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# FORMULATING A VECTOR WAVE EXPRESSION FOR POLARIMETRIC GNSS SURFACE SCATTERING

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**Abstract**—This paper formulates a simple vector integral expression for electromagnetic waves received after scattering from a surface. The derived expression is an alternative to the Stratton-Chu equation frequently used for polarimetric surface scattering. It is intended for use in polarimetric Global Navigation Satellite System (GNSS) ocean remote sensing, or any type of polarimetric remote sensing from surfaces, when the surface roughness pattern is known from simulation or data. This paper is intended to present a complete accounting of the steps leading to the simpler vector integral expression. It therefore starts with the scalar case, using Maxwell's equations and Green's theorem. It principally treats the case of a transmitter within the integration volume, but discusses how the formalism changes if the transmitter is outside of the integration volume, as with plane waves. It then shows how the scalar expression can be extended to a vector expression for the component of the electric field in an arbitrary receive-polarization direction due to scattering from a rough surface of an incident wave with an arbitrary transmit polarization. It uses the Kirchhoff, or tangent-plane, approximation in which each facet on the ocean is considered to specularly reflect the incoming signal. The derived vector expression is very similar to that for a scalar wave, but it includes all vector properties of the scattering. Equivalence is demonstrated between the Stratton-Chu equation and the derived, simpler expression, which is operationally easier to code than the Stratton-Chu equation in many modeling applications.

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## 1. INTRODUCTION

The ocean-scattered electromagnetic waves from Global Navigation Satellite System (GNSS) signals are being explored for their potential to estimate characteristics of the ocean surface and inferred wind vectors, e.g., [1–9]. In addition to the scattered power received from GNSS signals, it is plausible that the polarimetric characteristics of the amplitude and phase of the received wave will also carry information about the surface roughness and consequent wind speed [10]. In the extreme, a completely smooth ocean surface will exhibit specular scattering with horizontal and vertical amplitudes and phases given by the smooth-surface Fresnel coefficients [11]. In contrast, a very rough surface will have no preferred direction and will therefore scatter horizontal and vertical waves similarly. In order to model the polarimetric signatures from ocean-scattered signals, when the surface normals and local reflection properties are specified via simulation from model surface power spectra [12] or data, fundamental expressions are required relating the field received at any polarization to the transmit polarization and ocean roughness.

The expressions in the literature relating received fields to surface roughness fall into two broad categories: 1) They describe a scalar version of the field based on Green's theorem [11] and the Helmholtz wave equation [2, 13], or 2) they describe a vector field using the Stratton-Chu equation [14]. Though scalar formulations have an appealing simplicity, they usually mention polarization but do not rigorously show the steps needed to treat polarimetric scattering. The Stratton-Chu equation is a full vector treatment, but is more calculationally cumbersome than the scalar formulation, requiring cross products and/or curls of surface fields. The Stratton-Chu equation has been repeatedly presented with the steps needed to treat polarimetric scattering in the tangent-plane approximation, including establishing a local coordinate system as in (14) and (15) [15–17]. The starting point for the simpler vector expression, (11) in this paper, was cited but not proven to be equivalent to the Stratton-Chu equation as early as 1949 [18]. Also, [18] did not discuss polarimetric scattering in the tangent-plane local coordinate system. Both the scalar and vector approaches characterize the incident wave as originating within an integration volume (spherical) [14, 19] or outside of the integration volume (spherical or plane) [13, 15, 18]. Direct and scattered fields in the final expressions have sometimes been misidentified [13], possibly because they manifest differently, depending on whether the transmitter is considered inside or outside the integration volume. All of the approaches cited in this paper use the

Kirchhoff tangent-plane approximation, as will we — that the field near the surface is equal to that specularly reflected from a local tangent plane.

Our aim in this paper is a complete presentation of the elements that lead to a polarimetric vector scattering expression (16), which is simpler than the Stratton-Chu equation, and therefore easier to code for simulations and analysis. This paper also proves that the simpler expression is rigorously and generally equivalent to the Stratton-Chu equation, and we formulate this simpler expression with the local coordinate system needed for polarimetric scattering in the tangent plane approximation ((14) and (15)). Particular attention is paid to the location and physical description of transmitter sources to clearly delineate their appearance in the final expressions for incident and scattered field.

In order to facilitate a heuristic approach to deriving the vector expression (16), the next section first derives the scattered field for a scalar wave as in [13]. It shows the assumptions made in that reference to treat only horizontal polarization, and it shows the importance of treating the location of the source and the resulting manifestation of the incident field in the final result. Section 3 derives the principal formulation of this paper, extending [13] to arbitrary polarization and retaining the source function considerations of Section 2. Section 4 shows the equivalence between the simple equation of Section 3 and the Stratton-Chu equation. Section 5 is a summary.

## 2. RECEIVED FIELD FOR A SCALAR WAVE

Figure 1 shows a GNSS satellite transmitting from the left toward a rough surface indicated by the wavy dotted line. The scattered wave is received on the right at the position of the antenna icon. Following conventions of [13] for incident and scattered monochromatic waves, $\vec{k}_1$ is the wave number — $k = 2\pi/\lambda$, with wavelength $\lambda$ — times the unit vector pointing from the transmitter to the indicated point on the surface $\vec{x}\,'$, while $\vec{k}_2$ is wave number times the unit vector from the surface to the receiver:

$$\begin{aligned} \vec{k}_1 &\equiv k\frac{\vec{x}\,' - \vec{x}_T}{|\vec{x}\,' - \vec{x}_T|} \\ \vec{k}_2 &\equiv k\frac{\vec{x}_R - \vec{x}\,'}{|\vec{x}_R - \vec{x}\,'|} \\ \vec{k}_{1,ref} &= \vec{k}_1 - 2\left(\vec{k}_1 \cdot \hat{n}\right)\hat{n} \end{aligned} \tag{1}$$

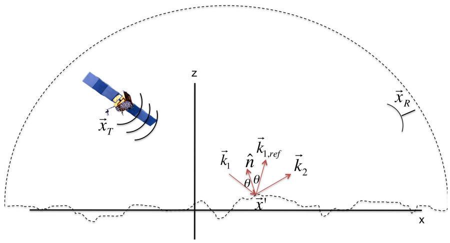

**Figure 1.** A GNSS transmitter shown on the left transmits with wave vectors $\vec{k}_1$ to the a surface point $\vec{x}'$, with normal $\hat{n}$. $\vec{k}_{1,ref}$ is the propagation direction of the wave that would be specularly reflected from a tangent plane at $\vec{x}'$, and $\vec{k}_2$ is the scattered wave vector. For the characterization of the scalar wave in Section 2, the field is assumed polarized in a direction into the paper ($\hat{y}$), and the surface is rough only in the $x$-direction. The dotted line indicates the integration volume. For most of the calculations in this paper, the transmitter is considered within the integration volume, as shown.

where $\hat{n}$ is the surface normal, and $\vec{k}_{1,ref}$ is the wave vector of the incident $\vec{k}_1$ field specularly reflected by a tangent plane (not shown), and is given by (1), due to equality of local incidence and reflection angle, $\theta$. Note that $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_{1,ref}$, $\hat{n}$, and $\theta$ — which is also the angle between $\vec{k}_{1,ref}$ and $\hat{n}$ — all depend on the selected surface coordinate $\vec{x}\,'$. The three dimensional positions of the transmitter and receiver are $\vec{x}_T$ and $\vec{x}_R$ respectively. Unlike [13] which posits plane waves from the transmitter — which places the transmitter infinitely far away and therefore outside of the volume — we place the transmitter within the volume, describe the source function, and later say in Section 2.3 how results would be different if the transmitter were outside of the integration volume.

### 2.1. Scalar Wave in Terms of Source Function and Surface Integral

A scalar wave — a single component of the electric field — will initially be considered as a simplified case of the general vector, polarimetric field treated in the next section. The scalar formulation of [13] must conform to the following restriction: The transmit and receive polarization, as well as the polarization of electromagnetic waves at all parts of the ocean surface contributing to the received field, are taken to be in the direction of a unit vector along the $y$-axis, $\hat{y}$ (implying horizontal polarization), perpendicular to the plane of the paper, or scattering plane. This restriction requires that the rough surface must be considered to be one dimensional, i.e., it only varies in the $x$-direction and cannot scatter the incident $\hat{y}$ field into polarizations other than horizontal. The restriction also implies that the incident, spherical wave must be considered to originate far enough away so that the direction of incident polarization is $\hat{y}$ at any point on the surface contributing to the received wave — i.e., the curvature of spherical wavefronts creates a negligible change in the polarization vector over the surface illuminated by the transmitter. If this restriction is met, then the scattering calculation can be seen as determining the $\hat{y}$ component of the received electric field due to scattering of the $\hat{y}$ field at the surface points $\vec{x}\,'$. In Section 3, the ocean will be allowed to be rough along both dimensions of the surface, and arbitrary transmit and receive polarizations will be treated.

Green's theorem — or "second identity" — [11, 13] is the starting point for calculating one Fourier component of the electric field in the $\hat{y}$ direction at the receiver, $E_y(\vec{x}_R)$. This field is at wavenumber $k = \omega/c$, where $\omega$ is the frequency of the Fourier component and $c$ is the speed of light. Applying Green's theorem to the Green's function at wavenumber $k$, $G(\vec{x}_R, \vec{x}\,')$, and to $E_y$, it relates the volume integral of these functions and their Laplacians to a surface integral of the same functions and normal derivatives:

$$\begin{aligned}&\int_V [G(\vec{x}_R, \vec{x}\,'')\nabla^2 E_y(\vec{x}'') - E_y(\vec{x}\,'')\nabla^2 G(\vec{x}_R, \vec{x}\,'')]d^3x''\\ &= \int_S [E_y(\vec{x}\,')\vec{\nabla} G(\vec{x}_R, \vec{x}\,') - G(\vec{x}_R, \vec{x}\,')\vec{\nabla} E_y\left(\vec{x}\,'\right)] \cdot \hat{n}\, d^2x' \qquad (2)\\ &\qquad \text{with} \quad G(\vec{x}_R, \vec{x}\,') \equiv \frac{e^{ik|\vec{x}_R - \vec{x}'|}}{|\vec{x}_R - \vec{x}\,'|}\end{aligned}$$

where the integration on the top line of (2) is taken over all points $\vec{x}\,''$ in the volume within the dashed line in Figure 1. The surface integral is taken along all points $\vec{x}\,'$ in the dashed line, and the normal $\hat{n}$ points inward, as it does throughout this paper. In (2) and equations that follow, it is understood that the gradients and Laplacians operate on the primed and double primed coordinates, respectively.

The Maxwell equations needed to solve (2) for $E_y(\vec{x}_R)$ are e.g., [11]

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= 4\pi \frac{\rho}{\epsilon}\\ \vec{\nabla} \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad (3)\\ \vec{\nabla} \times \vec{B} &= \frac{4\pi\mu}{c}\vec{J} + \frac{\mu\epsilon}{c}\frac{\partial \vec{E}}{\partial t}\end{aligned}$$

where $\vec{E}$ is the vector electric field, $\vec{B}$ is magnetic field. In (3), $\epsilon$ is the dielectric constant and $\mu$ is the magnetic permeability, both functions of $\vec{x}\,''$ in the volume or $\vec{x}\,'$ on the surface. In (3), $\rho$ is the charge density, $\vec{J}$ is the current density, and $\partial/\partial t$ indicates the time derivative.

It is frequently assumed, for example for the plane waves of [13], that $\rho = \vec{J} = 0$ within the volume in Figure 1 or 2, which means that the source of incident waves is either spherical or plane waves originating outside the volume. In the case of plane waves, they are always taken to be generated outside of the volume infinitely far away, as they are an idealization, laterally infinite in space, and have no curvature by which one might locate their origin.

Our derivation of the field at the receiver will assume that the transmitter is within the volume and will show in Section 2.3 how terms change if the transmitter is outside of the volume. The upper surface of the bounded volume can be taken to infinity to make its contribution negligible, but this will be shown to be unnecessary and will not be done. The transmitter's currents or charges occupy a small volume $V_T$ within the integration volume $V$, which contains a transmitter reference coordinate $\vec{x}_T$. Although transmitters usually produce currents, and not charge densities, both $\vec{J}$ and $\rho$ will be allowed to be nonzero within $V_T$ to completely illustrate the nature of the source function.

Taking the curl of the second equation in (3) and using the time derivative of the third for $\vec{\nabla} \times \partial\vec{B}/\partial t$ results in the following inhomogeneous Helmholtz equation, for which the dot product with $\hat{y}$ has been taken on both sides:

$$\begin{aligned} \nabla^2 E_y - \vec{\nabla}_y\left(\vec{\nabla}\cdot\vec{E}\right) &= \frac{1}{c}\left[\frac{4\pi\mu}{c}\dot{J}_y + \frac{\mu\epsilon}{c}\frac{\partial^2 E_y}{\partial t^2}\right] \\ \Longrightarrow \nabla^2 E_y + k^2 E_y &= \frac{4\pi\mu}{c^2}\dot{J}_y + 4\pi\vec{\nabla}_y\frac{\rho}{\epsilon} \end{aligned} \tag{4}$$

where the time dependence of $E_y$ is $\exp[-i\omega t]$, $k^2 = \mu\epsilon\omega^2/c^2$, $\vec{\nabla}_y \equiv \hat{y}\cdot\vec{\nabla}$, and the first line of (3) has been used to generate the term with $\rho$. The dot over the current density means time derivative. The second line of (4) shows that the initial assumption of the transmitted wave being polarized in the $\hat{y}$ direction is equivalent to there being only a $\hat{y}$-component to the time derivative of current density and the gradient of charge density.

By definition, Green's function satisfies

$$\nabla^2 G\left(\vec{x}_R, \vec{x}\,''\right) + k^2 G\left(\vec{x}_R, \vec{x}''\right) \equiv -4\pi\delta^3\left(\vec{x}_R - \vec{x}\,''\right) \tag{5}$$

Substituting the Laplacian of the electric field and that of the Green's function from (4) and (5), respectively, into the volume integral in (2) yields the electric field component as a surface integral and a source function:

$$E_y(\vec{x}_R) = -\int_{V_T} G(\vec{x}_R, \vec{x}\,'') \left[\frac{\mu}{c^2}\dot{J}_y + \vec{\nabla}_y \frac{\rho}{\epsilon}\right] d^3x'' + \frac{1}{4\pi}\int_S \left[E_y(\vec{x}\,')\vec{\nabla}G(\vec{x}_R, \vec{x}\,') - G(\vec{x}_R, \vec{x}')\vec{\nabla}E_y(\vec{x}\,')\right]\cdot \hat{n}\, d^2x' \quad (6)$$

where the first term is the field at the receiver due to the transmitter directly, without scattering, with integrand nonzero only in a small volume, $V_T$, defined by the transmitter current density and charge. The source term inside the square brackets is propagated from the location of the current and charge to the receiver at $\vec{x}_R$ by $G(\vec{x}_R, \vec{x}\,'')$. The first term inside the square brackets is the time derivative of the current density, and the second term, by the continuity equation, can be thought of as characterizing the spatial derivative of $J_y$. As mentioned, it is usually only the $\dot{J}_y$ term which is generated by transmitters, with no charge buildup.

Because the dimensions of $V_T$ in (6) are of order a few meters, and typical distances from transmit to reception points are tens to thousands of kilometers, (6) can be rewritten by explicitly putting in the Green's function and defining a transmitter location, $\vec{x}_T$ within $V_T$, as follows

$$\begin{aligned} E_y(\vec{x}_R) &\approx -\frac{e^{ik|\vec{x}_R-\vec{x}_T|}}{|\vec{x}_R-\vec{x}_T|}\int_{V_T} e^{ik(\vec{x}\,''-\vec{x}_T)\cdot\frac{(\vec{x}_R-\vec{x}_T)}{|\vec{x}_R-\vec{x}_T|}} \left[\frac{\mu}{c^2}\dot{J}_y + \vec{\nabla}_y\frac{\rho}{\epsilon}\right] d^3x'' \\ &\quad + \frac{1}{4\pi}\int_S \left[E_y\left(\vec{x}\,'\right)\vec{\nabla}G(\vec{x}_R, \vec{x}') - G(\vec{x}_R, \vec{x}\,')\vec{\nabla}E_y\left(\vec{x}\,'\right)\right]\cdot\hat{n}\, d^2x' \\ &\equiv E_{y0}(\hat{x}_{RT})\frac{e^{ik|\vec{x}_R-\vec{x}_T|}}{|\vec{x}_R-\vec{x}_T|} \\ &\quad + \frac{1}{4\pi}\int_S \left[E_y(\vec{x}')\,\vec{\nabla}G(\vec{x}_R, \vec{x}\,') - G(\vec{x}_R, \vec{x}\,')\,\vec{\nabla}E_y(\vec{x}\,')\right]\cdot\hat{n} d^2x' \quad (7) \end{aligned}$$

where $\hat{x}_{RT}$ is a unit vector pointing from the transmitter to the receive point, and $E_{y0}(\hat{x}_{RT})$ is defined by (7). From (7), the field at $\vec{x}_R$ is due to the first term, a source directly from the transmitter, and a term from a surface integral. If the source is outside of the integration volume, this first term will be zero, and the transmitted field will show along with the scattered field in the surface fields $E_y\,(\vec{x}\,')$, as described in Subsection 2.3. The next subsection will evaluate the surface fields needed in the integral.

## 2.2. The Kirchhoff Approximation to Evaluate the Surface Integral

The Kirchhoff approximation describes the surface fields needed for insertion in (7) as being due to the incident field plus a reflection, as though from a specular plane at the point $\vec{x}\,'$. In the simplified case where all polarizations are $\hat{y}$, the reflected field propagating along $\vec{k}_{1,ref}$ in Figure 1 will be in the same polarization direction as that incident along $\vec{k}_1$. Given a local reflection coefficient $R_y$ for the $\hat{y}$ polarization, the field at the surface point $\vec{x}\,'$ and its normal derivative needed in the surface integral of (7) are

$$\begin{aligned} E_y\left(\vec{x}\,'\right) &= (1+R_y)E_{y,inc}\left(\vec{x}\,'\right) \\ \vec{\nabla}E_y\left(\vec{x}\,'\right)\cdot\hat{n} &= i\left(\vec{k}_1+R_y\vec{k}_{1,ref}\right)\cdot\hat{n}E_{y,inc}\left(\vec{x}\,'\right) \\ &= i\vec{k}_1\cdot\hat{n}(1-R_y)E_{y,inc}\left(\vec{x}\,'\right) \end{aligned} \tag{8}$$

where $E_{y,inc}(\vec{x}\,')$ is the field at $\vec{x}\,'$ in the absence of any surface scattering, i.e., due solely to the transmitter. The last line of (8) follows from the relationship between $\vec{k}_1$ and $\vec{k}_{1,ref}$ in (1) and from the identification of the first term in (7) as the incident field. That this term is the incident field can be formally demonstrated by evaluating (7) at a particular surface point, $\vec{x}_s$, and inserting fields from (8) into the surface integral with $R_y = 0$:

$$\begin{aligned} E_{y,inc}(\vec{x}_s) &= E_{y0}\frac{e^{ik|\vec{x}_s-\vec{x}_T|}}{|\vec{x}_s-\vec{x}_T|}+\frac{1}{4\pi}\int_S\Big[E_{y,inc}\left(\vec{x}\,'\right)\vec{\nabla}G\left(\vec{x}_s,\vec{x}\,'\right) \\ &\quad -G\left(\vec{x}_s,\vec{x}\,'\right)\vec{\nabla}E_{y,inc}\left(\vec{x}\,'\right)\Big]\cdot\hat{n}d^2x' \\ &= E_{y0}\frac{e^{ik|\vec{x}_s-\vec{x}_T|}}{|\vec{x}_s-\vec{x}_T|}-\frac{1}{4\pi}\int_V\vec{\nabla}\cdot\Big[E_{y,inc}\left(\vec{x}\,'\right)\vec{\nabla}G\left(\vec{x}_s,\vec{x}'\right) \\ &\quad -G\left(\vec{x}_s,\vec{x}'\right)\vec{\nabla}E_{y,inc}\left(\vec{x}\,'\right)\Big]d^3x'' \\ &= E_{y0}\frac{e^{ik|\vec{x}_s-\vec{x}_T|}}{|\vec{x}_s-\vec{x}_T|}-\frac{1}{4\pi}\int_V\Big[E_{y,inc}\left(\vec{x}'\right)\vec{\nabla}^2G\left(\vec{x}_s,\vec{x}'\right) \\ &\quad -G\left(\vec{x}_s,\vec{x}'\right)\vec{\nabla}^2E_{y,inc}\left(\vec{x}\,''\right)\Big]d^3x'' \\ &= E_{y0}\frac{e^{ik|\vec{x}_s-\vec{x}_T|}}{|\vec{x}_s-\vec{x}_T|}-\frac{1}{4\pi}\int_V\Big[E_{y,inc}\left(\vec{x}'\right)(-4\pi)\delta^3(\vec{x}_s-\vec{x}') \\ &\quad -G(\vec{x}_s,\vec{x}')\left(\frac{4\pi\mu}{c^2}\dot{J}_y+4\pi\vec{\nabla}_y\frac{\rho}{\epsilon}\right)\Big]d^3x'' \\ &= E_{y0}\frac{e^{ik|\vec{x}_s-\vec{x}_T|}}{|\vec{x}_s-\vec{x}_T|}-\left[-E_{y,inc}(\vec{x}_s)+E_{y,inc}(\vec{x}_s)\right] \end{aligned}$$

$$\Longrightarrow E_{y,inc}(\vec{x}_s) = E_{y0} \frac{e^{ik|\vec{x}_s - \vec{x}_T|}}{|\vec{x}_s - \vec{x}_T|} \tag{9}$$

where the divergence theorem with inward pointing normals (17) was used in the second equation. In the fourth equation of (9), the Laplacian of the incident field due to the transmitter from the second line of (4) is used along with the definition of the Green's function in (5). In (9), the variation of $E_{y0}$ due to its dependence on the unit vector in the direction $\vec{x}\,' - \vec{x}_s$ is assumed negligible over the surface contributing to the integral. When the field at the surface is taken to be just the incident field due to a source within the volume, with no reflected wave ($R_y = 0$), (9) demonstrates that the surface integral vanishes. Therefore, identifying the incident field as in the last line of (9) is consistent with its being the first term in (7) for the wave received anywhere in the volume. Note that the surface integral with incident fields only vanishes regardless of the shape of the surface or boundary, as long as the sources are within the volume. There is no need to take the upper surface to infinity to make the incident field vanish on the surface.

Because the surface integral in (9) vanishes, following [19], the incident field can be subtracted from $E_y\left(\vec{x}\,'\right)$ in (7), and, using (8), the field at the receiver is

$$\begin{aligned}
E_y(\vec{x}_R) =& E_{y0} \frac{e^{ik|\vec{x}_R - \vec{x}_T|}}{|\vec{x}_R - \vec{x}_T|} + \frac{1}{4\pi} \int_S \Big[ \left( E_y\left(\vec{x}\,'\right) - E_{y,inc}\left(\vec{x}\,'\right)\right) \vec{\nabla} G\left(\vec{x}_R, \vec{x}\,'\right) \\
& - G\left(\vec{x}_R, \vec{x}\,'\right) \vec{\nabla}\left( E_y\left(\vec{x}\,'\right) - E_{y,inc}\left(\vec{x}\,'\right)\right)\Big] \cdot \hat{n}\, d^2x' \\
=& E_{y0} \frac{e^{ik|\vec{x}_R - \vec{x}_T|}}{|\vec{x}_R - \vec{x}_T|} + \frac{1}{4\pi} \int_S \Big[ -i\vec{k}_2 R_y\left(\vec{x}\,'\right)\, E_{y,inc}\left(\vec{x}\,'\right) G(\vec{x}, \vec{x}\,') \\
& - i\vec{k}_1 \left( -R_y\left(\vec{x}\,'\right)\right) E_{y,inc} G\left(\vec{x}, \vec{x}\,'\right)\Big] \cdot \hat{n} d^2x' \\
=& E_{y0} \frac{e^{ik|\vec{x}_R - \vec{x}_T|}}{|\vec{x}_R - \vec{x}_T|} \\
& + \frac{iE_{y0}}{4\pi} \int_S \frac{e^{ik|\vec{x}\,' - \vec{x}_T| + |\vec{x}_R - \vec{x}\,'|}}{|\vec{x}\,' - \vec{x}_T||\vec{x}_R - \vec{x}\,'|} R_y\left(\vec{x}'\right)\left(\vec{k}_1 - \vec{k}_2\right) \cdot \hat{n} d^2x'
\end{aligned} \tag{10}$$

As shown in going from the first equation of (10) to the second, using (1), (8), and (9), the term involving $\vec{k}_2$ in (10) results from taking the gradient of the Green's function in the limit that $|\vec{x}_R - \vec{x}\,'| \gg \lambda$ (ignoring derivatives of the denominators of in the Green's function). The term involving $\vec{k}_1$ comes from the gradient of $E_y$ as in (8), again ignoring derivatives of the denominator in the electric field. The term in the last line of (10) agrees with one of the terms in (18a) of [13], but that reference also has a term dependent on $\vec{k}_1 + \vec{k}_2$. As can be demonstrated by taking the derivatives of the surface $E_{y,inc}$ and $G$ in the first equation in (9), this $\vec{k}_1 + \vec{k}_2$ term in [13] derives from considering only incident ($R_y = 0$) fields in the surface integral, and evaluating the field at the receiver, i.e., $\vec{x}_s \to \vec{x}_R$, due to those incident fields. As mentioned above, this term is zero for sources producing spherical waves within the volume, but is nonzero for plane or spherical waves originating outside the volume, as indicated in the next subsection.

### 2.3. Changes in the Formalism for Transmitted Sources Outside of the Integration Volume

The following differences should be noted when considering plane waves (or spherical waves) originating outside of the volume:

1) The right side of the second line of (4) is zero, as there is no current or charge within the volume.
2) The source term, the first term to the right of the equal sign in (6), (7), and (9), is zero and is no longer the incident field, and the total field $E_y(\vec{x}_R)$ is given by the surface integral only.
3) The second term in the integral in the fourth line of (9), involving currents and charge densities, is zero.
4) The entire surface integral in (9) is therefore no longer zero as it was in the previous section. Instead it is the incident, transmitted plane-wave field itself at $\vec{x}_s$, or the spherical-wave field originating outside of the volume.
5) The first term of (10) is zero (as in (6), (7), and (9)). The term in the denominator of the surface integral, $|\vec{x}\,' - \vec{x}_T|$ will only appear if the source is spherical waves outside the volume. If the source is plane waves, this term will not appear.

The $\vec{k}_1 + \vec{k}_2$ term in [13] is the incident-field surface integral of (9) evaluated at the receiver, as $\vec{x}_s \to \vec{x}_R$, (with $R_y = 0$). As in (9), if [13] had used a source inside the volume, this extra integral would have been zero. Since they used plane waves, this integral is the incident field propagated directly to the receiver, and the entire expression is misidentified as the "scattered field". Instead, it is the incident plus the scattered field. As in the second line of (10), the term $E_{y,inc}\left(\vec{x}\,'\right)$ should have been subtracted from $E_y\left(\vec{x}\,'\right)$ in the integral to arrive at the scattered field only.

Thus the last line of (10) is the correct form for the scattered field for sources either inside or outside the integration volume. For sources inside the volume, subtracting $E_{y,inc}(\vec{x}\,')$ in the second line of (10), as is done by [19], removes a term equal to zero,which would be confusing if it were retained. For sources outside the volume, subtracting $E_{y,inc}(\vec{x}\,')$ removes the incident field propagated directly to the receiver. In the next section on the vector generalization of (10) and for the rest of the paper, the source is taken to be inside the integration volume.

## 3. RECEIVED FIELD FOR A VECTOR WAVE

In this section, the field received at any specified polarization, $\hat{p}_{rec}$, is derived in terms of surface properties and an arbitrary transmit polarization, $\hat{p}_{inc}$. The polarization vectors are no longer taken to be in the $\hat{y}$ direction, and the surface is arbitrary, i.e., no longer varying in only the $x$-direction. This means, for example, that a horizontally transmitted ($\hat{H}_{inc}$, see Figure 2) wave can have receive components in both the $\hat{H}_{rec}$ and the vertical ($\hat{V}_{rec}$) directions.

The derivation starts by noting that the relationship between volume and surface integrals in (2) is true for any two scalar functions. The $y$-component of the electric field can then be replaced by the component in an arbitrary polarization direction corresponding to the receive polarization, $\hat{p}_{rec}$:

$$\begin{aligned}&\int_V \left[G\left(\vec{x}_R,\vec{x}\,''\right)\nabla^2\left(\vec{E}\left(\vec{x}\,''\right)\cdot\hat{p}_{rec}\right)-\vec{E}\left(\vec{x}\,''\right)\cdot\hat{p}_{rec}\nabla^2 G\left(\vec{x}_R,\vec{x}\,''\right)\right]d^3x''\\&=\int_S\left[\vec{E}\left(\vec{x}\,'\right)\cdot\hat{p}_{rec}\vec{\nabla}G\left(\vec{x}_R,\vec{x}\,'\right)-G\left(\vec{x}_R,\vec{x}\,'\right)\vec{\nabla}\left(\vec{E}\left(\vec{x}\,'\right)\cdot\hat{p}_{rec}\right)\right]\cdot\hat{n}\,d^2x' \quad (11)\end{aligned}$$

The field received with polarization $\hat{p}_{rec}$ results from (7) with the $y$ component of the electric field replaced by its component in the $\hat{p}_{rec}$ direction:

$$\begin{aligned}&\vec{E}(\vec{x}_R)\cdot\hat{p}_{rec}\approx-\frac{e^{ik|\vec{x}_R-\vec{x}_T|}}{|\vec{x}_R-\vec{x}_T|}\int_{V_T}e^{ik(\vec{x}\,''-\vec{x}_T)\cdot\frac{(\vec{x}_R-\vec{x}_T)}{|\vec{x}_R-\vec{x}_T|}}\left[\frac{\mu}{c^2}\dot{\vec{J}}+\vec{\nabla}\frac{\rho}{\epsilon}\right]\cdot\hat{p}_{rec}\,d^3x''\\&+\frac{1}{4\pi}\int_S\left[\vec{E}\left(\vec{x}\,'\right)\cdot\hat{p}_{rec}\vec{\nabla}G\left(\vec{x}_R,\vec{x}\,'\right)-G\left(\vec{x}_R,\vec{x}\,'\right)\vec{\nabla}\left(\vec{E}\left(\vec{x}\,'\right)\cdot p_{rec}\right)\right]\cdot\hat{n}\,d^2x'\\&\equiv\vec{E}_0(\hat{x}_{RT})\cdot\hat{p}_{rec}\frac{e^{ik|\vec{x}_R-\vec{x}_T|}}{|\vec{x}_R-\vec{x}_T|}+\frac{1}{4\pi}\int_S\left[\vec{E}\left(\vec{x}\,'\right)\cdot\hat{p}_{rec}\vec{\nabla}G\left(\vec{x}_R,\vec{x}\,'\right)\right.\\&\left.-G(\vec{x}_R,\vec{x}\,')\vec{\nabla}\left(\vec{E}\left(\vec{x}\,'\right)\cdot\hat{p}_{rec}\right)\right]\cdot\hat{n}\,d^2x' \qquad (12)\end{aligned}$$

where $\vec{E}_0$ is the negative of the integral involving transmitter currents and charge gradients in the first line of (12), with $\hat{p}_{rec}$ factored out of the integral.

In order to evaluate the surface fields in (12), the vector field incident on a surface point (in the absence of surface effects themselves) is needed. By the same arguments as in (9), the surface integral in (12) vanishes if the incident field only is inserted for the surface fields, keeping in mind that we treat only sources within the volume in this section. The vector incident field used below to calculate fields to insert in (12) is therefore the first term in (12) dotted into $\hat{p}_{rec}$ evaluated at $\vec{x}\,'$

$$\vec{E}_{inc}\left(\vec{x}\,'\right) = \vec{E}_0 \frac{e^{ik|\vec{x}\,' - \vec{x}_T|}}{|\vec{x}\,' - \vec{x}_T|} \equiv A_{inc}(\vec{x}')\hat{p}_{inc} \tag{13}$$

where the incident polarization $\hat{p}_{inc}$ is the unit-vector direction of $\vec{E}_0$, determined by the currents and charges associated with the transmitter, and $A_{inc}$ is defined on the right side of (13) for use in (14) below.

Generalizing (8), a vector Kirchhoff approximation is assumed in which the fields and normal derivatives at the surface are based on the vector sum of the incident fields and those specularly reflected by the facet. The approach to calculating the surface fields is to project an arbitrary incoming $\hat{p}_{inc}$ polarization, which is usually expressed in the $\hat{H}_{inc} - \hat{V}_{inc}$ transmit-specific basis in Figure 2, onto local, facet-specific horizontal $\hat{h}_l$ and vertical $\hat{v}_{l_{inc}}$ polarization vectors. The usual Fresnel coefficients for horizontal $(R_h)$ and vertical $(R_v)$

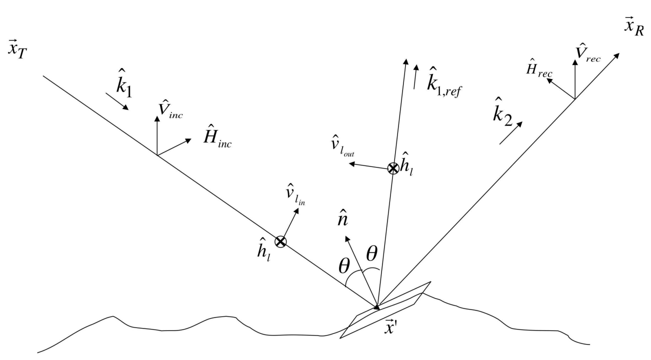


**Figure 2.** A local facet is shown tilted in an arbitrary direction on the ocean's surface, with normal $\hat{n}$. $\hat{k}_1$ is the unit vector along the incident direction to the point on the surface $\vec{x}'$ from the transmitter at $\vec{x}_T$. The transmit and receive polarization coordinate systems $(\hat{H}_{inc}, \hat{V}_{inc})$ and$(\hat{H}_{rec}, \hat{V}_{rec})$ are shown as rotated from the local-facet coordinate systems, $(\hat{h}_l, \hat{v}_{l_{in}})$ and $(\hat{h}_l, \hat{v}_{l_{out}})$. $\hat{k}_2$ is the direction of propagation of the received, scattered field, and $\theta$ is the angle between $\hat{k}_1$ and $\hat{n}$.

polarization are applied [11]. The outgoing specularly reflected field has polarization $\hat{p}_{ref}$, determined from the incoming polarization and the Fresnel coefficients. The outgoing $\hat{p}_{ref}$ is projected onto the facet-specific, specular outgoing polarizations for horizontal ($\vec{h}_l$ — equal to incoming) and vertical ($\vec{v}_{l_{out}}$). The quantity $R\hat{p}_{ref}$ defined in (14) below is then the vector field at the surface due to local specular reflection by the facet. The surface fields and normal derivatives for insertion in (12) are (also see [15–17])

$$\begin{aligned} \vec{E}\left(\vec{x}\,'\right) =& A_{inc}\hat{p}_{inc} + RA_{inc}\,\hat{p}_{ref} \\ (\hat{n}\cdot\vec{\nabla})\vec{E}\left(\vec{x}\,'\right) =& i\vec{k}_1\cdot\hat{n}A_{inc}\left(\hat{p}_{inc} - R\hat{p}_{ref}\right) \\ \text{where } R\hat{p}_{ref}\left(\vec{x}\,'\right) \equiv& (\hat{p}_{inc}\cdot\hat{h}_l(\vec{x}'))R_h(\theta\left(\vec{x}\,'\right))\hat{h}_l\left(\vec{x}\,'\right) \\ &+ \left(\hat{p}_{inc}\cdot\hat{v}_{l_{inc}}\left(\vec{x}\,'\right)\right)R_v\left(\theta\left(\vec{x}'\right)\right)\hat{v}_{l_{out}}\left(\vec{x}\,'\right) \end{aligned} \quad (14)$$

where $A_{inc}$ is from (13), and, as in Figure 2, $\theta$ is the angle between the normal at $\vec{x}\,'$ and the incident $\vec{k}_1$ vector (with unit vector $\hat{k}_1$). In the third line of (14), the incident polarization, $\hat{p}_{inc}$, is expressed in terms of its components along the local-facet horizontal and vertical polarization vectors as described above. In Figure 2, the facet's specular scattering plane is now in the plane of the paper, the incoming and reflected $\hat{h}_l$ horizontal polarization vectors are equal and perpendicular to the paper, and the local incoming wave along $\hat{k}_1$ is shown specularly reflected by the facet into a wave propagating along $\hat{k}_{1,ref}$. The local polarization vectors in (14) are dependent on the unit incident vector $\hat{k}_1$ and the angle between $\hat{k}_1$ and $\hat{n}$, $\theta$, as follows:

$$\begin{aligned} &\hat{h}_l\left(\vec{x}\,'\right) = \frac{\hat{n}\left(\vec{x}\,'\right)\times\hat{k}_1(\vec{x}\,')}{\sin\theta\left(\vec{x}\,'\right)} \\ &\hat{v}_{l_{inc}}(\vec{x}\,') = \hat{k}_1\times\hat{h}_l\left(\vec{x}\,'\right) = \hat{k}_1\times\frac{\hat{n}(\vec{x}\,')v\times\hat{k}_1}{\sin\theta\left(\vec{x}\,'\right)} = \frac{\hat{n}(\vec{x}\,') - \hat{k}_1\left(\hat{k}_1\cdot\hat{n}(\vec{x}\,')\right)}{\sin\theta(\vec{x}')} \\ &\hat{v}_{l_{out}}(\vec{x}\,') = \hat{k}_{1,ref}\left(\vec{x}'\right)\times\hat{h}_l\left(\vec{x}\,'\right) \equiv \left(\hat{k}_1 - 2\left(\hat{k}_1\cdot\hat{n}\left(\vec{x}\,'\right)\right)\hat{n}(\vec{x}\,')\right)\times\hat{h}_l\left(\vec{x}\,'\right) \end{aligned} \quad (15)$$

Subtracting the incident field in (13) from the fields in the surface integral in (12), as in the scalar case, and using (14) and (15) gives the vector generalization of (10):

$$\begin{aligned} \vec{E}(\vec{x}_R)\cdot\hat{p}_{rec} = \vec{E}_0\cdot\hat{p}_{rec}\frac{e^{ik|\vec{x}_R-\vec{x}_T|}}{|\vec{x}_R-\vec{x}_T|} + \frac{i}{4\pi}\int_S \frac{A_{inc}(\vec{x}')e^{ik|\vec{x}_R-\vec{x}\,'|}}{|\vec{x}_R-\vec{x}'|} \\ \left(R\hat{p}_{ref}\left(\vec{x}\,'\right)\cdot\hat{p}_{rec}\right)\left(\vec{k}_1(\vec{x}\,') - \vec{k}_2\left(\vec{x}\,'\right)\right)\cdot\hat{n}\left(\vec{x}'\right)d^2x' \end{aligned} \quad (16)$$

Equation (16) is the principal formula of this paper. It is a simple vector formulation of ocean-surface scattering, given a set of $\hat{n}$ vectors and local reflection properties characterizing the surface, and a transmitter located within the integration volume. The changes in (16) required if the transmitter is outside of the integration volume are the same as those for the scalar case discussed in Subsection 2.3. As mentioned in the introduction, [18] shows and uses the rightmost term of (12) as a useful alternative to the Stratton-Chu equation. But this reference does not directly demonstrate the equivalence to the Stratton-Chu equation, as in the next section. Also, [18] does not use the tangent plane approximation or show the coordinate system of (15), nor the complete expression of (16). References [15–17] do show the tangent plane coordinate system but do not use the surface integral of (16); instead they use the more complicated Stratton-Chu equation. Reference [2] shows an essentially scalar formulation, and though Fresnel reflection coefficients are mentioned, the coordinate system of (15) is not mentioned. In [2], the "stationary phase" approximation appears to be used, which considers only facets oriented so that specular rays point from the transmitter to the surface, then from the surface to the receiver, whereas (16) of this paper considers all possible orientations of facets. Thus, aside from the "stationary phase" assumption of [2], which may give slightly different numerical answers than (16), other presentations of scattering equations should be numerically equivalent to (16). The contribution of (16) is that it is operationally simpler to use than Stratton-Chu based approaches, that it has been derived with attention to source functions, explicitly proving (in the scalar case) the nature of the surface integral when only source fields are incident, and that the Stratton-Chu equivalence is rigorously established, in the next section.

## 4. EQUIVALENCE OF VECTOR WAVE EXPRESSION TO STRATTON-CHU EQUATION

This section shows the equivalence of (12), from which (16) is derived, to the Stratton-Chu equation [14] . The Stratton-Chu equation, with a source term, will first be shown, then the surface integral in (12) will be shown to be equal to the Stratton-Chu equation.

The following 3 mathematical identities will be used for vectors or vector fields $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$, and for any scalar fields $a$ and $b$, where the product of the scalars goes to zero at the extremum of the volume (e.g., on a bounding surface):

$$-\int \vec{A} \cdot \hat{n}\, d^2x' = \int \vec{\nabla} \cdot \vec{A}\, d^3$$

$$\begin{aligned}&\int_V \vec{\nabla}(ab)\,d^3x'' = 0 = \int_V (a\vec{\nabla}b + b\vec{\nabla}a)\,d^3x'' \\ &\vec{A}\times\left(\vec{B}\times\vec{C}\right)\cdot\vec{D} = (\vec{A}\cdot\vec{C})\left(\vec{B}\cdot\vec{D}\right) - \left(\vec{A}\cdot\vec{B}\right)(\vec{C}\cdot\vec{D})\end{aligned} \tag{17}$$

where the minus sign on the first line for the divergence theorem is because all unit normals in this paper point inward.

### 4.1. The Stratton-Chu Equation with a Source Term

If current densities and charge gradients are considered, [14] shows that

$$\begin{aligned}\vec{E}(\vec{x}_R) =& \int_V \left[-G\frac{\mu\dot{\vec{J}}}{c^2} + \frac{\rho}{\epsilon}\vec{\nabla}G\right] d^3x'' \\ &+1/4\pi \int_S \left[\left(\hat{n}\times\vec{E}\right)\times\vec{\nabla}G + \left(\hat{n}\times\frac{i\omega}{c}\vec{B}\right)G + \left(\vec{E}\cdot\hat{n}\right)\vec{\nabla}G\right] d^2x'\end{aligned} \tag{18}$$

where the surface term is usually considered the "Stratton-Chu" equation. Noting that $\rho/\epsilon \to 0$ outside of the transmitter volume, using the second line of (17) on the term with $\rho$ shows that the first integral is identical to the source term in (6), generalized for arbitrary components as in (12).

### 4.2. The Surface Stratton-Chu Equation Surface Integrals

Turning now to the surface integral in (18), first take the dot product of both sides of (18) with the receive polarization, $\hat{p}_{rec}$. The integrand of the first surface integral term in (18) can be rewritten, using the third line of (17), as

$$\begin{aligned}\left(\left(\hat{n}\times\vec{E}\right)\times\vec{\nabla}G\right)\cdot\hat{p}_{rec} &= \vec{\nabla}G\times\left(\vec{E}\times\hat{n}\right)\cdot\hat{p}_{rec} \\ &= \hat{n}\cdot\vec{\nabla}G\left(\vec{E}\cdot\hat{p}_{rec}\right) - \left(\vec{\nabla}G\cdot\vec{E}\right)(\hat{n}\cdot\hat{p}_{rec})\end{aligned} \tag{19}$$

The integral of the second term of (19) can be written as a volume integral using the first line of (17) and can be shown to be

$$\begin{aligned}-\int\left(\vec{\nabla}G\cdot\vec{E}\right)(\hat{p}_{rec}\cdot\hat{n})\,d^2x' &= \int\vec{\nabla}\cdot\left(\vec{\nabla}G\cdot\vec{E}\right)\hat{p}_{rec}d^3x'' \\ =\int\Big[\vec{\nabla}\cdot\left[\left(\hat{p}_{rec}\cdot\vec{\nabla}G\right)\vec{E}\right] &- \left(\vec{\nabla}\cdot\vec{E}\right)\left(\vec{\nabla}G\cdot\hat{p}_{rec}\right) + \vec{\nabla}G\cdot\left[\left(\hat{p}_{rec}\cdot\vec{\nabla}\right)\vec{E}\right]\Big]d^3x''\end{aligned} \tag{20}$$

The second line of (20) was expanded from the first by manipulating indices of differential operators, as shown in the Appendix. The first surface integral term of (18) is therefore

$$\int\left(\left(\hat{n}\times\vec{E}\right)\times\vec{\nabla}G\right)\cdot\hat{p}_{rec}d^2x' = \int\hat{n}\cdot\vec{\nabla}G\left(\vec{E}\cdot\hat{p}_{rec}\right)d^2x'$$

$$
-\int \left(\hat{p}_{rec} \cdot \vec{\nabla} G\right) \left(\vec{E} \cdot \hat{n}\right) \, d^2x' - \int \left(\vec{\nabla} \cdot \vec{E}\right) \left(\vec{\nabla} G \cdot \hat{p}_{rec}\right) d^3x''
+ \int \vec{\nabla} G \cdot \left[\left(\hat{p}_{rec} \cdot \vec{\nabla}\right) \vec{E}\right] d^3x'' \tag{21}
$$

where the second term to the right of equal sign derived from the first term on the right of (20) by the divergence theorem (first line of (17)). By using the second Maxwell equation in (3), expressing (18) only in terms of the electric field, the second surface integrand in (18) can be written as

$$
(\hat{n} \times \vec{\nabla} \times \vec{E}) G \cdot \hat{p}_{rec} = [G(\hat{p}_{rec} \cdot \vec{\nabla})\vec{E}] \cdot \hat{n} - G\hat{n} \cdot \vec{\nabla}(\vec{E} \cdot \hat{p}_{rec}) \tag{22}
$$

By changing the surface integral for the first integrand in (22) to a volume integral, as was done in (20) using the first line of (17), the surface integral of the left side of (22) can be written as

$$
\begin{aligned}
&\int \left(\hat{n} \times \vec{\nabla} \times \vec{E}\right) G \cdot \hat{p}_{rec} \, d^2x' \\
&= -\int \vec{\nabla} \cdot \left[G \left(\hat{p}_{rec} \cdot \vec{\nabla}\right) \vec{E}\right] d^3x'' - \int G\hat{n} \cdot \vec{\nabla} \left(\vec{E} \cdot \hat{p}_{rec}\right) \, d^2x' \\
&= -\int \vec{\nabla} G \cdot \left(\hat{p}_{rec} \cdot \vec{\nabla}\right) \vec{E} d^3x'' - \int G\hat{p}_{rec} \cdot \vec{\nabla} \left(\vec{\nabla} \cdot \vec{E}\right) \, d^3x'' \\
&\quad -\int G\hat{n} \cdot \vec{\nabla} \left(\vec{E} \cdot \hat{p}_{rec}\right) \, d^2x' \\
&= -\int \vec{\nabla} G \cdot \left(\hat{p}_{rec} \cdot \vec{\nabla}\right) \vec{E} \, d^3x'' + \int \vec{\nabla} G \cdot \hat{p}_{rec} \left(\vec{\nabla} \cdot \vec{E}\right) \, d^3x'' \\
&\quad -\int G\hat{n} \cdot \vec{\nabla} \left(\vec{E} \cdot \hat{p}_{rec}\right) d^2x'
\end{aligned} \tag{23}
$$

where the third line of (23) uses the second line of (17).

Adding the first surface integral in (18), (dotted into $\hat{p}_{rec}$) from (21), to the second surface integral as in (22), to the third surface integral in (18) yields, for the Stratton-Chu surface integrals

$$
\begin{aligned}
&\int_S \left[\left[\left(\hat{n} \times \vec{E}\right) \times \vec{\nabla} G\right] \cdot \hat{p}_{rec} + \left(\hat{n} \times \frac{i\omega}{c} \vec{B}\right) G \cdot \hat{p}_{rec} + \left(\vec{E} \cdot \hat{n}\right) \vec{\nabla} G \cdot \hat{p}_{rec}\right] d^2x' \\
&= \int_S \left[\hat{n} \cdot \vec{\nabla} G \left(\vec{E} \cdot \hat{p}_{rec}\right) - G\hat{n} \cdot \vec{\nabla} \left(\vec{E} \cdot \hat{p}_{rec}\right)\right] d^2x'
\end{aligned} \tag{24}
$$

which is the same as the surface integral in (12) from which (16) is derived. The equivalence of the surface integrals and the source terms demonstrates the equivalence of the Stratton-Chu equation with a transmitter source, and the vector wave formulation of (16). For insight into why the two should be equivalent, Appendix B shows that the vector generalization of Green's theorem [14], used as the starting point to derive the Stratton-Chu equation, is mathematically identical to the scalar version of Green's theorem using field components as in (11) or [13]. The expressions for surface fields in (14) and (15) would be used in both approaches, but for the vector wave formulation here, the normal derivatives of (14) are also needed. While for the Stratton-Chu approach, cross products and curls of the surface fields are needed.

## 5. CONCLUSION

A scalar expression (10) for the scattering of scalar waves off a 1-dimensional ocean by GNSS waves was described using the Kirchhoff tangent-plane scattering approximation and considering the location of source terms. It was shown that the transmitted incident field appears differently in the final equations for the incident and scattered field, depending on whether the transmitter is inside or outside of the integration volume. A simple expression for the vector, polarimetric scattering of a GNSS wave off a rough ocean surface is presented in (16). This vector wave expression is useful to evaluate the polarimetric response of GNSS signals to ocean waves, if the surface topography (surface normals) and local reflection coefficients are known via modeling and simulation, or via data. It requires vector electric fields and their normal derivatives at the ocean surface. The expression will be simpler to use in most coding applications than the Stratton-Chu equation, which requires fields, cross products of fields, and their curls. The equivalence was demonstrated between the vector wave expression derived here in (16) and the Stratton-Chu equation with a transmitting source term.

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## APPENDIX A. THE SECOND LINE OF (19)

The second line of (19) is obtained from the first as follows:

$$\begin{aligned}
&\int \vec{\nabla}\cdot\left(\vec{\nabla}G\cdot\vec{E}\right)\hat{p}_{rec}\,d^3x'' = \int \frac{\partial}{\partial x_i}\left[\frac{\partial G}{\partial x_j}E_j\right]\hat{p}_{rec_i}\,d^3x'' \\
&= \int\left[\frac{\partial^2 G}{\partial x_i\partial x_j}E_j\hat{p}_{rec_i} + \frac{\partial G}{\partial x_j}\hat{p}_{rec_i}\frac{\partial E_j}{\partial x_i}\right]d^3x'' \\
&= \int\left[E_j\frac{\partial}{\partial x_j}\left(\hat{p}_{rec}\cdot\vec{\nabla}G\right) + \vec{\nabla}G\cdot\left(\hat{p}_{rec}\cdot\vec{\nabla}\right)\vec{E}\right]d^3x'' \\
&= \int\left[\frac{\partial E_j\left(\hat{p}_{rec}\cdot\vec{\nabla}G\right)}{\partial x_j} - \left(\hat{p}_{rec}\cdot\vec{\nabla}G\right)\frac{\partial E_j}{\partial x_j} + \vec{\nabla}G\cdot\left(\hat{p}_{rec}\cdot\vec{\nabla}\right)\vec{E}\right]d^3x'' \\
&= \int\left[\vec{\nabla}\cdot\left(\hat{p}_{rec}\cdot\vec{\nabla}G\right)\vec{E} - \left(\vec{\nabla}\cdot\vec{E}\right)\left(\hat{p}_{rec}\cdot\vec{\nabla}G\right) + \vec{\nabla}G\cdot\left(\hat{p}_{rec}\cdot\vec{\nabla}\right)\vec{E}\right]d^3x'' \qquad \text{(A1)}
\end{aligned}$$

where repeated indices are summed.

## APPENDIX B. THE EQUIVALENCE OF THE VECTOR GENERALIZATION OF GREEN'S THEOREM AND THE SCALAR GREEN'S THEOREM

This appendix shows that the vector generalization of Green's Theorem applied to vector fields is mathematically identical to the scalar version applied to a field component and any other scalar function, as in (11). The vector generalization starts by using the divergence theorem (17) for any arbitrary vector fields $\vec{P}$ and $\vec{Q}$ [14]:

$$\begin{aligned}
&\int_V \vec{P}\cdot\left(\vec{\nabla}\times\vec{\nabla}\times\vec{Q}\right) - \vec{Q}\cdot\left(\vec{\nabla}\times\vec{\nabla}\times\vec{P}\right)d^3x'' \\
&= \int_S\left(\vec{P}\times\vec{\nabla}\times\vec{Q} - \vec{Q}\times\vec{\nabla}\times\vec{P}\right)\cdot\hat{n}\,d^2x' \qquad \text{(B1)}
\end{aligned}$$

where the surface normal $\hat{n}$ is pointing inward, and $\vec{P}$ and $\vec{Q}$ are arbitrary vector functions. Using the expression for the curl of the curl, as was done to derive (4), and allowing $\vec{Q} \to Q\vec{a}$ where $\vec{a}$ is an arbitrary vector in the direction of $\vec{Q}$, yields

$$\begin{aligned}
&\int_V \vec{P}\cdot\left[\vec{\nabla}\left(\vec{a}\cdot\vec{\nabla}Q\right) - \nabla^2 Q\vec{a}\right] - Q\left[\left(\vec{a}\cdot\vec{\nabla}\right)\left(\vec{\nabla}\cdot\vec{P}\right) - \nabla^2\vec{P}\cdot\vec{a}\right]d^3x'' \\
&= \int_S\left[\left(\left(\vec{P}\cdot\vec{a}\right)\vec{\nabla}Q - \left(\vec{P}\cdot\vec{\nabla}\right)Q\vec{a}\right) - \left(Q\vec{\nabla}\left(\vec{P}\cdot\vec{a}\right) - Q\left(\vec{a}\cdot\vec{\nabla}\right)\vec{P}\right)\right]\cdot\hat{n}d^2x'' \quad \text{(B2)}
\end{aligned}$$

Using the divergence theorem (17) again on two terms in (B2),

$$\begin{aligned}
&\int_S \left[Q\left(\vec{a}\cdot\vec{\nabla}\right)\vec{P}-\left(\vec{P}\cdot\vec{\nabla}\right)Q\vec{a}\right]\cdot\hat{n}\,d^2x' \\
&=\int_V \left[\vec{\nabla}\cdot\left(\vec{P}\cdot\vec{\nabla}\right)Q\vec{a}-\vec{\nabla}\cdot Q\left(\vec{a}\cdot\vec{\nabla}\right)\vec{P}\right]d^3x'' \\
&=\int_V \left[\vec{P}\cdot\vec{\nabla}\left(\vec{a}\cdot\vec{\nabla}Q\right)-Q\left(\vec{a}\cdot\vec{\nabla}\right)\left(\vec{\nabla}\cdot\vec{P}\right)\right]d^3x'' \qquad \text{(B3)}
\end{aligned}$$

Applying (B3) to (B2) gives

$$\begin{aligned}
&\int_V \left[Q\nabla^2\left(\vec{P}\cdot\vec{a}\right)-\left(\vec{P}\cdot\vec{a}\right)\nabla^2 Q\right]d^3x'' \\
&=\int_S \left[\left(\vec{P}\cdot\vec{a}\right)\vec{\nabla}Q-Q\vec{\nabla}\left(\vec{P}\cdot\vec{a}\right)\right]\cdot\hat{n}d^2x' \qquad \text{(B4)}
\end{aligned}$$

which is the scalar Green's theorem applied to $Q$ and $\vec{P}\cdot\vec{a}$. Thus the vector generalization of Green's theorem which leads to the Stratton-Chu equation is in every way equivalent to the scalar Green's theorem of (11) — with $\vec{P}=\vec{E}$, $\vec{a}=\hat{p}_{rec}$, and $Q=G$ — which leads to the simpler (16).

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