Progress In Electromagnetics Research Letters, Vol. 24, 119–128, 2011

REFLECTION AND TRANSMISSION OF ELECTROMAGNETIC WAVE DUE TO A QUASI-FRACTIONAL-SPACE SLAB

A. M. Attiya*

Faculty of Engineering, King Saud University, Riyadh, Saudi Arabia

Abstract—A new method is introduced to construct a slab that has electric fields with propagation properties which are equivalent to a fractional-space wave equation in two-coordinate system. While its magnetic fields have propagation properties which are equivalent to the complementary fractional-space wave equation. Analytical forms for the reflection and transmission coefficients of this slab are derived. Results of these reflection and transmission coefficients show that such quasi-fractional-space slab has spatial and frequency selectivity properties.

1. INTRODUCTION

Fractional calculus is found to be a good tool for presenting homogenous models for fractal boundaries in different physical problems including electromagnetic wave propagation [1, 2]. This was the motivation for many researchers to introduce extensive studies of formulating and solving fractional dimensional problems in electromagnetics [3–13]. Engheta [3, 4] introduced an early investigation of fractional solution of wave equation. This analysis was mainly based on introducing source distributions which are equivalent to fractional-dimensional Dirac delta functions. This analysis was more investigated by Naqvi and Rizvi [5] to introduce dual solutions and corresponding sources for this fractional electromagnetic wave. More recently, another view for fractional-space wave was introduced based on the propagating medium instead of the source. As examples, chiral media and perfect electromagnetic (PEMC) boundary are found to be good candidates that can be modeled by using fractional calculus [6–8]. Other problems related to propagation, reflection and diffraction

Received 11 May 2011, Accepted 31 May 2011, Scheduled 8 June 2011
* Corresponding author: Ahmed M. Attiya (aattiya@ksu.edu.sa).

of electromagnetic waves in fractional space medium have also been studied [9–13].

On the other hand, recent advances in synthesized materials, like metamaterials, artificial magnetic conductors and perfect electromagnetic conductors, open the door for investigating the features of non-standard materials which may be available in near future. These synthesized materials were the key for introducing new applications like electromagnetic cloaking [14, 15] by controlling the constitutive parameters of the cloaking structure. This is the motivation in this research to study the possibility of constructing a slab that has fractional space properties by controlling its permittivity and permeability.

The present analysis is based on assuming that both the permittivity and permeability of the slab are general functions of the normal direction on the slab. By following conventional steps of deriving the wave equations for electric and magnetic field, one can obtain a general form for these wave equations for general constitutive-parameters distributions. The relations between these constitutive parameters are chosen such that the wave equation of the electric fields would correspond to a fractional space wave equation in two-coordinate system. These relations are not found to be satisfying the same fractional wave equation for the magnetic field components, but they satisfy the complementary fractional-space wave equation as it is shown in the following section. Reflection and transmission coefficients of this quasi-fractional-space slab are derived analytically in Section 3. In Section 4, sample results for these reflection and transmission coefficients are presented and discussed.

2. EQUIVALENT PARAMETERS OF A QUASI-FRACTIONAL-SPACE MEDIUM

The slab is assumed to be infinite in x and y directions and it is bounded in the region $z_i \leq z \leq z_b$. Assuming that the fields in all regions are transverse magnetic with respect to the normal direction on the slab and the xz plane is the plane of incidence. Thus, the field components inside or outside the slab can be presented as:

$$\mathbf{E} = E_x(x,z)\mathbf{a}_x + E_z(x,z)\mathbf{a}_z \tag{1a}$$

$$\mathbf{H} = H_y(x,z)\mathbf{a}_y \tag{1b}$$

The permittivity and permeability of the slab are assumed to be depending only on the normal z direction. The following analysis is used to determine the required dependence which makes the wave propagation equation inside the slab is equivalent to fractional-dimensional wave equation. The rotation of the differential form of Faraday's law inside the slab can be presented as:

$$\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}=\left(j\omega\frac{\partial\mu_1}{\partial z}H_y+j\omega\mu_1\frac{\partial H_y}{\partial z}\right)\mathbf{a}_x-j\omega\mu_1\frac{\partial H_y}{\partial x}\mathbf{a}_z \quad (2)$$

where

$$\frac{\partial}{\partial z}H_y = -j\omega\varepsilon_1 E_x \quad \text{(From Ampere's Law)} \quad (3a)$$

$$H_y = -\frac{1}{j\omega\mu_1}\left(\frac{\partial}{\partial z}E_x-\frac{\partial}{\partial x}E_z\right) \quad \text{(From Faraday's Law)} \quad (3b)$$

$$\nabla\cdot\mathbf{E} = -\frac{1}{\varepsilon_1}\frac{\partial\varepsilon_1}{\partial z}E_z \quad \text{(From Gauss' Law)} \quad (3c)$$

By using Eq. (3) into Eq. (2), one can obtain the generalized wave equation for a slab with varying permittivity and permeability along its normal direction for 2-D TM wave as follows:

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial z^2}+\frac{1}{\varepsilon_1}\frac{\partial\varepsilon_1}{\partial z}\frac{\partial E_z}{\partial x}+\frac{1}{\mu_1}\frac{\partial\mu_1}{\partial z}\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right)+\omega^2\mu_1\varepsilon_1 E_x=0 \quad (4)$$

Assuming that the permittivity and the permeability of the slab are related as follows:

$$\frac{1}{\varepsilon_1}\frac{\partial\varepsilon_1}{\partial z}=-\frac{1}{\mu_1}\frac{\partial\mu_1}{\partial z}=\frac{D-2}{z} \quad (5)$$

where D is a constant and $0 \leq D \leq 2$. Thus, the resulting wave equation of the electric field in Eq. (4) would be:

$$\frac{\partial^2 E_x}{\partial z^2}+\frac{\partial^2 E_x}{\partial x^2}+\frac{D-2}{z}\frac{\partial E_x}{\partial z}+\omega^2\mu_1\varepsilon_1 E_x=0 \quad (6)$$

which is exactly the form of the fractional-dimensional wave Equation for two-coordinate system [9].

By using Eq. (5), one can obtain the permittivity and the permeability of the corresponding slab are given by:

$$\varepsilon_1(z) = \varepsilon_0\varepsilon_{r1}z^{D-2} \quad (7a)$$

$$\mu_1(z) = \mu_0\mu_{r1}z^{-(D-2)} \quad (7b)$$

It can be noted that this slab has a constant propagation wave number where $k_1 = \omega\sqrt{\mu_1\varepsilon_1} = k_0\sqrt{\varepsilon_{r1}\mu_{r1}}$ while its intrinsic impedance is varying as $\eta_1 = \eta_0\sqrt{\mu_{r1}/\varepsilon_{r1}}z^{-(D-2)}$.

It is interesting to note that the wave equation for the magnetic field in this case is the complementary for the wave equation of the electric field as follows:

$$\frac{\partial^2}{\partial z^2}H_y+\frac{\partial^2}{\partial x^2}H_y+\frac{2-D}{z}\frac{\partial H_y}{\partial z}+\omega^2\mu_1\varepsilon_1 H_y=0 \quad (8)$$

Analytical solutions of Eqs. (6) and (8) can be obtained by using separation of variables to get the fields inside the fractional-space slab as follows [10]:

$$E_x = (k_{z1}z)^n \left(C_1 J_n(k_{z1}z) + C_2 Y_n(k_{z1}z)\right) e^{-jk_x x} \quad (9a)$$

$$H_y = (k_{z1}z)^{n_h} \left(D_1 J_{n_h}(k_{z1}z) + D_2 Y_{n_h}(k_{z1}z)\right) e^{-jk_x x} \quad (9b)$$

where $n = |3 - D|/2$, $n_h = |D - 1|/2$ and $\omega^2 \mu_1 \varepsilon_1 = k_1^2 = k_x^2 + k_{z1}^2$.

It can be noted that for the limit where $D = 2$ which corresponds to the conventional 2-D case, both n and n_h would be 0.5. In this case the limits of the Bessel functions in Eq. (9) would be simply combination of $\sin(k_{z1}z)$ and $\cos(k_{z1}z)$ which are the basic functions of standing wave-function inside a 2-D slab. It should also be noted that the relation between the magnitudes of the electric and magnetic field components in this case is not still the simple characteristic impedance as in the case of the conventional fractional space electromagnetic field solution [10]. To obtain this relation it would be required to apply Maxwell's curl equations on Eq. (9). This step would introduce only two equations. Additional relations based on the boundary conditions are used combined with these two equations to obtain the amplitudes of the electric and magnetic field components as it is discussed in the following section.

By following the same steps, one can obtain the same wave equations of Eqs. (6) and (8) for the case of a transverse electric fields. The only difference in this case is exchanging the transverse field components to be E_y and H_x. The remaining parts would be exactly the same as the previous case.

Before ending this section, it should be mentioned that the case where the permittivity and permeability are functions of x has also been studied. In this case we could obtain wave equations which are similar to Eqs. (6) and (8) while the fractional-space term is function of x. However, the propagation wave number k_1 is found to be also a function of x, not a constant like in the present case. Thus, the resulting equations in this case do not coincide with the conventional definition of the fractional-dimensional wave equation.

3. REFLECTION AND TRANSMISSION OF ELECTROMAGNETIC WAVES DUE TO A FRACTIONAL-SPACE SLAB

For the case of a TM oblique incident plane wave in free space side, the incident fields can be presented as:

$$E_{xi} = \cos\theta_i \exp\left(-jk_x x - jk_{zo}z\right) \quad (10a)$$

$$E_{zi} = -\sin\theta_i \exp\left(-jk_x x - jk_{zo}z\right) \quad (10b)$$

$$H_{yi} = \frac{1}{\eta_0} \exp\left(-jk_x x - jk_{zo} z\right) \tag{10c}$$

where $k_x = k_0 \sin\theta_i$ and $k_{zo} = k_0 \cos\theta_i$. On the other hand, the reflected and transmitted fields can be presented as:

$$E_{xr} = \Gamma \cos\theta_i \exp\left(-jk_x x + jk_{zo} z\right) \tag{11a}$$

$$E_{zr} = \Gamma \sin\theta_i \exp\left(-jk_x x + jk_{zo} z\right) \tag{11b}$$

$$H_{yr} = -\frac{\Gamma}{\eta_0} \exp\left(-jk_x x + jk_{zo} z\right) \tag{11c}$$

$$E_{xt} = \tau \cos\theta_i \exp\left(-jk_x x - jk_{zo} z\right) \tag{12a}$$

$$E_{zr} = -\tau \sin\theta_i \exp\left(-jk_x x - jk_{zo} z\right) \tag{12b}$$

$$H_{yt} = \frac{\tau}{\eta_0} \exp\left(-jk_x x - jk_{zo} z\right) \tag{12c}$$

where Γ and τ in this case are the TM reflection and transmission coefficients respectively. Thus, it is required to determine six unknown quantities which correspond to the four amplitude values of Eq. (9) and the transmission and the reflection coefficients. These unknowns can be obtained by applying continuity of the tangential electric and magnetic fields on the two sides of the fractional-space slab. These continuity equations introduce four equations. To obtain the other two equations, one can use Faraday's and Ampere's laws to obtain the relations between the electric and magnetic fields inside the slab. Following these steps, one can obtain these six unknown quantities as follows:

$$\begin{bmatrix} \Gamma \\ (k_{z1} z_i)^n C_1 \\ (k_{z1} z_i)^n C_2 \\ \eta_0 (k_{z1} z_i)^{n_h} D_1 \\ \eta_0 (k_{z1} z_i)^{n_h} D_2 \\ \tau \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} \cos\theta_i \exp(-jk_{z0} z_i) \\ \exp(-jk_{z0} z_i) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{13a}$$

where

$$\mathbf{A} = \left[\begin{matrix} -\cos\theta_i \exp(jk_{z0} z_i) & J_n(k_{z1} z_i) & Y_n(k_{z1} z_i) \\ \exp(jk_{z0} z_i) & 0 & 0 \\ 0 & J_n(k_{z1} z_i) & Y_n(k_{z1} z_i) \\ 0 & J_{n-1}(k_{z1} z_i) & Y_{n-1}(k_{z1} z_i) \\ 0 & \left(\frac{z_b}{z_i}\right)^n J_n(k_{z1} z_b) & \left(\frac{z_b}{z_i}\right)^n Y_n(k_{z1} z_b) \\ 0 & 0 & 0 \end{matrix} \right.$$

$$\left.\begin{matrix} 0 & 0 & 0 \\ J_{n_h}(k_{z1}z_i) & Y_{n_h}(k_{z1}z_i) & 0 \\ \frac{k_{z1}}{j\eta_0\omega\varepsilon_1(z_i)}J_{n_h-1}(k_{z1}z_i) & \frac{k_{z1}}{j\eta_0\omega\varepsilon_1(z_i)}Y_{n_h-1}(k_{z1}z_i) & 0 \\ \frac{-k_{z1}}{j\eta_0\omega\varepsilon_1(z_i)}J_{n_h}(k_{z1}z_i) & \frac{-k_{z1}}{j\eta_0\omega\varepsilon_1(z_i)}Y_{n_h}(k_{z1}z_i) & 0 \\ 0 & 0 & -\cos\theta_i\exp(-jk_{z0}z_b) \\ \left(\frac{z_b}{z_i}\right)^{n_h}J_{n_h}(k_{z1}z_b) & \left(\frac{z_b}{z_i}\right)^{n_h}Y_{n_h}(k_{z1}z_b) & -\exp(-jk_{z0}z_b) \end{matrix}\right] \tag{13b}$$

By following similar steps, one can obtain the reflection and transmission coefficients of TE oblique incident plane wave due to a fractional space slab as follows:

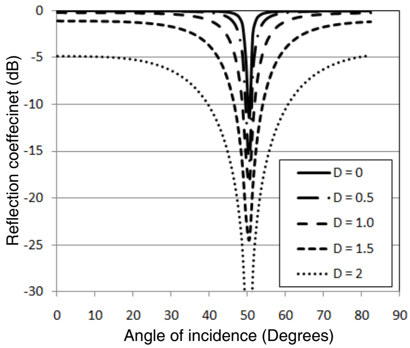
$$\begin{bmatrix} \Gamma \\ (k_{z1}z_i)^n C_1 \\ (k_{z1}z_i)^n C_2 \\ \eta_0(k_{z1}z_i)^{n_h} D_1 \\ \eta_0(k_{z1}z_i)^{n_h} D_2 \\ \tau \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} \exp(-jk_{z0}z_i) \\ -\cos\theta_i\exp(-jk_{z0}z_i) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14a}$$

$$\mathbf{B} = \left[\begin{matrix} -\exp(jk_{z0}z_i) & J_n(k_{z1}z_i) & Y_n(k_{z1}z_i) \\ -\cos\theta_i\exp(jk_{z0}z_i) & 0 & 0 \\ 0 & J_n(k_{z1}z_i) & Y_n(k_{z1}z_i) \\ 0 & J_{n-1}(k_{z1}z_i) & Y_{n-1}(k_{z1}z_i) \\ 0 & \left(\frac{z_b}{z_i}\right)^n J_n(k_{z1}z_b) & \left(\frac{z_b}{z_i}\right)^n Y_n(k_{z1}z_b) \\ 0 & 0 & 0 \end{matrix}\right.$$

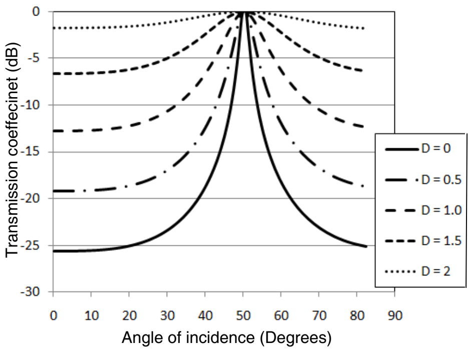
$$\left.\begin{matrix} 0 & 0 & 0 \\ J_{n_h}(k_{z1}z_i) & Y_{n_h}(k_{z1}z_i) & 0 \\ \frac{j\omega\mu_1(z_i)}{\eta_0 k_{z1}}J_{n_h-1}(k_{z1}z_i) & \frac{j\omega\mu_1(z_i)}{\eta_0 k_{z1}}Y_{n_h-1}(k_{z1}z_i) & 0 \\ -\frac{j\omega\mu_1(z_i)}{\eta_0 k_{z1}}J_{n_h}(k_{z1}z_i) & -\frac{j\omega\mu_1(z_i)}{\eta_0 k_{z1}}Y_{n_h}(k_{z1}z_i) & 0 \\ 0 & 0 & -\exp(-jk_{z0}z_b) \\ \left(\frac{z_b}{z_i}\right)^{n_h}J_{n_h}(k_{z1}z_b) & \left(\frac{z_b}{z_i}\right)^{n_h}Y_{n_h}(k_{z1}z_b) & \cos\theta_i\exp(-jk_{z0}z_b) \end{matrix}\right] \tag{14b}$$

4. RESULTS AND DISCUSSIONS

In this section, we present sample results for the reflection and transmission coefficients of a quasi-fractional-space slab. The present



(a)



(b)

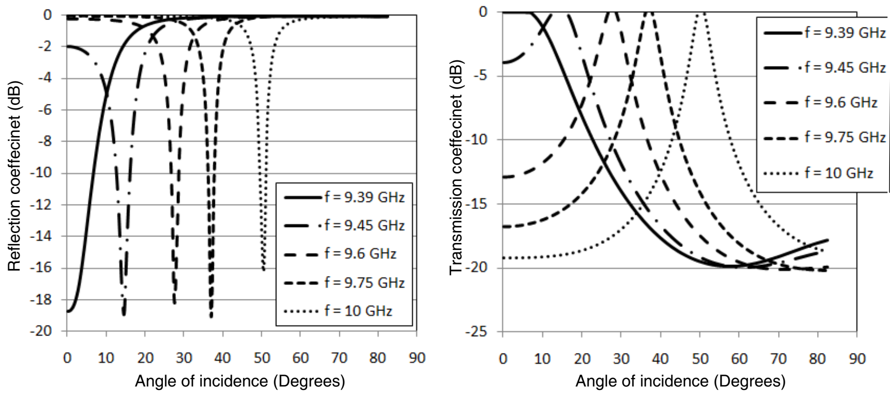
Figure 1. TM Reflection and transmission coefficients of a quasi-fractional-space slab as functions of incident angle for different values of fractional-space parameter D. The parameters of the slab are $\varepsilon_{r1} = 5$, $\mu_{r1} = 1$, $z_i = 0.2\,\text{m}$ and $z_b = 0.25\,\text{m}$. The frequency of the incident wave is 10 GHz. (a) TM reflection coefficient. (b) TM transmission coefficient.

results are based on TM fields. Similar results are obtained for TE fields, thus these results are not presented here.

Figure 1 shows the TM reflection and transmission coefficients for a quasi-fractional-space slab as functions of the incident angle. The parameters of the slab are $\varepsilon_{r1} = 5$, $\mu_{r1} = 1$, $z_i = 0.2\,\text{m}$ and $z_b = 0.25\,\text{m}$. The frequency of the incident plane wave is fixed at 10 GHz. It can be noted that the transmission coefficient in this case has a maximum at the angle on incidence of nearly 50°. By decreasing the fractional-space parameter D, it can be noted that the transmission coefficient would have a sharp decrease around the peak transmission angle. This corresponds to spatial selectivity property.

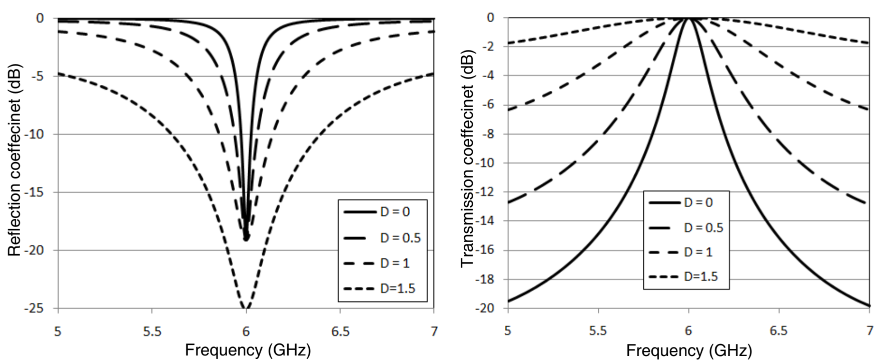
Figure 2 shows the dependence of this spatial selectivity property on frequency. In this case the fractional-space parameter D is fixed to be 0.5. The remaining parameters are the same as in Fig. 1. It can be noted from Fig. 2 that the maximum transmission angle depends on the operating frequency and it can be spanned over wide scanning angles starting from 0°.

Figure 3 shows the dependence of these reflection and transmission coefficients on frequency. The fields are assumed to be normal incident. The parameters of the quasi-fractional-space slab in this case are $\varepsilon_{r1} = 1$, $\mu_{r1} = 1$, $z_i = 0.2\,\text{m}$ and $z_b = 0.25\,\text{m}$. It can be noted that the slab in this case represents a frequency selective surface with a center frequency of nearly 6 GHz. By decreasing the fractional-space factor



(a) (b)

Figure 2. TM Reflection and transmission coefficients of a quasi-fractional-space slab as functions of incident angle for different frequencies. The parameters of the slab are $D = 0.5$, $\varepsilon_{r1} = 5$, $\mu_{r1} = 1$, $z_i = 0.2\,\mathrm{m}$ and $z_b = 0.25\,\mathrm{m}$. (a) TM reflection coefficient. (b) TM transmission coefficient.



(a) (b)

Figure 3. TM Reflection and transmission coefficients of a quasi-fractional-space slab as functions of frequency for different values of fractional-space parameter D. The parameters of the slab are $\varepsilon_{r1} = 1$, $\mu_{r1} = 1$, $z_i = 0.2\,\mathrm{m}$ and $z_b = 0.25\,\mathrm{m}$. (a) TM reflection coefficient. (b) TM transmission coefficient.

D, this frequency selectivity increases as it can be noted from Fig. 3.

It can be concluded that by controlling the fractional-space factor one can control both the spatial and frequency selectivity of the quasi-fractional-space slab.

5. CONCLUSION

In this paper we presented a detailed analysis of the parameters for a slab to be equivalent to a quasi-fractional-space slab where the electric fields inside the slab are following fractional space wave equation and the magnetic fields are following the complementary fractional space wave equation. It is shown that these properties can be obtained if the relative permittivity and permeability of the slab are proportional to z^{D-2} and z^{2-D} respectively where z is the normal direction on the slab and D is the fractional space factor. These fractional space wave equations are used to obtain closed forms for the fields inside the fractional space slab based on Bessel functions. These solutions are used combined with boundary conditions on the two sides of the fractional-space slab and Maxwell's curl equations to obtain the reflection and transmission coefficients of a quasi-fractional-space slab. The results of these reflection and transmission coefficients show interesting spatial and frequency selectivity properties of these fractional space slabs.

ACKNOWLEDGMENT

The author would like to thank M. Zubair for the useful discussions and the references related to the subject of the present research.

REFERENCES

1. Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific Publishing, 2000.
2. Sabatier, J., O. P. Agrawal, and J. A. Tenreiro Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, 2007.
3. Engheta, N., "Use of fractional integration to propose some 'fractional' solutions for the scalar helmholtz equation," *Progress In Electromagnetics Research*, Vol. 12, 107–132, 1996.
4. Engheta, N., "On the role of fractional calculus in electromagnetic theory," *IEEE Antennas and Propagation Magazine*, Vol. 39, No. 4, 35–46, 1997.
5. Naqvi, Q. A. and A. A. Rizvi, "Fractional dual solutions and corresponding sources," *Progress In Electromagnetics Research*, Vol. 25, 223–238, 2000.
6. Lakhtakia, A., "A representation theorem involving fractional

derivatives for linear homogeneous chiral media," *Microwave and Optical Technology Letters*, Vol. 28, No. 6, 385–386, 2001.

7. Naqvi, Q. A., "Fractional dual interface in chiral nihility medium," *Progress In Electromagnetics Research Letters*, Vol. 8, 135–142, 2009.
8. Hussain, A., Q. A. Naqvi, and M. Abbas, "Fractional duality and perfect electromagnetic conductor (PEMC)," *Progress In Electromagnetics Research*, Vol. 71, 85–94, 2007.
9. Zubair, M., M. J. Mughal, Q. A. Naqvi, and A. A. Rizvi, "Differential electromagnetic equations in fractional space," *Progress In Electromagnetics Research*, Vol. 114, 255–269, 2011.
10. Zubair, M., M. J. Mughal, and Q. A. Naqvi, "The wave equation and general plane wave solutions in fractional space," *Progress In Electromagnetics Research Letters*, Vol. 19, 137–146, 2010.
11. Zubair, M., M. J. Mughal, and Q. A. Naqvi, "An exact solution of the cylindrical wave equation for electromagnetic field in fractional dimensional space," *Progress In Electromagnetics Research*, Vol. 114, 443–455, 2011.
12. Baleanu, D., A. K. Golmankhaneh, and A. K. Golmankhaneh, "On electromagnetic field in fractional space," *Nonlinear Analysis: Real World Applications*, Vol. 11, No. 1, 288–292, 2010.
13. Ivakhnychenko, M. V. and E. I. Veliev, "Fractional operators approach in reflection and diffraction problems," *8th International Conference on Telecommunications in Modern Satellite, Cable and Broadcasting Services, TELSIKS*, Vol. 254–257, No. 26–28, Sep. 2007.
14. Cheng, X., H. Chen, X.-M. Zhang, B. Zhang, and B.-I. Wu, "Cloaking a perfectly conducting sphere with rotationally uniaxial nihility media in monostatic radar system," *Progress In Electromagnetics Research*, Vol. 100, 285–298, 2010.
15. Zhai, Y.-B. and T.-J. Cui, "Three-dimensional axisymmetric invisibility cloaks with arbitrary shapes in layered-medium background," *Progress In Electromagnetics Research B*, Vol. 27, 151–163, 2011.