0}\left\{a_0 + b_0 \ln r' + \sum_{n>0}\left(a_n r'^n + b_n r'^{-n}\right)\cos n\theta' \right. \\ \left. +\left(c_n r'^n + d_n r'^{-n}\right)\sin n\theta'\right\} \text{ on } \hat{\Omega}_+, \\ -\frac{1}{2\pi\varepsilon_0}\left\{a'_0 + b'_0 \ln r' + \sum_{n>0}\left(a'_n r'^n + b'_n r'^{-n}\right)\cos n\theta' \right. \\ \left. +\left(c'_n r'^n + d'_n r'^{-n}\right)\sin n\theta'\right\} \text{ on } \hat{\Omega}_-, \end{cases}$$

where

$$(r', \theta') = \left(\sqrt{\left(\frac{2r_0^2}{r}\right)^2 + r_0^2 - \frac{4r_0^3}{r}\cos\theta}, \arccos\frac{(2r_0^2/r)\cos\theta - r_0}{\sqrt{(2r_0^2/r)^2 + r_0^2 - (4r_0^3/r)\cos\theta}}\right),$$

and all the coefficients were defined in Lemma 3.3.

In particular, the solution $\varphi|D$ may be approximated by φ_ℓ such that $\forall\delta > 0$, $\exists\ell > 0$ satisfying $|\varphi|D - \varphi_\ell| < \delta$, where

$$\psi_\ell = \begin{cases} \sum_{n>0} \sin\frac{2n\pi}{\ell}x\left(c_n \cosh\frac{2n\pi}{\ell}y + d_n \sinh\frac{2n\pi}{\ell}y\right) \text{ on } D_\ell, \\ 0 \quad \text{on } D - D_\ell, \end{cases}$$

and $D_\ell = [-\frac{1}{2}\ell, \frac{1}{2}\ell] \times [-\frac{1}{2}h, \frac{1}{2}h]$.

Proof. Apply Lemma 3.4 to the case wherein $\varepsilon_+ = \varepsilon_0$ and $\varepsilon_- = \varepsilon$. Then, the solution φ_1 obtained applies to the region $\hat{\Omega}_+$. Likewise, on setting $\varepsilon_+ = \varepsilon$ and $\varepsilon_- = \varepsilon_0$, the solution φ_2 is obtained for $\hat{\Omega}_-$. Finally, the last assertion follows immediately from Lemma 3.4, where $h_\pm = \varphi|\partial D_\pm$.

3.6. Corollary

Given the conditions of Theorem 3.5, the potential difference across D is given by $\delta\varphi = h_+ - h_-$. Explicitly,

$$\delta\varphi = -\tfrac{1}{2\pi\varepsilon}\Big\{a_0 - a_0' + b_0 \ln\tfrac{r'_+}{r'_-} + \textstyle\sum_{n>0}\Big(\Big(a_n r'^n_+ + b_n r'^{-n}_+\Big) - \Big(a'_n r'^n_- + b'_n r'^{-n}_-\Big)\Big)$$
$$\cos n\theta' + \textstyle\sum_{n>0}\Big(\Big(c_n r'^n_+ + d_n r'^{-n}_+\Big) - \Big(c'_n r'^n_- + d'_n r'^{-n}_-\Big)\Big)\sin n\theta'\Big\}.$$

Repeating the proof of Corollary 2.4 *mutatis mutandis*, the result $j = \frac{\varepsilon}{h} v \partial_x \delta\varphi$ below follows from Corollary 3.6 and by noting that

$$\frac{\mathrm{d}}{\mathrm{d}x}\theta' = -\tfrac{1}{4r_0}\left(1 - \left(\tfrac{1}{4r_0}\,\tfrac{2r_0x - x^2 - \frac{1}{4}h^2}{r_0^2 - r_0x + x^2 + \frac{1}{4}h^2}\right)^2\right)^{-\frac{1}{2}}$$
$$\times \tfrac{1}{r_0^2 + x^2 + \frac{1}{4}h^2}\left(2(r_0 - x) + \frac{(2x - r_0)\left(x^2 + \frac{1}{4}h^2 - 2r_0x\right)}{r_0^2 - r_0x + x^2 + \frac{1}{4}h^2}\right).$$

3.7. Corollary

Suppose D is moving at a constant velocity $\mathbf{v} = v\mathbf{e}_x$. Then, the potential φ induces a displacement current density through D given by $j(x) = \frac{\varepsilon}{h} v \partial_x \delta\varphi(x)$. Explicitly,

$$j(x) = -\frac{v}{2\pi h}\left\{a_0 - a_0' + b_0 \ln\frac{r'_+}{r_-} + \sum_{n>0}\frac{n}{4r_0}\left(\left(a_n r'^n_+ + b_n r'^{-n}_+\right)\right.\right.$$
$$-\left(a'_n r'^n_- + b'_n r'^{-n}_-\right)\Big) \times \left(1 - \left(\tfrac{1}{4r_0}\,\tfrac{2r_0x - x^2 - \frac{1}{4}h^2}{r_0^2 - r_0x + x^2 + y^2}\right)^2\right)^{-\frac{1}{2}} \frac{1}{r_0^2 + x^2 + \frac{1}{4}h^2}$$
$$\times\left(2(r_0 - x) + \frac{(2x - r_0)\left(x^2 + \frac{1}{4}h^2 - 2r_0x\right)}{r_0^2 - r_0x + x^2 + \frac{1}{4}h^2}\right)\sin\left(n \arccos\tfrac{1}{4r_0}\,\frac{2r_0x - x^2 - \frac{1}{4}h^2}{r_0^2 - r_0x + x^2 + \frac{1}{4}h^2}\right)$$
$$-\textstyle\sum_{n>0}\tfrac{n}{4r_0}\left(\left(c_n r'^n_+ + c_n r'^{-n}_+\right) - \left(c'_n r'^n_- + d'_n r'^{-n}_-\right)\right)$$

$$\times\cos\left(n\arccos\frac{1}{4r_0}\frac{2r_0x-x^2-\frac{1}{4}h^2}{r_0^2-r_0x+x^2+\frac{1}{4}h^2}\right)\left(1-\left(\frac{1}{4r_0}\frac{2r_0x-x^2-\frac{1}{4}h^2}{r_0^2-r_0x+x^2+\frac{1}{4}h^2}\right)^2\right)^{-\frac{1}{2}}$$
$$\times\frac{1}{r_0^2+x^2+\frac{1}{4}h^2}\left(2(r_0-x)+\frac{(2x-r_0)(x^2+\frac{1}{4}h^2-2r_0x)}{r_0^2-r_0x+x^2+\frac{1}{4}h^2}\right)\Bigg\}.$$

4. CONCLUSION

It is clear from Section 2 that the potential field and hence the electric field are symmetric about the y-axis, wherein the cylinders were not rotating. In particular, when the dielectric plane is moving at a constant velocity, the symmetry of the field profile remains unchanged.

It is equally clear from Section 3 that when the cylinders are PEC, introducing a constant rotation does not impact the symmetry of the potential and electric fields in the domain of definition. However, a moving dielectric plane will experience a displacement current as the fields are non-uniform along the x-axis; that is, the fields tend to zero in the limit as $x \to \pm\infty$. Hence, along the moving dielectric plane, each differential element sees a changing electric field.

Finally, it is evident that the dielectric constant of the plane and that of the tubular medium surrounding the cylinders impact the field profile. From an application perspective, changing the dielectric constants will change the dynamics of charged particles on the cylinder moving onto the plane via convection. This has strong implications in certain research and development industries, one of which is the printer industry mentioned in the Introduction. For instance, it furnishes a theoretical basis for modelling toner transfer in laser printers.

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